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Closed ideals in a new class of algebras of holomorphic functions on the disc

Abstract. We define a new class of Banach algebras of holomorphic functions on the unit disc $\mathbb{D}$ which contains the algebras studied in [GMR2] and [GW]. To a function G of the class $\mathcal{A}^1(\mathbb{D})$ nowhere vanishing in $\mathbb{D}$ we associate a Banach algebra $\mathcal{A}^n_G(\mathbb{D})$ contained in the disc algebra $\mathcal{A}(\mathbb{D})$. We prove that all closed ideals of $\mathcal{A}^n_G(\mathbb{D})$ of at most countable hull are of the standard form.

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1. Introduction. The algebras of holomorphic functions on the disc which appeared in the paper [GMR2] attract attention both by their relation with convolution algebras of functions on the half-line $\mathbb{R}^+$ studied in [GMR1] as well as examples of non-homogeneous algebras with interesting ideals structure.

Recall that these algebras denoted by $\mathcal{A}^n_{\pm 1}(\mathbb{D})$ ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) are subalgebras of the disc algebra $\mathcal{A}(\mathbb{D})$ of functions holomorphic on the unit disc $\mathbb{D} \subset \mathbb{C}$ and continuous on $\overline{\mathbb{D}}$. Moreover, every element f of the space $\mathcal{A}^n_{\pm 1}(\mathbb{D})$ is a function of the class $C^{(n)}$ on $\overline{\mathbb{D}} \setminus \{1, -1\}$, vanishes at 1 and satisfies $\lim_{z \to \pm 1} (z^2 - 1)^k f^{(k)}(z) = 0$, for $k = 1, \ldots, n$.

The natural norm in $\mathcal{A}^n_{\pm 1}(\mathbb{D})$ is given by the formula

$$\|f\|_n = \sup_{z \in \mathbb{D}} |f(z)| + \sum_{k=1}^{n} \sup_{z \in \mathbb{D}} |(z^2 - 1)^k f^{(k)}(z)|.$$

While most of the subalgebras of the algebra $\mathcal{A}(\mathbb{D})$ studied from the point of view of the ideal structure are homogeneous, the algebras $\mathcal{A}^n_{\pm 1}(\mathbb{D})$ are not invariant under rotations.

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Nevertheless, all closed ideals of $\mathcal{A}^n_{\pm1}(\mathbb{D})$ can be described as the most natural ones which can be constructed in these algebras. So, we can say that all closed ideals in the algebras $\mathcal{A}^n_{\pm1}(\mathbb{D})$ are standard.

Denote by ζ the identity function $z \to z$.

The description of closed ideals in $\mathcal{A}^n_{\pm1}(\mathbb{D})$ was done in [GW] and it is based on a simple observation that the mapping $\phi\colon \mathcal{A}^n_{\pm1}(\mathbb{D}) \ni f \to (\zeta^2-1)^n f$ sends $\mathcal{A}^n_{\pm1}(\mathbb{D})$ onto a closed ideal of the Korenblyum algebra $\mathcal{A}^n(\mathbb{D})$ of elements of $\mathcal{A}(\mathbb{D})$ which are of the class $C^{(n)}$ on $\overline{\mathbb{D}}$. The ideals of the latter algebra are well known (see [Ko]) what permits to describe also the ideals of the algebra $\mathcal{A}^n_{\pm1}(\mathbb{D})$.

Every closed ideal $I \subset \mathcal{A}^n_{\pm1}(\mathbb{D})$ is uniquely described by a pair $(U_I, \mathfrak{h}(I))$, where U_I is an inner function which divides all elements of the ideal and $\mathfrak{h}(I) = (h_0(I), h_1(I), \ldots, h_n(I))$ where $h_0(I) = \{z \in \mathbb{T}\setminus\{1\} : f(z) = 0,\ f \in I\}$ is called the hull of I and $h_j(I) = \{z \in \mathbb{T}\setminus\{-1,1\} : f(z) = f'(z) = \cdots = f^{(j)}(z) = 0,\ f \in I\}$ for $1 \leqslant j \leqslant n$.

Recall that a function U holomorphic on $\mathbb{D}$ is inner if $\lim_{t\nearrow 1} |U(te^{i\theta})| = 1$ for $d\theta$-almost all $e^{i\theta}$. The inner function U_I divides f (denoted by $U|f$) if the quotient f/U_I is a bounded holomorphic function on $\mathbb{D}$.

