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Copies of the sequence space ω in F-lattices with applications to Musielak–Orlicz spaces

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Summary. Let E be a fixed real function F-space, i.e., E is an order ideal in $L_0(S, \Sigma, \mu)$ endowed with a monotone F-norm $\|\,\|$ under which E is topologically complete. We prove that E contains an isomorphic (topological) copy of ω, the space of all sequences, if and only if E contains a lattice-topological copy W of ω. If E is additionally discrete, we obtain a much stronger result: W can be a projection band; in particular, E contains a complemented copy of ω. This solves partially the open problem set recently by W. Wnuk.

The property of containing a copy of ω by a Musielak–Orlicz space is characterized as follows. (1) A sequence space ℓ_Φ, where $\Phi = (\varphi_n)$, contains a copy of ω iff $\inf_{n\in\mathbb{N}} \varphi_n(\infty) = 0$, where $\varphi_n(\infty) = \lim_{t\to\infty} \varphi_n(t)$. (2) If the measure μ is atomless, then ω embeds isomorphically into $L_{\mathcal{M}}(\mu)$ iff the function $\mathcal{M}_\infty$ is positive and bounded on some set $A \in \Sigma$ of positive and finite measure, where $\mathcal{M}_\infty(s) = \lim_{n\to\infty} \mathcal{M}(n, s)$, $s \in S$. In particular, (1)' ℓ_φ does not contain any copy of ω, and (2)' $L_\varphi(\mu)$, with μ atomless, contains a copy W of ω iff φ is bounded, and every such copy W is uncomplemented in $L_\varphi(\mu)$.

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1. Introduction

In this paper we deal mainly with *real* vector lattices (i.e., Riesz spaces), but it is easy to check that our results remain true in the complex case as well. For basic facts undefined

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here, we refer the reader to the monographs [1,2,8,10,12]; some notions are also presented in the following section. Throughout this paper, (S, Σ, μ) denotes a fixed σ-finite measure space and $L_0(\mu)$ denotes the Dedekind complete lattice of all (classes of) μ-measurable real functions on S endowed with its natural μ-a.e. algebraic operations and ordering.

The classical result of Lozanovskii asserts that a σ-Dedekind complete Banach lattice E contains a copy of the space ℓ_∞ if and only if E contains a lattice copy of ℓ_∞, and a similar equivalence, due to Lozanovskii and Meyer–Nieberg, holds for the class of all Banach lattices where ℓ_∞ is replaced by c_0 (see [2]; Theorems 14.9 and 14.12, and Remarks on pp. 124 and 127). (Here and in what follows the term "copy" means "linear-isomorphic (i.e., homeomorphic) copy", and "lattice copy" means "both linear-lattice and isomorpic copy" (cf. [1, Theorem 16.6]).)

The purpose of this paper is to prove an analogue of the above-cited Lozanovskii / Meyer–Nieberg theorems for a large class of F-lattices, including the class of all (non--Banach) Musielak–Orlicz spaces, for the Frèchet lattice ω of all real sequences instead of c_0 and ℓ_∞ (Main Theorem below). In 2013, this result was suggested by W. Wnuk [16, Remark 1.2], who put the following question:

(W) *If an F-lattice contains a copy of ω, does it also contain a lattice copy of ω?*

A complete metrizable locally solid Riesz space E is said to be an F-lattice. Its topology is determined by a monotone F-norm (for details, see [1, p. 111] and [8]). The space ω endowed with the pointwise ordering and the lattice F-norm

$$\|(t_n)\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|t_n|}{1+|t_n|}$$

is an example of a Frèchet lattice.

If E is an order ideal in $L_0(\mu)$ endowed with a monotone F-norm $\|\cdot\|$ under which E is topologically complete, then E is said to be *a function F-space.*

Now we are in a position to present our main result. Its atomless and purely atomic versions are given in Section 5, Theorems 3.1 and 3.2, respectively.

1.1. Main theorem. *Let E be a function F-space. The following two conditions are equivalent.*

(a) *E contains a copy of ω.*

(b) *E contains a lattice copy of ω.*

Hence, in the class of function F-spaces the property of containing a lattice copy of ω is invariant under linear homeomorphisms.

The paper is organized as follows. In Section 2 we list some notions and facts concerning F-spaces and Riesz spaces. The main technique used in proofs of Theorems 3.1 and 3.2 are the Bessaga–Pełczyński–Rolewicz theorem on arbitrarily short straight lines in an F-space, presented in Proposition 2.1, and the result on the construction of a disjoint refinement of a sequence (A_n) of elements of Σ with positive measure, given in Lemma 2.4. In Section 5 we present more detailed versions of the Main Theorem, and in Section 6 we show how our theory works for the class of Musielak–Orlicz spaces (Theorems 4.1 and 4.4); in particular, ℓ_φ does not contain any copy of ω (Corollary 4.2), and if the Orlicz function φ is bounded and the measure μ is atomless, then $L_\varphi(\mu)$ contains a (lattice) copy W of ω, although any such copy is uncomplemented in $L_\varphi(\mu)$ (Corollary 4.5). The proofs of the theorems and lemmas are given in the last section.

2. Preliminaries

The vector lattice (i.e., Riesz spaces) terminology and notation used in this paper are standard (see [1, 2]). In what follows, E is an Archimedean vector lattice.

