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Uniform non-ℓ_1^n-ness of ℓ_1-sums of Banach spaces

Abstract. We shall characterize the uniform non-ℓ_1^n-ness of the ℓ_1-sum $(X_1 \oplus \cdots \oplus X_m)_1$ of a finite number of Banach spaces $X_1, \cdots, X_m$. Also we shall obtain that $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^{m+1} if and only if all $X_1, \ldots, X_m$ are uniformly non-square (note that $(X_1 \oplus \cdots \oplus X_m)_1$ is not uniformly non-ℓ_1^m). Several related results will be presented.

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1. Introduction and Preliminaries. As it is well known, the ℓ_1-sum $X \oplus_1 Y$ cannot be uniformly non-square for any Banach spaces X and Y (recall that the 2-dimensional space ℓ_1^2, which is regarded as a subspace of $X \oplus_1 Y$, is not uniformly non-square; see also [13]). In the recent paper Kato-Saito-Tamura [15] it is shown that $X \oplus_1 Y$ is uniformly non-ℓ_1^3 (uniformly non-octahedral) if and only if X and Y are uniformly non-square.

In the present paper we shall show that for a finite number of Banach spaces $X_1, \cdots, X_m$ the ℓ_1-sum $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^n if and only if there exist positive integers $n_1, \cdots, n_m$ with $n_1 + \cdots + n_m = n-1$ such that X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$ (Theorem 2.3), where we shall use the sharp triangle inequality with n elements presented recently in [14] (see also [17]). This yields the previous result for $m = 2$ in [15] with a much simpler proof. Theorem 2.3 also implies the fundamental fact that the ℓ_1-sum $(X_1 \oplus \cdots \oplus X_m)_1$ of m Banach spaces $X_1, \cdots, X_m$ cannot be uniformly non-ℓ_1^m.

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In Theorem 2.6 we shall obtain that $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^{m+1} if and only if $X_1, \cdots, X_m$ are all uniformly non-square. This extends the above-mentioned result concerning the uniform non-ℓ_1^3-ness of $X \oplus_1 Y$.

It is known that uniformly non-square spaces are super-reflexive ([11]), while uniformly non-ℓ_1^3 spaces need not to be so (cf. [10, 11]). Our Corollary 2.8 asserts that any uniformly non-ℓ_1^n space which is (isomorphic to) an ℓ_1-sum of $(n-1)$ Banach spaces, $n \geq 3$, is super-reflexive. Finally we shall consider the fixed point property. Recently Garcia-Falset et al. [6] showed that all uniformly non-square Banach spaces have the fixed point property for nonexpansive mappings. Our final result, Corollary 2.9 states that if an ℓ_1-sum $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^{m+1}, all $X_1, \cdots, X_m$ have the fixed point property.

Now we shall recall some definitions and preliminary results. A Banach space X is said to be *uniformly non-ℓ_1^n* (cf. [2, 16]) provided there exists ϵ $(0 < \epsilon < 1)$ such that for any $x_1, \cdots, x_n \in S_X$, the unit sphere of X, there exists an n-tuple of signs $\theta = (\theta_j)$ for which

$$\left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1-\epsilon) : \tag{1}$$

Here one can take $x_1, \cdots, x_n$ from the unit ball B_X of X instead of the unit sphere S_X (this is readily seen by Lemma 2.1 below; see [14, Corollary 4]). When $n = 2$, resp. $n = 3$, X is called *uniformly non-square* ([10]; cf. [2, 16]), resp. *uniformly non-octahedral* ([12]). Though we can consider the case $n = 1$ formally, no Banach space is uniformly non-ℓ_1^1. The following fact was proved in Brown [3] (see also Hudzik [8])

Proposition 1.1 ([3, 8]) *If a Banach space X is uniformly non-ℓ_1^n, X is uniformly non-ℓ_1^{n+1} for every $n \geq 2$.*

A Banach space X is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive self-mapping of a nonempty bounded closed convex subset of X has a fixed point.