The ideal I is then given by the formula

$$I(U_I;\mathfrak{h}(I)) = \{f \in \mathcal{A}^n_{\pm1}(\mathbb{D}) : U_I|f \text{ and } f^{(j)}|h_j(I) \equiv 0,\ 0 \leqslant j \leqslant n\}. \tag{1}$$

The algebras $\mathcal{A}^n_{\pm1}(\mathbb{D})$ are not unital. They appeared in the paper [GMR2] as completions of the spaces of Gelfand transforms of several non-unital convolution algebras of integrable functions on $\mathbb{R}_+$.

Obviously, $\mathcal{A}^n_{\pm1}(\mathbb{D})$ embeds in a natural way in the unital algebra $\mathcal{A}^n_{\pm1}(\mathbb{D}) \oplus \mathbb{C}$ and the problem of description of closed ideals of the latter algebra is even more general than that of the ideals of $\mathcal{A}^n_{\pm1}(\mathbb{D})$.

The algebras introduced and studied in the present paper are generalizations of the unital algebras $\mathcal{A}^n_{\pm1}(\mathbb{D}) \oplus \mathbb{C}$. We substitute the function $\zeta^2 - 1$ by a function G of the class $\mathcal{A}(\mathbb{D})$ nowhere vanishing in $\mathbb{D}$.

Our principal motivation is the question if the similar description of ideals is possible in algebras consisting of functions whose derivatives are "controlled" not only at points ±1 but at an infinite number of element of the circle $\mathbb{T}$.

After introducing the algebra $\mathcal{A}^n_G(\mathbb{D})$ we distinguish the class of its standard ideals and we look for conditions assuring that a given closed ideal $I \subset \mathcal{A}^n_G(\mathbb{D})$ is standard.

In this paper, which should be considered as the first step in the study of algebras $\mathcal{A}^n_G(\mathbb{D})$, we prove that at least for $G \in \mathcal{A}^1(\mathbb{D})$ the ideals of the algebra $\mathcal{A}^n_G(\mathbb{D})$ with countable hull $h_0(I)$ are of the standard form. This result is a consequence of the principal theorem from the paper [SW], which in turn generalizes the results from [AZ].

If the function G is of the class $\mathcal{A}^n(\mathbb{D})$, stronger results about ideals of the algebra $\mathcal{A}^n_G(\mathbb{D})$ can be obtained. However the arguments used in the proofs are of a different nature hence this case will be treated in a separate paper [MW].

2. Algebras $\mathfrak{B}^n_G(\mathbb{D})$ and $\mathcal{A}^n_G(\mathbb{D})$. Although principal objects of our interest are the algebras denoted by $\mathcal{A}^n_G(\mathbb{D})$, we define at the beginning the family of algebras

$\mathfrak{B}^n_G(\mathbb{D})$ which contain each $\mathcal{A}^n_G(\mathbb{D})$ as a closed subalgebra.

Definition 1 Let $G \in \mathcal{A}(\mathbb{D})$ and suppose that $h_0(G) = \{z \in \overline{\mathbb{D}} : G(z) = 0\} \subset \mathbb{T}$. Denote by $\mathfrak{B}^n_G(\mathbb{D})$ $(n \in \mathbb{N})$ the space of all functions $f \in \mathcal{A}(\mathbb{D})$ such that

1. f is of class $C^{(n)}$ on $\overline{\mathbb{D}} \setminus h_0(G)$.
2. For all $1 \leqslant j \leqslant n$ the function $G^j f^{(j)}$ on $\mathbb{D}$ extends to an element of $\mathcal{A}(\mathbb{D})$.

The space $\mathfrak{B}^n_G(\mathbb{D})$ is endowed with the norm

$$\|f\|_{G,n} = \sum_{0\leqslant k\leqslant n} \sup_{z\in\mathbb{T}} |G^k(z) f^{(k)}(z)|.$$

By $\mathcal{A}^n_G(\mathbb{D})$ we denote the closure in $\mathfrak{B}^n_G(\mathbb{D})$ of the space of polynomials $\mathbb{C}[z]$. □

Observe that by the maximal module principle

$$\|f\|_{G,n} = \sum_{0\leqslant k\leqslant n} \sup_{z\in\mathbb{D}} |G^k(z) f^{(k)}(z)|.$$

for elements of the space $\mathfrak{B}^n_G(\mathbb{D})$.