Basic properties of vector lattices, F-lattices, and the space ω. Let E be a vector lattice. The set of all $0 \leqslant x \in E$ is denoted by E^+. Let $x \in E$. The symbol A_x denotes the ideal in E generated by x, i.e., $A_x = \{y \in E,\ \exists_{\lambda>0} : |y| \leqslant \lambda|x|\}$. If B is a nonempty subset E, then the symbol B^d stands for the orthogonal complement of B, i.e., $B^d = \{x \in E : |x| \wedge |y| = 0$ for every $y \in B\}$. A linear projection Q on the lattice E (i.e., an endomorphism of E such that $Q^2 = Q$) is said to be an *order projection* if $0 \leqslant Qx \leqslant x$ for all $x \in E^+$.

An order ideal B in E of the form $B = B^{dd}$ is said to be a band. The band B is said to be *projection band* if $E = B + B^d$.

If B is a projection band in E, then there exists an order projection $P_B : E \to B$ such that $B = P_B(E)$ and $B^d = P'_B(E)$, where $P'_B = I - P_B$ and I is the identity on E. It is well known (and can be checked directly) that if E is an order ideal in $L_0(\mu)$ then every order projection Q in E is of the form

$$Qx = \mathbb{1}_A \cdot x, \quad x \in E,$$

where A is an element of Σ and $\mathbb{1}_A$ denotes the characteristic function of A.

The lattice E is called [σ-]Dedekind complete if every [countable] nonempty subset A of E bounded from above has the least upper bound in E. Every order ideal J of $L_0(\mu)$ is Dedekind complete. If E is Dedekind complete, then every band in E is a projection band [1, Theorem 2.12].

An element $e \in E^+ \setminus \{0\}$ is said to be discrete (or, an atom) if the ideal generated by e coincides with the subspace generated by e, i.e., $A_e = \mathrm{lin}\{e\}$. The lattice E are said to be *discrete* if every element $x \in E^+ \setminus \{0\}$ majorizes a discrete element, i.e., for every $x \in E^+ \setminus \{0\}$ there exists a discrete element $y_x \in E^+$ such that $x \geqslant y_x$.

An order ideal J in E is said to be order dense if for every $x \in E^+ \setminus \{0\}$ there is $y \in J^+ \setminus \{0\}$ such that $y \leqslant x$; equivalently [1, Theorem 1.11], $J^{dd} = E$.

A subadditive function $x \mapsto \|x\|$ defined on a linear space X such that $\|x\| = 0$ if and only if $x = 0$, and for every $x \in X$ the function $R^+ \ni t \mapsto \|tx\|$ is continuous at zero, is called an F-norm. If X is $\|\cdot\|$-complete, then $(X, \|\cdot\|)$ is said to be an F-space [8, pp. 2–3]. If, additionally, X is a Riesz space and the F-norm $\|\cdot\|$ is monotone, it is called an F-lattice (cf. Section 1). For $(E, \|\cdot\|)$ an F-lattice, the symbol E_a denotes the order continuous part of E:

$$E_a := \{x \in E : |x| \geqslant |u_\alpha| \downarrow 0 \text{ implies } \|u_\alpha\| \to 0\}.$$

We say that the F-space X contains *arbitrarily short straight lines* if for every $\varepsilon > 0$ there exists an element $0 \neq x \in X$ such that $r(x) < \varepsilon$, where

$$r(x) = \sup_{t \in \mathbb{R}} \|tx\|. \tag{1}$$

Set

$$\delta(X) = \inf_{0 \neq x \in X} r(x).$$

Then, according to the above definition, X contains arbitrarily short straight lines if and only if $\delta(X) = 0$.

For example, it is easy to check that for the space ω endowed with its natural F-norm (see Section 1), $\delta(\omega) = 0$; thus ω contains arbitrarily short straight lines. In 1957, C. Bessaga, A. Pełczyński and S. Rolewicz [4, Theorem 9], cf. [12, Proposition 4.2.7], proved the following theorem, showing the role of ω in the latter notion.

2.1. Proposition (B–P–R). *An F-space X contains arbitrarily short straight lines* (*i.e.,* $\delta(X) = 0$) *if and only if X contains a closed subspace X_0 isomorphic to ω.*

Since the direct calculation of the number $\delta(E)$ is difficult, in the lemma below we give a method of determining $\delta(E)$ in a few typical cases.

2.2. Lemma. *Let E be an F-lattice, and let Y be a subset of $E^+ \setminus \{0\}$ satisfying the condition: for every $x \in E^+ \setminus \{0\}$ there exists a number $\lambda_x > 0$ and $y_x \in Y$ such that*

$$x \geqslant \lambda_x y_x. \tag{2}$$

Then

$$\delta(E) = \delta(Y) := \inf_{x \in Y} r(x). \tag{3}$$

where the function r is defined in (1). *In particular,*

(i) *If E is a discrete F-lattice and $D = \{d_\gamma : \gamma \in \Gamma\}$ denotes a maximal set of discrete and pairwise disjoint elements of E, then*

$$\delta(E) = \inf_{d_\gamma \in D} r(d_\gamma). \tag{4}$$

(ii) *If E is a function F-lattice in $L_0(\mu)$, then*

$$\delta(E) = \delta(E^0) = \inf\{r(\mathbb{1}_A) : \mathbb{1}_A \in E^0\},$$

where $E^0 = \{\mathbb{1}_A \in E : A \in \Sigma$ and $0 < \mu(A) < \infty\}$.
Additionaly, if the ideal E_a (of order continuous elements of E) is order dense in E, then

$$\delta(E) = \delta(E_a) = \delta\big((E_a)^0\big). \tag{5}$$

The following property of ω allows us to shorten the proof of Theorem 3.1.

2.3. Lemma.

(a) *The space ω is a minimal F-space, i.e., ω admits no strictly weaker linear Hausdorff topology* [6, Theorem 4.1].