2. Uniform non-ℓ_1^n-ness.

We begin with a couple of lemmas. First we shall state the sharp triangle inequality (and its reverse one) presented in [14] which will be powerful in the geometry of Banach spaces (see [17] for a stronger version).

Lemma 2.1 (Kato-Saito-Tamura [14]) *For all nonzero elements $x_1, x_2, \ldots, x_n$ in a Banach space X*

$$\begin{aligned} &\left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leq j \leq n} \|x_j\| \\ \leq\ & \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\| + \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \max_{1 \leq j \leq n} \|x_j\|. \end{aligned} \tag{2}$$

Lemma 2.2 *Let $\{x_1^{(k)}\}_k, \ldots, \{x_n^{(k)}\}_k$ be n sequences in a Banach space X for which $\{\|x_j^{(k)}\|\}_k$ converges for every $1 \le j \le n$. Let m be a positive integer with $m < n$. Then*

$$\lim_{k\to\infty}\left\|\sum_{j=1}^{n} x_j^{(k)}\right\| = \lim_{k\to\infty}\sum_{j=1}^{n}\|x_j^{(k)}\| \quad \textit{implies} \quad \lim_{k\to\infty}\left\|\sum_{j=1}^{m} x_j^{(k)}\right\| = \lim_{k\to\infty}\sum_{j=1}^{m}\|x_j^{(k)}\|.$$

Further, if $\|x_j^{(k)}\| \le 1$ for all $1 \le j \le n$ and $k \ge 1$

$$\lim_{k\to\infty}\left\|\sum_{j=1}^{n} x_j^{(k)}\right\| = n \quad \textit{implies} \quad \lim_{k\to\infty}\left\|\sum_{j=1}^{m} x_j^{(k)}\right\| = m.$$

Proof Assume that $\lim_{k\to\infty}\|\sum_{j=1}^{n} x_j^{(k)}\| = \lim_{k\to\infty}\sum_{j=1}^{n}\|x_j^{(k)}\|$. Then since

$$\left\|\sum_{j=1}^{n} x_j^{(k)}\right\| \quad \le \quad \left\|\sum_{j=1}^{m} x_j^{(k)}\right\| + \sum_{j=m+1}^{n}\|x_j^{(k)}\| \le \sum_{j=1}^{n}\|x_j^{(k)}\|,$$

we have $\lim_{k\to\infty}\|\sum_{j=1}^{m} x_j^{(k)}\| = \lim_{k\to\infty}\sum_{j=1}^{m}\|x_j^{(k)}\|$. Next assume that $\|x_j^{(k)}\| \le 1$ for all $1 \le j \le n$, $k \ge 1$ and $\lim_{k\to\infty}\|\sum_{j=1}^{n} x_j^{(k)}\| = n$. Then $\lim_{k\to\infty}\|\sum_{j=1}^{n} x_j^{(k)}\| = \lim_{k\to\infty}\sum_{j=1}^{n}\|x_j^{(k)}\| = n$, whence we have $\lim_{k\to\infty}\|x_j^{(k)}\| = 1$ for all j. Therefore by our first assertion we obtain that $\lim_{k\to\infty}\|\sum_{j=1}^{m} x_j^{(k)}\| = \lim_{k\to\infty}\sum_{j=1}^{m}\|x_j^{(k)}\| = m$. ∎

Now we shall present the main theorem.

Theorem 2.3 *Let $X_1, \cdots, X_m$ be Banach spaces. Then the following are equivalent:*
(i) *$(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^n.*
(ii) *There exist positive integers $n_1, \ldots, n_m$ with $n_1 + n_2 + \cdots + n_m = n-1$ such that X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$.*