For $n = 0$ this norm coincides with $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Proposition 2.1 *There exists a constant $C > 0$ such that for arbitrary functions f, g on $\mathbb{T}$ for which $\|f\|_{G,n}$, $\|g\|_{G,n}$ makes sense*

$$\|fg\|_{G,n} \leqslant C\|f\|_{G,n}\|g\|_{G,n}. \tag{2}$$

The space $\mathfrak{B}^n_G(\mathbb{D})$ is a unital Banach algebra and $\mathcal{A}^n_G(\mathbb{D})$ is a unital Banach subalgebra of $\mathfrak{B}^n_G(\mathbb{D})$.

Proof By Leibniz formula we obtain

$$\begin{aligned} \sup_{z\in\mathbb{T}} |G^k(z)(fg)^{(k)}(z)| &= \sup_{z\in\mathbb{T}} \sum_{j=0}^{k} \binom{k}{j} |G^k(z) f^{(j)}(z) g^{(k-j)}(z)| \leqslant \\ &\leqslant \sum_{j=0}^{k} \binom{k}{j} \sup_{z\in\mathbb{T}} |G^j f^{(j)}(z)| \sup_{z\in\mathbb{T}} |G^{(k-j)} g^{(k-j)}(z)|. \end{aligned}$$

This implies the formula (2) immediately.

The completeness of the space $\mathfrak{B}^n_G(\mathbb{D})$ is obvious. The space $\mathcal{A}^n_G(\mathbb{D})$ is the closure in $\mathfrak{B}^n_G(\mathbb{D})$ of the algebra of polynomials, hence it is a unital subalgebra of $\mathfrak{B}^n_G(\mathbb{D})$. ■

The algebra $\mathcal{A}^n(\mathbb{D})$ provided with its usual norm $\sum_{j=1}^n \sup_{z\in\mathbb{D}} |f^{(j)}(z)|$ is continuously immersed in $\mathfrak{B}^n_G(\mathbb{D})$. The polynomials are dense in $\mathcal{A}^n(\mathbb{D})$, hence $\mathcal{A}^n(\mathbb{D})$ is contained and dense in $\mathcal{A}^n_G(\mathbb{D})$.

Proposition 2.2 *All elements of the algebra* $\mathcal{A}^n_G(\mathbb{D})$ *satisfy*

$$G^j f^{(j)}(z) = 0$$

for $z \in h_0(G)$, $j = 1, \dots, n$.

Proof For every $z_0 \in h_0(G)$ and $1 \leqslant j \leqslant n$ the functional $f \to G^j(z_0) f^{(j)}(z_0)$ is continuous on $\mathfrak{B}^n_G(\mathbb{D})$ and vanishes on the space $\mathcal{A}^n(\mathbb{D})$ hence it vanishes also on $\mathcal{A}^n_G(\mathbb{D})$. ∎

It is natural to ask if $\mathfrak{B}^n_G(\mathbb{D}) = \mathcal{A}^n_G(\mathbb{D})$. Proposition 2.3 causes serious doubts with respect to this. By definition, for $z_0 \in h_0(I)$ and $f \in \mathfrak{B}^n_G(\mathbb{D})$ the limit of $G^j(z) f^{(j)}(z)$ at z_0 does exist but not necessarily it is zero.

Nevertheless, it can be proved that $\mathfrak{B}^n_G(\mathbb{D}) = \mathcal{A}^n_G(\mathbb{D})$ if G is a polynomial nowhere vanishing on $\mathbb{D}$ and with zeros of multiplicity 1 in $\mathbb{T}$. This fact is not obvious and it will be studied in the mentioned paper [MW].

The following result is a slight modification of a well known fact.