(b) *Every continuous injection acting from ω to a fixed F-lattice is an isomorphism* (*this follows from the minimality of ω*).

The next result concerns the existence of a disjoint refinement of a sequence of nontrivial elements of Σ and will be used in the proof of Theorem 3.2. It appears in [3, Example 1.5, p. 338]; however, the authors do not include a proof, referring the reader to [3, Proposition 1.4], which describes a more general situation. Our proof is constructive.

2.4. Lemma. *Let (S, Σ, μ) be an atomless measure space. If $(A_l) \subset \Sigma$ is a sequence of sets of positive and finite μ-measure, then there exists a sequence $(G_l) \subset \Sigma$ with $\mu(G_l) > 0$ for all l and such that:*

(i) *$G_l \subset A_l$, $l = 1, 2, \ldots$, and*

(ii) *$G_l \cap G_k = \varnothing$, for $l \neq k$, $l, k \in \mathbb{N}$.*

The sequence (G_l) satisfying the two conditions of the above lemma is said to be *a disjoint refinement of* (A_l).

Musielak–Orlicz spaces. Below we recall basic notions regarding Musielak–Orlicz spaces (non-Banach, in general); more exhaustive information on this topic the reader will find in the monograph by J. Musielak [11] and in the paper by W. Wnuk [13] (cf. [12]; for the Banach space-case, see [9]).

A function $\mathcal{M}: [0, \infty) \times S \to [0, \infty)$ is said to be a *Musielak–Orlicz function* if:

(C1) for every $s \in S$, the function $\mathcal{M}(\cdot, s): [0, \infty) \to [0, \infty)$ is non-decreasing, left-continuous, continuous at zero, and such that $\mathcal{M}(t, s) = 0$ if and only if $t = 0$; and

(C2) for every $t \in [0, \infty)$, the function $\mathcal{M}_t := \mathcal{M}(t, \cdot) : S \to [0, \infty)$ is Σ-measurable.

Notice that, in contrast to the classical case [9], in the above definition the functions $\mathcal{M}(\cdot, s)$ are *not* convex in general. To simplify the proofs we also assume that *these functions are continuous*.

If $\mathcal{M}(r, s_1) = \mathcal{M}(r, s_2)$ for all $s_1, s_2 \in S$, then the function $\mathcal{M}$ is said to be an *Orlicz function* and will be denoted by φ or ψ.

If $S = \mathbb{N}$ and the measure μ is counting, it is more convenient to consider the function $\mathcal{M}$ as a sequence of Orlicz functions φ_n, $n = 1, 2, \ldots$, i.e., $\mathcal{M} = (\varphi_n)_{n=1}^{\infty}$. In this case, the symbol $\mathcal{M}$ will be replaced by Φ or Ψ.

The Musielak–Orlicz function $\mathcal{M}$ determines a function $\rho_{\mathcal{M}}: L_0(\mu) \to [0, \infty]$, called *a modular*, of the form:

$$\rho_{\mathcal{M}}(x) = \int_S \mathcal{M}(|x(s)|, s) d\mu.$$

The order ideal $L_{\mathcal{M}}(\mu)$ of the space $L_0(\mu)$ defined as

$$L_{\mathcal{M}}(\mu) = \left\{x \in L^0(\mu) \, : \, \rho_{\mathcal{M}}(\lambda x) < \infty \text{ for some number } \lambda > 0\right\}$$

is called *a Musielak–Orlicz space*. The formula

$$\|x\|_{\mathcal{M}} := \inf\left\{\lambda > 0 \, : \, \rho_{\mathcal{M}}(x/\lambda) \leqslant \lambda\right\} \tag{6}$$

defines a monotone F-norm on $L_{\mathcal{M}}(\mu)$ under which $L_{\mathcal{M}}(\mu)$ is a function F-lattice [13, p. 144]. Notice that $\|\cdot\|_{\mathcal{M}}$ is *not* a norm (however, if every function $\mathcal{M}(\cdot, s)$ in condition (C1) is convex, then $L_{\mathcal{M}}(\mu)$, endowed with the equivalent norm $\|x\| := \inf\{\lambda > 0 : \rho_{\mathcal{M}}(x/\lambda) \leqslant 1\}$, becomes a Banach function space).

Since, by hypothesis, $\mathcal{M}$ takes finite values only, we have the following description of the order continuous part $(L_{\mathcal{M}}(\mu))_a$ of $L_{\mathcal{M}}(\mu)$:

$$\left(L_{\mathcal{M}}(\mu)\right)_a = E_{\mathcal{M}}(\mu) := \left\{x \in L_0(\mu) \, : \, \rho_{\mathcal{M}}(\lambda x) < \infty \text{ for all } \lambda > 0\right\}, \tag{7}$$

and $(L_{\mathcal{M}}(\mu))_a$ *is a topologically closed and an order dense ideal of* $L_{\mathcal{M}}(\mu)$ (see [14, Theorem 1.3], cf. [13, p. 145]).

In particular, if $S = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$, the measure μ is counting, and $\Phi = (\varphi_n)$, where φ_n is an Orlicz function for every $n \in \mathbb{N}$, then the modular ρ_Φ on the space $\omega = L_0(\mu)$ is of the form

$$\rho_\Phi\big((t_n)\big) = \sum_{n=1}^{\infty} \varphi_n\big(|t_n|\big), \tag{8}$$

and the space $\ell_\Phi := L_\Phi(\mu)$ is said to be *a Musielak–Orlicz sequence space*. If $\varphi_n = \varphi$ for all $n \in \mathbb{N}$, then we write $\ell_\Phi = \ell_\varphi$ and ℓ_φ is said to be an Orlicz sequence space. The symbol h_Φ denotes the order continuous part of ℓ_Φ, and h_φ has a similar meaning.