Proof (i) $\Rightarrow$ (ii). Assume that $(X_1 \oplus \cdots \oplus X_m)_1$ is uniformly non-ℓ_1^n. Since X_i is regarded as a subspace of $(X_1 \oplus \cdots \oplus X_m)_1$, there exist $N_1, \ldots, N_m$ such that X_i is uniformly non-$\ell_1^{N_i+1}$ but not uniformly non-$\ell_1^{N_i}$ for all $1 \le i \le m$. Then for each i there exist N_i sequences $\{x_{i1}^{(k)}\}_k, \ldots, \{x_{iN_i}^{(k)}\}_k$ in S_{X_i}, the unit sphere of X_i, such that

$$\lim_{k\to\infty}\left\|\sum_{j=1}^{N_i}\theta_j x_{ij}^{(k)}\right\| = N_i \tag{3}$$

for all (θ_j) of N_i signs. First assume that $\sum_{i=1}^{m} N_i < n$. Let

$$L = (n-1) - \sum_{i=1}^{m} N_i \text{ and } n_1 = N_1 + L,\ n_i = N_i\ (2 \le i \le m).$$

Then we have $n_1+\cdots+n_m = n-1$ and X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$ by Proposition 1.1, as is desired.

Next we shall see the other cases do not in fact occur. We define the sequences $\{z_1^{(k)}\}_k, \dots, \{z_{\sum_{i=1}^m N_i}^{(k)}\}_k$ in the unit sphere $S_{(X_1\oplus\cdots\oplus X_m)_1}$ of $(X_1\oplus\cdots\oplus X_m)_1$ by

$$\begin{aligned}
&z_1^{(k)} = (x_{11}^{(k)}, 0, \dots, 0), \dots, z_{N_1}^{(k)} = (x_{1N_1}^{(k)}, 0, \dots, 0),\\
&z_{N_1+1}^{(k)} = (0, x_{21}^{(k)}, 0, \dots, 0), \dots, z_{N_1+N_2}^{(k)} = (0, x_{2N_2}^{(k)}, 0, \dots, 0),\\
&\dots,\\
&z_{(\sum_{i=1}^{m-1} N_i)+1}^{(k)} = (0, \dots, 0, x_{m1}^{(k)}), \dots, z_{\sum_{i=1}^m N_i}^{(k)} = (0, \dots, 0, x_{mN_m}^{(k)}).
\end{aligned}$$

If $\sum_{i=1}^m N_i = n$, we have

$$\begin{aligned}
\left\|\sum_{j=1}^n \theta_j z_j^{(k)}\right\|_1 &= \left\|\left(\sum_{j=1}^{N_1} \theta_j x_{1j}^{(k)}, \cdots, \sum_{j=1}^{N_m} \theta_i x_{mj}^{(k)}\right)\right\|_1 = \sum_{i=1}^m \left\|\sum_{j=1}^{N_i} \theta_i x_{ij}^{(k)}\right\|\\
&\to \sum_{i=1}^m N_i = n \quad \text{as } k\to\infty
\end{aligned}$$

for all $\theta_j = \pm 1$ by (3), which contradicts (i).

Suppose that $\sum_{i=1}^m N_i > n$. Then there exist natural numbers s and M with $1 < s < m$ and $1 \le M \le N_{s+1}$ such that $n = \sum_{i=1}^s N_i + M$ and

$$\left\|\sum_{j=1}^n \theta_j z_j^{(k)}\right\|_1 = \left\|\left(\sum_{j=1}^{N_1} \theta_j x_{1j}^{(k)}, \dots, \sum_{j=1}^{N_s} \theta_j x_{sj}^{(k)}, \sum_{j=1}^{M} \theta_j x_{s+1\,j}^{(k)}, 0, \cdots, 0\right)\right\|_1$$

(note that $1 \le M < N_m$ for $s = m-1$). Hence by (3) and Lemma 2.2 we obtain

$$\begin{aligned}
\left\|\sum_{j=1}^n \theta_j z_j^{(k)}\right\|_1 &= \sum_{i=1}^s \left\|\sum_{j=1}^{N_i} \theta_j x_{ij}^{(k)}\right\| + \left\|\sum_{j=1}^{M} \theta_j x_{s+1\,j}^{(k)}\right\|\\
&\to \sum_{i=1}^s N_i + M = n \quad \text{as } k\to\infty
\end{aligned}$$

for all $\theta_j = \pm 1$, which is a contradiction.