Lemma 2.3 *Let* X *be a compact topological space. Suppose that* $C(X) \ni f_m \to f$ *uniformly on* X. *Denote by* S *the set of zeros of* f. *Let* ψ_m *be a bounded sequence in* $C(X)$ *such that all elements of the sequence vanish on* S *and* $\psi_m \to 1$ *uniformly on the complement of arbitrary neighbourhood of* S. *Then* $f_m \psi_m \to f$ *uniformly.*

Proof Let $M = \sup_{m\in\mathbb{N},\, z\in X} |\psi_n(z)|$. For given $\varepsilon > 0$ let $O = \{z \in X : |f(z)| < \varepsilon/2\}$. There is N such that $|f_m(z)| \leqslant \varepsilon/2M$ for $m > N$, $z \in O$. Then $|f_m\psi_m(z) - f(z)| < \varepsilon$ for $z \in O$. On the other hand by supposition $f_m\psi_m(z) \to f(z)$ uniformly on the complement of O. The proof follows. ∎

The functions of the form $(z - z_0)^p$, $0 < p < 1$ are examples of elements of the space $\mathcal{A}^n_G(\mathbb{D})$ which are not smooth.

Let $z_0 \in \mathbb{T}$. Let $\log z$ be the branch of logarithm defined $\mathbb{C}\setminus\mathbb{R}^+ z_0$ and $(z-z_0)^p = e^{p\log(z-z_0)}$.

Lemma 2.4 *Suppose that* $G \in \mathcal{A}^1(\mathbb{D})$ *and let* $z_0 \in h_0(G)$, $p > 0$. *Then* $(\zeta - z_0)^p \in \mathcal{A}^n_G(\mathbb{D})$.

Proof For every $1 < t$ the function $(z - tz_0))^p$ is of the class $\mathcal{A}^n(\mathbb{D})$ hence it belongs to $\mathcal{A}^n_G(\mathbb{D})$. It suffice to prove that $\lim_{t\searrow 1}(z - (1 + tz_0))^p = (z - z_0))^p$ in the norm of $\mathcal{A}^n_G(\mathbb{D})$.

We must prove prove that for every $0 \leqslant k \leqslant n$

$$\lim_{t\searrow 1}((\zeta - tz_0)^p - (\zeta - z_0))^p)^{(k)} G^k = 0$$

uniformly on $\mathbb{D}$.

It is obvious in case of $k = 0$. For $k \in \mathbb{N}$ we obtain

$$\begin{aligned}((\zeta - tz_0))^p)^{(k)} G^k &= p(p-1)\dots(p-k+1)(\zeta - tz_0))^{p-k} G^k = \\ &= p(p-1)\dots(p-k+1)(\zeta - tz_0)^p \left(\frac{\zeta - z_0}{\zeta - tz_0}\right)^k \frac{G^k}{(\zeta - z_0)^k}.\end{aligned}$$

The factors $f_t := (\zeta - tz_0)^p$ tend uniformly to $f = (\zeta - z_0)^p$ for $t \searrow 1$. The factors $\psi_t := \left(\frac{\zeta - z_0}{\zeta - tz_0}\right)^k$ consist of continuous functions which vanish at z_0, are uniformly bounded and tend to 1 almost uniformly on $\overline{\mathbb{D}} \setminus \{z_0\}$. By Lemma 2.3 $f_t\psi_t$ tends to $f = (\zeta - z_0)^p$ uniformly. The function $\frac{G^k}{(\zeta - z_0)^k}$ is continuous by the differentiability of G.

Finally we obtain

$$\begin{aligned} \lim_{t \searrow 1} ((\zeta - tz_0))^p)^{(k)} G^k &= p(p-1)\dots(p-k+1)(\zeta - z_0)^p \frac{G^k}{(\zeta - z_0)^k} = \\ &= ((\zeta - z_0))^p)^{(k)} G^k, \end{aligned}$$

the limit being uniform.

The proof follows. ■

The supposition of differentiability of G can we weakened. Notice that we have used the differentiability of G only at $z_0 \in h_0(G)$.

On the other hand, for $G(z) = (z-1)^{\frac{1}{2}}$ the functions $(z-1)^p$, $0 < p < 1$ do not belong to the space $\mathfrak{B}_G^n(\mathbb{D})$ for $n \in \mathbb{N}$.

The continuity of G is not sufficient for the functions $(z - z_0)^p$, $0 < p < 1$ to be elements of $\mathcal{A}_G^n(\mathbb{D})$.

These functions play an important rôle in the proof of the principal theorem.