For $x \in E_{\mathcal{M}}(\mu)$, we have a useful identity:

2.5. Lemma. *For* $x \in L_{\mathcal{M}}(\mu)$, *set* $p(x) := \rho_{\mathcal{M}}(x)$. *Then, if* $p(x) < \infty$ (*in particular, if* $x \in E_{\mathcal{M}}(\mu)$),

$$\|p(x) \cdot x\|_{\mathcal{M}} = p(x).$$

For a Musielak–Orlicz sequence space, formula (4) in Lemma 2.2 takes a simple form.

2.6. Lemma. *Let Φ denote a sequence (φ_n) of increasing and continuous Orlicz functions. Set $\varphi_n(\infty) := \lim_{t\to\infty} \varphi_n(t)$. Then*

$$\delta(\ell_\Phi) = \inf_{n\in\mathbb{N}} \varphi_n(\infty).$$

3. Copies of ω in function F-lattices

Let us notice first that *the proof of the Main Theorem can be reduced to one of the following cases: E is discrete (i.e., atomic), or E is continuous (i.e., atomless)*. Indeed, if E contains an infinite (maximal) set D of discrete and pairwise disjoint elements then the band $E_D := D^{dd}$ in E, generated by D, is a maximal discrete sublattice of E (cf. [1, p. 154]). Assuming the continuous part $E_c := E_D^d$ is nontrivial (hence, of infinite dimension), we have a decomposition $E = E_D + E_c$ with $E_D \cap E_c = \{0\}$. Since E is Dedekind complete, there is an order projection P from E onto E_D such that its complement $P' := I - P$ maps E onto E_c (see [2, Theorem 1.45 (1)]). Hence, if E contains a copy of ω, by Proposition 2.1 and Lemma 2.2, there is a sequence (x_n) in $E^+ \setminus \{0\}$ such that $r(x_n) \to 0$. Setting $u_n = Px_n$ and $v_n = P'x_n$, $n = 1, 2, \ldots$, we obtain that either (1) $u_n \neq 0$ infinitely often, or (2) $v_n \neq 0$ infinitely often. Since for every $t \in \mathbb{R}$, $\|tu_n\| = \|P(tx_n)\| \leqslant \|tx_n\|$ and, similarly, $\|tv_n\| \leqslant \|tx_n\|$, we obtain that $r(u_n) \leqslant r(x_n)$ and $r(v_n) \leqslant r(x_n)$ for all n. Hence, in case (1), $\delta(E_D) = 0$, and, in case (2), $\delta(E_c) = 0$. By Proposition 2.1, either E_D or E_c contains a copy of ω.

In the theorem below, we consider the atomless case. Notice that, by the assumption, E is Dedekind complete.

3.1. Theorem. *Let (S, Σ, μ) be an atomless σ-finite measure space, and let $E \subset L_0(S, \Sigma, \mu)$ be a function F-lattice. Then the following conditions are equivalent.*

(i) *E contains a copy of ω.*

(ii) *E contains a lattice copy of ω.*

If the ideal E_a is order dense in E, then conditions (i) and (ii) are equivalent to

(i') *E_a contains a copy of ω.*

(ii') *E_a contains a lattice copy of ω.*

One should note that in contrast to the discrete case (see below), there exist examples of atomless F-lattices containing uncomplemented copies of ω only (Corollary 4.5).

The next theorem deals with the discrete case. Formally, E is Dedekind complete (see the form of the Main Theorem) but in the proof of this case we do not apply this property.

3.2. Theorem. *Let E be a discrete F-lattice, and let D denote a fixed maximal set of discrete and pairwise disjoint elements of E. Then the following conditions are equivalent.*

(i) *E contains a copy of ω.*

(ii) *E contains a nontrivial projection band B that is order-topologically isomorphic to ω and spanned by a sequence $(d_n) \subset D$ (i.e., $B = \overline{\operatorname{lin}\{d_n : n \in \mathbb{N}\}}$).*

(ii') *E_a contains a nontrivial projection band B that is order-topologically isomorphic to ω and spanned by a sequence $(d_n) \subset D$.*

In particular, a discrete F-lattice contains a copy of ω if and only if it contains a complemented copy of ω.

4. Applications to Musielak–Orlicz spaces

In this section, we present an application of the results of Section 5 to Musielak–Orlicz spaces. We show that the fact of $L_{\mathcal{M}}(\mu)$ containing of a copy of ω can be expressed by a property of $\mathcal{M}$.

We shall first consider the sequence case.

4.1. Theorem. *Let Φ denote a sequence (φ_n) of increasing and continuous Orlicz functions. Then the following three conditions are equivalent.*

(i) *ℓ_Φ contains a copy of ω.*

(ii) *ℓ_Φ contains a projection band order-topologically isomorphic to the lattice ω.*

(iii) $\inf_{n\in\mathbb{N}} \varphi_n(\infty) = 0$.

Moreover, in conditions (i) and (ii), the space ℓ_Φ can be replaced by its order continuous part h_Φ.

If the sequence $\Phi = (\varphi_n)$ of Orlicz functions is constant, then from Theorem 4.1 (iii) we obtain

4.2. Corollary. *If φ is a continuous and increasing (non-convex) Orlicz function, then the F-lattice ℓ_φ does not contain any copy of ω.*

The next corollary concerns the problem studied earlier in the class of Musielak–Orlicz Banach spaces [15]:

Is a Musielak–Orlicz F-space ℓ_Φ isomorphic to an Orlicz space ℓ_φ?