(ii) $\Rightarrow$ (i). We assume that there exist positive integers $n_1, \dots, n_m$ with $n_1 + n_2 + \cdots + n_m = n-1$ such that X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$. Let

$$K = \sup\left\{\min_{\theta_j=\pm1}\left\|\sum_{j=1}^n \theta_j w_j\right\|_1 : w_1, \dots, w_n \in S_{(X_1\oplus\cdots\oplus X_m)_1}\right\}. \tag{4}$$

We show that $K < n$. Take n sequences $\{w_1^{(k)}\}_k, \cdots, \{w_n^{(k)}\}_k$ in the unit sphere of $(X_1\oplus\cdots\oplus X_m)_1$ so that

$$K = \lim_{k\to\infty}\min_{\theta_j=\pm1}\left\|\sum_{j=1}^n \theta_j w_j^{(k)}\right\|_1. \tag{5}$$

Let $w_j^{(k)} = (x_{1j}^{(k)}, \cdots, x_{mj}^{(k)})$. As $\|x_{ij}^{(k)}\| \le 1$ for $1 \le j \le n$ and $k \ge 1$, we may assume that all mn sequences $\{\|x_{ij}^{(k)}\|\}_k$ have limits in k (take subsequences necessary times if needed). Now let

$$L_i = \{j : \lim_{k\to\infty} \|x_{ij}^{(k)}\| > 0,\ 1 \le j \le n\},\ 1 \le i \le m.$$

We first consider the case $\mathrm{card}(L_i) \le n_i$ for all $1 \le i \le m$. Then we obtain

$$\begin{aligned} K &= \lim_{k\to\infty} \min_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j w_j^{(k)} \right\|_1 \\ &= \lim_{k\to\infty} \min_{\theta_j=\pm 1} \left\| \left(\sum_{j=1}^n \theta_j x_{1j}^{(k)}, \cdots, \sum_{j=1}^n \theta_j x_{mj}^{(k)} \right) \right\|_1 \\ &= \lim_{k\to\infty} \min_{\theta_j=\pm 1} \sum_{i=1}^m \left\| \sum_{j=1}^n \theta_j x_{ij}^{(k)} \right\| \\ &\le \lim_{k\to\infty} \sum_{i=1}^m \sum_{j=1}^n \|x_{ij}^{(k)}\| \\ &= \sum_{i=1}^m \sum_{j\in L_i} \lim_{k\to\infty} \|x_{ij}^{(k)}\| \\ &\le \sum_{i=1}^m n_i = n-1. \end{aligned}$$

Next assume that $\mathrm{card}(L_{i_0}) \ge n_{i_0}+1$ for some $i_0, 1 \le i_0 \le m$. Then since X_{i_0} is uniformly non-$\ell_1^{n_{i_0}+1}$ and hence uniformly non-$\ell_1^{\mathrm{card}(L_{i_0})}$, there exists $0 < \varepsilon_{i_0} < 1$ such that for each $k \ge 1$ we have

$$\left\| \sum_{j\in L_{i_0}} \theta_j^{(k)} \frac{x_{i_0 j}^{(k)}}{\|x_{i_0 j}^{(k)}\|} \right\| \le \mathrm{card}(L_{i_0})(1-\varepsilon_{i_0}) \tag{6}$$

with some $\theta_j^{(k)} = \pm 1$ $(j \in L_{i_0})$. Therefore by virtue of Lemma 2.1 we have in X_{i_0}

$$\begin{aligned} \left\| \sum_{j\in L_{i_0}} \theta_j^{(k)} x_{i_0 j}^{(k)} \right\| &\le \sum_{j\in L_{i_0}} \|x_{i_0 j}^{(k)}\| - \left(\mathrm{card}(L_{i_0}) - \left\| \sum_{j\in L_{i_0}} \theta_j^{(k)} \frac{x_{i_0 j}^{(k)}}{\|x_{i_0 j}^{(k)}\|} \right\| \right) \min_{j\in L_{i_0}} \|x_{i_0 j}^{(k)}\| \\ &\le \sum_{j\in L_{i_0}} \|x_{i_0 j}^{(k)}\| - \mathrm{card}(L_{i_0})\varepsilon_{i_0} \min_{j\in L_{i_0}} \|x_{i_0 j}^{(k)}\| \\ &\le \sum_{j\in L_{i_0}} \|x_{i_0 j}^{(k)}\| - (n_{i_0}+1)\varepsilon_{i_0} \min_{j\in L_{i_0}} \|x_{i_0 j}^{(k)}\|. \end{aligned}$$