3. The ideals of at most countable hull. The following theorem was proved in [SW].

THEOREM 3.1 *Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach algebra and subalgebra of the disc algebra $\mathcal{A}(\mathbb{D})$. Suppose that*

(H1) The space of polynomials is dense in $\mathcal{B}$.

(H2) $\lim_{n\to\infty} \|\zeta^n\|_{\mathcal{B}}^{\frac{1}{n}} = 1$.

(H3) There exist $k \geqslant 0$ and $C > 0$ such that

$$|1 - |\lambda||^k \|f\|_{\mathcal{B}} \leqslant C \|(\zeta - \lambda) f\|_{\mathcal{B}}, \; f \in \mathcal{B}, \; |\lambda| < 2.$$

(D) For every $z_0 \in \mathbb{T}$ there exists $N(z_0) \in \mathbb{N}_0$ with the following properties:

(1) all functionals $\mathcal{B} \ni f \to f^{(j)}(z_0)$, $0 \leqslant j \leqslant N(z_0)$, are well defined and continuous,

(2) there exists a sequence (φ_m) in the algebra $\mathcal{B}$ such that $\varphi_m(z_0) = 0$ and

$$\lim_{n\to\infty} \|(\zeta - z_0)^{N(z_0)+1} \varphi_m - (\zeta - z_0)^{N(z_0)+1}\|_{\mathcal{B}} = 0.$$

Then for every closed ideal I of $\mathcal{B}$ with at most countable hull $h_0(I)$ there exists an inner function U_I such that for

$$h_m(I) := \{z \in \mathbb{T} : \; N(z) \geqslant m, \; \forall f \in I, \; 0 \leqslant j \leqslant m, \; f^{(j)}(z) = 0\}$$

it holds

$$I = \{f \in \mathcal{B} : \; U_I | f \textit{ and } f^{(m)}(z) = 0 \textit{ for } z \in h_m(I), \; m \in \mathbb{N}_0\}.$$

Theorem 3.1 applies to algebra $\mathcal{A}^n_G(\mathbb{D})$.

THEOREM 3.2 *Let G be a function of class $\mathcal{A}^1(\mathbb{D})$ nowhere vanishing in $\mathbb{D}$. The algebra $\mathcal{A}^n_G(\mathbb{D})$ satisfies conditions (H2), (H3) and (D)(1) with $N(z) = 0$ if $z \in h_0(G)$ and $N(z) = n$ otherwise. Every closed ideal I of $\mathcal{A}^n_G(\mathbb{D})$ such that $h_0(I)$ is at most countable has the form*

$$I = I(U_I; h_0(I), \ldots, h_n(I)).$$

PROOF We verify first condition (H2). Denote $M = \sup_{z\in\mathbb{T}} |G(z)|$. For $m > n$

$$\begin{aligned}
\|\zeta^m\|_{G,n} &= \sum_{0\leqslant k\leqslant n} \sup_{z\in\mathbb{T}} |G^k(z)(z^m)^{(k)}| = \\
&= \sum_{0\leqslant k\leqslant n} \sup_{z\in\mathbb{T}} |G^k(z)z^{m-k})m(m-1)\ldots(m-k+1) \leqslant \\
&\leqslant nm^n \sum_{0\leqslant k\leqslant n} M^k.
\end{aligned}$$

Then $\lim_{m\to\infty} \|\zeta^m\|^{\frac{1}{m}}_{G,n} = 1$.

Condition (H3) is satisfied thanks to Proposition 2.2.

$$\|f\|_{G,n} = \|(\zeta-\lambda)f(\zeta-\lambda)^{-1}\|_{G,n} \leqslant C\|(\zeta-\lambda)f\|_{G,n}\|(\zeta-\lambda)^{-1}\|_{G,n}.$$

Next,

$$\begin{aligned}
\|(\zeta-\lambda)^{-1}\|_{G,n} &= \sum_{k=0}^{n} \sup_{z\in\mathbb{T}} |((z-\lambda)^{-1})^{(k)} G^k(z)| = \\
&= \sum_{k=0}^{n} \sup_{z\in\mathbb{T}} k!|(z-\lambda)^{-(k+1)}G^k(z)| \leqslant \\
&\leqslant (n+1)!(1-|\lambda|^{n+1})^{-1}.
\end{aligned}$$

We have proved the inequality

$$\|f\|_{G,n} \leqslant C\|(\zeta-\lambda)f\|_{G,n}(n+1)!(1-|\lambda|^{n+1})^{-1},$$

hence the algebra $\mathcal{A}^n_G(\mathbb{D})$ satisfies the condition $(H3)$ with $k = n+1$.