Since containing of a copy of ω is a linear-topological property, from Theorem 4.1 and Corollary 4.2 we obtain a partial solution to the above problem in the class of Musielak–Orlicz sequence spaces.

4.3. Corollary. *Let $\Phi = (\varphi_n)$ be a sequence of continuous Musielak–Orlicz functions. If $\inf_{n\in\mathbb{N}} \varphi_n(\infty) = 0$, then the space ℓ_Φ is non-isomorphic to any Orlicz space ℓ_φ.*

The result below is a particular version of Theorem 3.1 for the case when $E = L_{\mathcal{M}}(\mu)$ and μ is atomless. Let us note that here $\delta(E)$ can take two values only: 0 or ∞.

4.4. Theorem. *Let (S, Σ, μ) be a σ-finite and atomless measure space, and let $L_{\mathcal{M}}(\mu)$ be a Musielak–Orlicz space. Set $\mathcal{M}_\infty(s) = \lim_{n\to\infty} \mathcal{M}(n, s)$, $s \in S$. Then the following five conditions are equivalent.*

(i) *$L_{\mathcal{M}}(\mu)$ contains a copy of ω.*

(ii) *$L_{\mathcal{M}}(\mu)$ contains a lattice copy of ω.*

(iii) *$\delta(L_{\mathcal{M}}(\mu)) = 0$.*

(iv) *$\delta(L_{\mathcal{M}}(\mu)) < \infty$.*

(v) *There exists $A \in \Sigma$ such that $\mathbb{1}_A \in E_{\mathcal{M}}(\mu)$, $0 < \mu(A) < \infty$, and*

$$0 < \mathcal{M}_{\infty|A} < \infty \quad \mu - a.e.$$

Additionally, in the above four conditions (i)–(iv) the F-lattice $L_{\mathcal{M}}(\mu)$ can be replaced by its order continuous part $E_{\mathcal{M}}(\mu)$.

Rolewicz's result [12, Theorem 4.2.2] states that if the measure space (S, Σ, μ) is atomless and the Orlicz function φ is bounded, then $L_\varphi(\mu)$ has a trivial dual. In this case, for $\mathcal{M} = \varphi$, we have $\mathcal{M}_{\infty|S} = \varphi(\infty) < \infty$, whence, by part (iv) of Theorem 4.4, $L_\varphi(\mu)$ contains a copy of ω. But since every (continuous) projection P from an F-space X onto a copy W of ω has the form $Px = \sum_{n=1}^\infty w_n^*(x) w_n$, for some $(w_n) \subset W$, with $(w_n^*) \subset X^*$ (the topological dual of X), from the above two remarks we immediately obtain:

4.5. Corollary. *Let (S, Σ, μ) be a σ-finite and atomless measure space, and let φ be a bounded Orlicz function. Then*

(a) *$L_\varphi(\mu)$ contains a copy of ω;*

(b) *every such copy of ω is uncomplemented in $L_\varphi(\mu)$.*

5. The proofs

In the proofs, we shall apply a few times the following obvious property of an atomless measure μ:

(M) *Let μ be a finite measure on a σ-algebra Σ of subsets of a nonempty set S. If μ is atomless then for every $\varepsilon > 0$ and every $A \in \Sigma$ with $0 < \mu(A)$, there is $B \in \Sigma$ with $B \subset A$ and $0 < \mu(B) < \varepsilon$.*

Proofs of the Theorems

Proof of Theorem 3.1. We shall prove the nontrivial implication (i) $\Rightarrow$ (ii) only. For the second part of the theorem, notice that if E_a is order dense in E then, by Proposition 2.1, condition (ii) implies (ii').

To shorten the notation, if $g\colon S \to \mathbb{R}$ is a Σ-measurable *function*, then the symbol $\overline{g}$ will denote the equivalence class in $L_0(S, \Sigma, \mu)$ determined by g.

By Proposition 2.1, E contains arbitrarily short straight lines. From the condition $\delta(E) = 0$ it follows that there exists a sequence (f_n) of Σ-measurable functions on S such that, for every $n \in \mathbb{N}$, we have $\overline{f_n} \in E$ and

$$\sum_{n=1}^{\infty} r(\overline{f_n}) < \infty, \tag{9}$$

and the sequence $(\overline{f_n})$ spans a closed subspace H in E, with Schauder basis $(\overline{f_n})$, isomorphic to ω: i.e., the series $\sum_{n=1}^{\infty} t_n \overline{f_n}$ is convergent in E if and only if $(t_n) \in \omega$ (see the proof of [12, Proposition 4.2.7]).

Now let $A_n := \operatorname{supp}(f_n)$, $n = 1, 2, \ldots$; then obviously $A_n \in \Sigma$ and $\mu(A_n) > 0$ for all n.

Since the measure space is σ-finite, for every $n \in \mathbb{N}$ there is a subset $A_n^0 \subset A_n$ such that $0 < \mu(A_n^0) < \infty$. By Lemma 2.4, there exists a sequence $(G_n) \subset \Sigma$ such that $G_n \subset A_n^0$, $\mu(G_n) > 0$, for $n = 1, 2, \ldots$, and $G_k \cap G_l = \varnothing$ for $k \neq l$.

Set $g_n := f_n \cdot \mathbb{1}_{G_n}$, $n = 1, 2, \ldots$. Then $\operatorname{supp}(g_n) = G_n$ (because $G_n \subset A_n^0 \subset A_n$, $n = 1, 2, \ldots$) and $|g_k| \wedge |g_l| = 0$ for $k \neq l$ and $n, k, l \in \mathbb{N}$.