Consequently we obtain that

$$\begin{aligned}
K &= \lim_{k\to\infty}\min_{\theta_j=\pm1}\left\|\sum_{j=1}^{n}\theta_j w_j^{(k)}\right\|_1 \\
&= \lim_{k\to\infty}\min_{\theta_j=\pm1}\sum_{i=1}^{m}\left\|\sum_{j=1}^{n}\theta_j x_{ij}^{(k)}\right\| \\
&= \lim_{k\to\infty}\min_{\theta_j=\pm1}\left[\sum_{i\neq i_0}\left\|\sum_{j=1}^{n}\theta_j x_{ij}^{(k)}\right\| + \left\|\sum_{j=1}^{n}\theta_j x_{i_0j}^{(k)}\right\|\right] \\
&\leq \limsup_{k\to\infty}\min_{\theta_j=\pm1}\left[\sum_{i\neq i_0}\left\|\sum_{j=1}^{n}\theta_j x_{ij}^{(k)}\right\| + \left\|\sum_{j\notin L_{i_0}}\theta_j x_{i_0j}^{(k)}\right\| + \left\|\sum_{j\in L_{i_0}}\theta_j x_{i_0j}^{(k)}\right\|\right] \\
&\leq \limsup_{k\to\infty}\min_{\theta_j=\pm1}\left[\sum_{i\neq i_0}\sum_{j=1}^{n}\|x_{ij}^{(k)}\| + \sum_{j\notin L_{i_0}}\|x_{i_0j}^{(k)}\| + \left\|\sum_{j\in L_{i_0}}\theta_j x_{i_0j}^{(k)}\right\|\right] \\
&= \limsup_{k\to\infty}\left[\sum_{i\neq i_0}\sum_{j=1}^{n}\|x_{ij}^{(k)}\| + \sum_{j\notin L_{i_0}}\|x_{i_0j}^{(k)}\| + \min_{\theta_j=\pm1}\left\|\sum_{j\in L_{i_0}}\theta_j x_{i_0j}^{(k)}\right\|\right] \\
&\leq \limsup_{k\to\infty}\left[\sum_{i\neq i_0}\sum_{j=1}^{n}\|x_{ij}^{(k)}\| + \sum_{j=1}^{n}\|x_{i_0j}^{(k)}\| - (n_{i_0}+1)\varepsilon_{i_0}\min_{j\in L_{i_0}}\|x_{i_0j}^{(k)}\|\right] \\
&= \limsup_{k\to\infty}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\|x_{ij}^{(k)}\| - (n_{i_0}+1)\varepsilon_{i_0}\min_{j\in L_{i_0}}\|x_{i_0j}^{(k)}\|\right] \\
&= \limsup_{k\to\infty}\left[\sum_{j=1}^{n}\|w_j^{(k)}\|_1 - (n_{i_0}+1)\varepsilon_{i_0}\min_{j\in L_{i_0}}\|x_{i_0j}^{(k)}\|\right] \\
&= \limsup_{k\to\infty}\left[n - (n_{i_0}+1)\varepsilon_{i_0}\min_{j\in L_{i_0}}\|x_{i_0j}^{(k)}\|\right] \\
&\leq n - (n_{i_0}+1)\varepsilon_{i_0}\liminf_{k\to\infty}\min_{j\in L_{i_0}}\|x_{i_0j}^{(k)}\| < n.
\end{aligned}$$

This completes the proof. ∎

Corollary 2.4 ([15]) *Let X and Y be Banach spaces. Then the following are equivalent:*

(i) *$X\oplus_1 Y$ is uniformly non-ℓ_1^n.*

(ii) *There exist positive integers n_1 and n_2 with $n_1+n_2=n-1$ such that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$.*

As mentioned in the head of the introduction the ℓ_1-sum $X\oplus_1 Y$ cannot be uniformly non-square for all Banach spaces X and Y. Theorem 2.3 yields in particular the following result, which is also a direct consequence of the fact that the space ℓ_1^m is not uniformly non-ℓ_1^m.