For every $n \in \mathbb{N}$, we have $|g_n| \leqslant |f_n|$ μ-a.e., whence $|\overline{g_n}| \leqslant |\overline{f_n}|$, and thus $\|\overline{g_n}\| = \||\overline{g_n}|\| \leqslant \||\overline{f_n}|\| = \|\overline{f_n}\|$. It follows that $r(\overline{g_n}) \leqslant r(\overline{f_n})$, $n = 1, 2, \ldots$. From this and condition (9) we obtain

$$\sum_{n=1}^{\infty} r(\overline{g_n}) < \infty. \tag{10}$$

For every sequence $(t_n) \in \omega$, we have $\sum_{n=1}^{\infty} \|t_n \overline{g_n}\|_E \leqslant \sum_{n=1}^{\infty} r(\overline{g_n})$; thus condition (10) implies that the series $\sum_{n=1}^{\infty} t_n \overline{g_n}$ is absolutely convergent in E.

Now from the fact that the elements $\overline{g_n}$ are pairwise disjoint it follows that the map $h\colon \omega \to E$ of the form $h((t_n)) = \sum_{n=1}^{\infty} t_n \overline{g_n}$ is a lattice isomorphism, and hence continuous [1, Theorem 16.6]. Applying Lemma 2.3 (b) we obtain that h is a linear homeomorphism.

We have thus proved that h is a homomorphism, and this means that h is a lattice-topological isomorphism from ω into E. The proof of the implication (i) $\Rightarrow$ (ii) is complete. $\square$

Proof of Theorem 3.2. Suppose that a discrete F-lattice E contains a copy of ω, and let f be a linear topological isomorphism from ω into E. Set $x_n := f(e_n)$, $n = 1, 2, \ldots$, where e_n denotes the nth unit vector in ω.

We shall only prove that (i) $\Rightarrow$ (ii), because the implications (ii) $\Rightarrow$ (ii') $\Rightarrow$ (i) are trivial: discrete elements always belong to E_a, and so the band B described in part (ii) is included in E_a.

Since f is an isomorphism, we easily obtain that

$$r(x_n) \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Now we apply the discreteness of E. Let $D = \{d_\gamma : \gamma \in \Gamma\}$ be a maximal set of discrete and pairwise disjoint elements of E. For every $n \in \mathbb{N}$ there exists a number $\lambda_n > 0$ and an element $d_{\gamma_n} \in D$ such that

$$|x_n| \geqslant \lambda_n d_{\gamma_n}. \tag{12}$$

From inequality (12) it follows that $r(|x_n|) = r(x_n) \geqslant r(d_{\gamma_n})$, whence, by (11), $r(d_{\gamma_n}) \to 0$ as $n \to \infty$. Without loss of generality we may assume that the sequence $r(d_{\gamma_n})$ is strictly decreasing to zero:

$$r(d_{\gamma_{n+1}}) < r(d_{\gamma_n}), \quad n = 1, 2, \ldots \tag{13}$$

Hence $d_{\gamma_n} \neq d_{\gamma_m}$, for $n \neq m$, and thus $d_{\gamma_n} \wedge d_{\gamma_m} = 0$ for $n \neq m$. For simplicity, we set $d_{\gamma_n} = d_n$, $n \in \mathbb{N}$.

By condition (13), we may assume that $\sum_{n=1}^{\infty} \sup_{t \in \mathbb{R}} \|t d_n\| < \infty$. Hence for every sequence $(t_n) \in \omega$,

$$\sum_{n=1}^{\infty} \|t_n d_n\| \leqslant \sum_{n=1}^{\infty} r(d_n) < \infty. \tag{14}$$

Since the F-lattice E is topologically complete, inequality (14) implies that the series

$$\sum_{n=1}^{\infty} t_n d_n$$

is absolutely convergent to an element $x \in E$. Because the elements d_n are pairwise disjoint we also obtain that the element x is uniquely represented by the sequence (t_n). From the uniqueness of the assignment $\omega \ni \xi = (t_n) \mapsto \sum_{n=1}^{\infty} t_n d_n$ it follows that map $h\colon \omega \to E$ of the form

$$h(\xi) = \sum_{n=1}^{\infty} t_n d_n, \quad \xi = (t_n),$$

is well defined, and h is a lattice isomorphism since the elements of the sequence (d_n) are positive and pairwise disjoint. In particular, h is continuous [1, Theorem 16.6]. By part (b) of Lemma 2.3, the sublattice $B := h(\omega)$ *is a closed subspace of the* F*-lattice* E. Hence h is a lattice-topological isomorphism from ω onto B.

In the final part of the proof we shall show that B *is a projection band in* E. It is quite obvious that

$$B^d = \{x \in E : |x| \wedge d_n = 0 \text{ for all } n\},$$

and thus $B^{dd} = B$, i.e, B is a band indeed. Moreover, the operator $P: E \to B$ defined by the formula

$$Px = (o) \sum_{n=1}^{\infty} \lambda_{\gamma_n} d_n,$$

where (o) denotes order convergence, is a band projection from E onto B.

The proof of the nontrivial implication (i) $\Rightarrow$ (ii) is complete. □

Proof of Theorem 4.1. We obtain the equivalence (i) $\Longleftrightarrow$ (ii) immediately from Theorem 3.2, parts (i) and (ii). Moreover, by Proposition 2.1 and Lemma 2.6, the equivalence (i) $\Longleftrightarrow$ (iii) holds true. Since $(\ell_\Phi)_a = h_\Phi$, the last part of the theorem follows from the equivalence (i) $\Longleftrightarrow$ (ii') in Theorem 3.2. □

Proof of Theorem 4.4. We shall prove the following implications: (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii).

The equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) follow from Theorem 3.1 and Proposition 2.1, and the implication (iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (v). We shall prove the implication non-(v) $\Rightarrow$ non-(iv) (i.e., $\delta(L_{\mathcal{M}}(\mu)) = \infty$). We claim that *the equality* $r(\mathbb{1}_A) = \int_A \mathcal{M}_\infty(s) d\mu(s)$ *holds*

$$\textit{for every } A \in \Sigma \textit{ with } \mathbb{1}_A \in E_{\mathcal{M}} \textit{ and } 0 < \mu(A) < \infty. \tag{15}$$

Indeed, let us fix such an A, and let $t \in \mathbb{R}$, $t \neq 0$. Since $t\mathbb{1}_A \in E_{\mathcal{M}}$, from Lemma 2.5 and by the Levi theorem [5, p. 112], we obtain

$$\begin{aligned} r(\mathbb{1}_A) &= \lim_{t\to\infty} \|t \cdot \mathbb{1}_A\|_{\mathcal{M}} = \lim_{t\to\infty} \|t \cdot \rho_{\mathcal{M}}(t \cdot \mathbb{1}_A) \cdot \mathbb{1}_A\|_{\mathcal{M}} \\ &= \lim_{t\to\infty} \rho_{\mathcal{M}}(t \cdot \mathbb{1}_A) = \lim_{n\to\infty} \rho_{\mathcal{M}}(n \cdot \mathbb{1}_A) \\ &= \lim_{n\to\infty} \int_A \mathcal{M}(n,s) d\mu(s) = \int_A \mathcal{M}_\infty(s) d\mu(s). \end{aligned}$$

Now assuming non-(v), by (15), for every $\mathbb{1}_A \in E_{\mathcal{M}}$ with $0 < \mu(A) < \infty$, we have $r(\mathbb{1}_A) = \infty$. But by (7) and equalities (5) in Lemma 2.2, we obtain

$$\delta(L_{\mathcal{M}}(\mu)) = \delta(E_{\mathcal{M}}(\mu)) = \inf\{r(\mathbb{1}_A) : \mathbb{1}_A \in E_{\mathcal{M}}(\mu) \text{ and } 0 < \mu(A) < \infty\},$$

thus $\delta(L_{\mathcal{M}}(\mu)) = \infty$, i.e., non-(iv) holds true.

(v) $\Rightarrow$ (iii). Let $\mathbb{1}_A \in E_{\mathcal{M}}(\mu)$, with $0 < \mu(A) < \infty$, be such that $0 < \mathcal{M}_\infty|_A < \infty$ μ-a.e. Without loss of generality we may assume that $0 < \mathcal{M}_\infty(s) < \infty$ for every $s \in A$. Set $A_j = \{s \in A : 1/j \leqslant \mathcal{M}_\infty(s) \leqslant j\}$, $j = 1, 2, \ldots$. Hence $A = \bigcup_{j=1}^{\infty} A_j$, and thus there is j_0 such that $\mu(A_{j_0}) > 0$. Setting $B = A_{j_0}$, $c_1 = 1/j_0$, $c_2 = j_0$, we obtain

$$0 < c_1 \leqslant \mathcal{M}_\infty|_B \leqslant c_2, \tag{16}$$

with $\mathbb{1}_B \in E_{\mathcal{M}}(\mu)$ because $E_{\mathcal{M}}(\mu)$ is an order ideal in $L_0(\mu)$ and $0 < \mathbb{1}_B \leqslant \mathbb{1}_A \in E_{\mathcal{M}}(\mu)$.

By property (M) applied to the set B, there is a sequence $(B_n) \in \Sigma$ such that $B \supset B_n \supset B_{n+1}$ for all $n \in \mathbb{N}$ and

$$0 < \mu(B_n) < \frac{1}{2^n}\mu(B) \downarrow 0 \quad \text{as } n \to \infty. \tag{17}$$

Moreover, by an argument as above (for $\mathbb{1}_{B_n} \leqslant \mathbb{1}_B$), we have that $\mathbb{1}_{B_n} \in E_{\mathcal{M}}(\mu)$ for all $n \in \mathbb{N}$. Now, by (15), (16), and (17), we obtain

$$0 < c_1\mu(B_n) \leqslant r(\mathbb{1}_{B_n}) \leqslant c_2\mu(B_n) \downarrow 0.$$

Hence, by part (ii) of Lemma 2.2, identity (5), we obtain $\delta(L_{\mathcal{M}}(\mu)) = \delta(E_{\mathcal{M}}(\mu)) = 0$.

The proof of the implication (v) $\Rightarrow$ (iii) is complete; at the same time, the preceding equalities show that the last part of the theorem is also true. □

Proofs of the Lemmas

Proof of Lemma 2.2. Since for every $x \in E$ we have $\|x\| = \||x|\|$, we obtain that $\delta(E) = \delta(E^+)$, so the following part of the proof is reduced to the elements of E^+.

From the inclusion $Y \subset E$ and from the definition of δ we immediately obtain

$$\delta(Y) \geqslant \delta(E). \tag{18}$$

On the other hand, by condition (2), for every $x \in E^+$ there exists $e_x \in Y$ and a number $\lambda_x > 0$ such that $x \geqslant \lambda_x \cdot e_x$, and thus $r(x) \geqslant r(e_x) \geqslant \delta(Y)$. Hence $\delta(E) \geqslant \delta(Y)$, and by (18) we obtain (4).