Corollary 2.5 *The* ℓ_1*-sum* $(X_1 \oplus \cdots \oplus X_m)_1$ *of* m *Banach spaces* $X_1, \cdots, X_m$ *is not uniformly non-*ℓ_1^m.

Thus we shall consider the uniform non-ℓ_1^{m+1}-ness of $(X_1 \oplus \cdots \oplus X_m)_1$.

Theorem 2.6 *Let* $X_1, \ldots, X_m$ *be Banach spaces. Then the following are equivalent:*
(i) $(X_1 \oplus \cdots \oplus X_m)_1$ *is uniformly non-*ℓ_1^{m+1}.
(ii) $X_1, \ldots, X_m$ *are uniformly non-square.*

Proof Assume that $(X_1 \oplus \cdots X_m)_1$ is uniformly non-ℓ_1^{m+1}. By Theorem 2.3 there exist positive integers $n_1, \ldots, n_m$ such that $\sum_{i=1}^m n_i = m$ and X_i is uniformly non-$\ell_1^{n_i+1}$ for all $1 \le i \le m$. Since $n_i \ge 1$, we have $n_i = 1$ for all $1 \le i \le m$. Thus $X_1, \cdots, X_m$ are uniformly non-square. Conversely, if $X_1, \cdots, X_m$ are uniformly non-square, $(X_1 \oplus \cdots X_m)_1$ is uniformly non-ℓ_1^{m+1} by Theorem 2.3. ■

Theorem 2.6 includes the next previous result as the case $m = 2$.

Corollary 2.7 ([15]) *Let* X *and* Y *be Banach spaces. Then the following are equivalent:*
(i) $X \oplus_1 Y$ *is uniformly non-*ℓ_1^3.
(ii) X *and* Y *are uniformly non-square.*

Theorem 2.6 (with Corollary 2.5) also implies the following basic fact: *The space* ℓ_1^n *is uniformly non-*ℓ_1^{n+1} *but not uniformly non-*ℓ_1^n.

Recall that a Banach space X is called *super-reflexive* if every Banach space Y which is finitely representable in X is reflexive, where Y is said to be *finitely representable* in X if for any $\epsilon > 0$ and for any finite dimensional subspace F of Y there is a finite dimensional subspace E of X with $\dim F = \dim E$ such that $d(F, E) := \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism of } F \text{ onto } E\} < 1 + \epsilon$ (see [11, 2]). As it is well known, uniformly non-square spaces are super-reflexive ([12]), whereas uniformly non-ℓ_1^3 spaces, *a fortiori*, uniformly non-ℓ_1^n spaces are not always reflexive ([12]). For the ℓ_1-sum spaces we obtain the following

Corollary 2.8 *Let* X *be a uniformly non-*ℓ_1^n *Banach space which is isomorphic to the* ℓ_1*-sum of* $n-1$ *Banach spaces* $X_1, \cdots, X_{n-1}$ $(n \ge 3)$*. Then* X *is super-reflexive.*

Indeed $X_1, \cdots, X_{n-1}$ are uniformly non-square by Theorem 2.6 and hence super-reflexive. Since an ℓ_1-sum of a finite number of super-reflexive spaces is super-reflexive, we obtain the conclusion.

Recently Garcia-Falset et al. [6] proved that all uniformly non-square Banach spaces have the fixed point property for nonexpansive mappings. Combining Theorem 2.6 and this result, we obtain the following

Corollary 2.9 *Let* $(X_1 \oplus \cdots \oplus X_m)_1$ *be uniformly non-*ℓ_1^{m+1}*. Then* $X_1, \ldots, X_m$ *have the fixed point property for nonexpansive mappings.*

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