Now condition (i) follows immediately from the definition of a discrete lattice: for every $x \in E^+ \setminus \{0\}$ there exists a number $\lambda > 0$ and a discrete element $d \in D$ such that $x \geqslant \lambda d$; thus D fulfils condition (2).

We shall prove condition (ii) in a more general case. Let J be an order ideal of $L_0(\mu)$ (in particular, $J = L_{\mathcal{M}}(\mu)$ or $J = E_{\mathcal{M}}(\mu)$), and let $S = \bigcup_{n=1} S_n$ with $\mu(S_n) < \infty$ for all $n \in \mathbb{N}$. Then, for every positive $x \in J$, there is $m \in \mathbb{N}$ such that $0 < y := x \wedge \mathbb{1}_{S_m}$ (and, obviously, $y \leqslant \mathbb{1}_{S_m}$). Since y is a pointwise limit of a nondecreasing sequence of positive simple functions, there is $B \in \Sigma$ with $B \subset S_m$, $\mu(B) > 0$, and $\lambda > 0$ such that

$$x \geqslant y \geqslant \lambda \mathbb{1}_B, \tag{19}$$

which implies that $\mathbb{1}_B \in J$ (because J is an order ideal of $L_0(\mu)$). We also have $\mu(B) \leqslant \mu(S_m) < \infty$, thus, by (19), the set

$$J^0 := \left\{\mathbb{1}_B \in J : B \in \Sigma,\ 0 < \mu(B) < \infty\right\}$$

fulfils condition (2); now (3) applies.

If, additionally, E_a is order dense in E, then E_a fulfils condition (2), and we apply (3). □

Proof of Lemma 2.4. The following proof has been suggested to the authors by the referee; it is shorter than our previous proof.

Let $(A_k) \subset \Sigma$ be a sequence of sets of positive and finite measure. Starting with $B_1 = A_1$ and using property (M), for every $k = 1, 2, \ldots$ we can find $B_k \subset A_k$ satisfying $0 < \mu(B_k) < \frac{1}{3}\mu(B_k)$, and so

$$\mu(B_{k+l}) < \frac{1}{3^l}\mu(B_k), \quad l = 1, 2, \ldots \tag{20}$$

Put

$$G_l = B_l \setminus \bigcup_{k=l+1}^{\infty} B_k, \quad l = 1, 2, \ldots \tag{21}$$

Clearly, $G_l \subset A_l$ (since $B_l \subset A_l$) and

$$G_l \subset S \setminus B_{l+j} \quad \text{for all } j = 1, 2, \ldots \tag{22}$$

The sets G_l are pairwise disjoint because, by (21) and (22),

$$G_{l+m} \cap G_l \subset B_{l+m} \cap (S \setminus B_{l+m}) = \varnothing, \quad \text{for all } l, m = 1, 2, \ldots$$

Moreover, the sets G_l are of positive measure. Indeed, by (20) and (21),

$$\begin{aligned}\mu(G_l) &= \mu(B_l) - \mu\Big(B_l \cap \bigcup_{k=l+1}^{\infty} B_k\Big) \geqslant \mu(B_l) - \sum_{k=l+1}^{\infty} \mu(B_k)\\ &= \mu(B_l) - \sum_{m=1}^{\infty} \mu(B_{l+m}) \geqslant \mu(B_l) - \sum_{m=1}^{\infty} \frac{1}{3^m}\mu(B_l) = \frac{1}{2}\mu(B_l) > 0.\end{aligned}$$

Hence, the sequence (G_l) satisfies the required conditions (i) and (ii) of our lemma. □

Proof of Lemma 2.5. The identity is true for $x = 0$. For $x \in L_{\mathcal{M}}(\mu) \setminus \{0\}$ with $p(x) < \infty$, set $y = p(x) \cdot x$ and notice that $\rho_{\mathcal{M}}(y/p(x)) = p(x)$. Hence, by (6), $\|y\|_{\mathcal{M}} \leqslant p(x)$. On the other hand, using the assumption that $\mathcal{M}(\cdot, s)$ is nondecreasing, for every positive $\lambda < p(x)$ we obtain $\rho_{\mathcal{M}}(y/\lambda) \geqslant p(x) > \lambda$, whence, by (6) again, $\|y\|_{\mathcal{M}} = p(x)$, as claimed. □

Proof of Lemma 2.6. Let $D = \{e_n : n \in \mathbb{N}\}$, where e_n denotes the nth unit vector. Obviously, D is a maximal subset of discrete and pairwise disjoint elements of ℓ_Φ. By Lemma 2.2,

$$\delta(\ell_\Phi) = \inf_{n\in\mathbb{N}} r(e_n). \tag{23}$$

We have $r(e_n) = \varphi_n(\infty)$, $n = 1, 2, \dots$ Indeed, by (8) and Lemma 2.5, for every $t \in \mathbb{R}$ we have $\|te_n\rho_\Phi(te_n)\| = \rho_\Phi(te_n) = \varphi_n(t)$, and thus $\|t\varphi_n(t)e_n\| = \varphi_n(t)$. Consequently,

$$r(e_n) = \lim_{t\to\infty} \|te_n\|_\Phi = \lim_{t\to\infty} \|t\varphi_n(t)e_n\|_\Phi = \varphi_n(\infty).$$

Now, by (23), we obtain $\delta(\ell_\Phi) = \inf_{n\in\mathbb{N}} \varphi_n(\infty)$. □

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