The elements of the algebra $\mathcal{A}^n_G(\mathbb{D})$ are continuous at $z \in h(G)$ and there exist elements of the algebra which are not derivable at these points (like the functions $(\zeta - z)^p$, $0 < p < 1$). Using the notation of Theorem 3.1 we have $N(z) = 0$ for $z \in h_0(G)$. The restriction of elements of $\mathcal{A}^n_G(\mathbb{D})$ to $\overline{\mathbb{D}} \setminus h(G)$ are uniform limits of functions analytic in $\mathbb{D}$ and of the class $C^{(n)}$ on $\mathbb{T} \setminus h_0(G)$, hence $N(z) = n$ for $z \in \mathbb{T} \setminus h_0(G)$.

The condition (D)(2) is satisfied with $\varphi_m(z) = (z - z_0)^{\frac{1}{m}}$ for $z_0 \in h_0(G)$ and $\varphi_m(z) = \frac{z-z_0}{z-(1+\frac{1}{m})z_0}$ when $z_0 \in \mathbb{T} \setminus h_0(G)$. In fact, in the case of $z_0 \in h_0(G)$ the uniform convergence of $(\zeta - z_0)^{\frac{1}{m}+1}$ to $(\zeta - z_0)$ follows by Lemma 2.3. Next,

$((z-z_0)^{1+\frac{1}{m}}-(z-z_0))'G(z) = (1+\frac{1}{m})((z-z_0)^{\frac{1}{m}}-1)G(z)$ tends uniformly to zero by the same argument. For $j>1$ the derivative $((z-z_0)^{1+\frac{1}{m}}-(z-z_0))^{(j)}$ multiplied by G^j has the form

$$(1+\tfrac{1}{m})\tfrac{1}{m}(\tfrac{1}{m}-1)\dots(\tfrac{1}{m}-j+2)(z-z_0)^{\frac{1}{m}-j+1}G^j(z).$$

Thanks to the differentiability of G the function $G(z)/(z-z_0)$ is bounded hence the coefficient $\frac{1}{m}$ assures the uniform convergence of the sequence to zero.

In the case of $z_0 \in \mathbb{T}\setminus h_0(G)$ we must prove the convergence to zero in $\mathcal{A}^n_G(\mathbb{D})$ of

$$\psi_m(z) = \varphi_m(z)(z-z_0)^{n+1}-(z-z_0)^{n+1} = \frac{z_0}{m}\cdot\frac{(z-z_0)^{n+1}}{z-(1+\frac{1}{m})z_0}.$$

It suffice to prove the uniform convergence of all derivatives of ψ_m up to the order n because the topology of $\mathcal{A}^n$ is stronger than that of $\mathcal{A}^n_G(\mathbb{D})$.

We obtain for $0\leqslant k\leqslant n$:

$$((z-z_0)^{n+1}(z-(1+\tfrac{1}{m})z_0)^{-1})^{(k)} =$$

$$= \textstyle\sum_{j=0}^{k} c_{nkj}(z-z_0)^{n+1-j}(z-(1+\frac{1}{m})z_0)^{-(k+1-j)},$$

where the constants c_{nkj} do not depend of m. All terms of the sum are bounded hence the coefficient $\frac{1}{m}$ assures that $\psi_m^{(k)}$ converges uniformly to zero for $0\leqslant k\leqslant n$.

The proof follows by Theorem 3.1. ∎

Let us apply Theorem 3.2 in the case of a closed ideal I_f generated by a function $0\neq f\in\mathcal{A}^n_G(\mathbb{D})$.

According to Factorization Theorem (see e. g. [H] p. 67) the function f is uniquely expressible in the form $f=U_f g$ where U_f is an inner function and g is an outer function, what means that it of the form

$$g(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|g(\theta)|d\theta(\theta)\right).$$

. The outer factor nowhere vanishes on $\mathbb{D}$.

Corollary 3.3 *Let G be a function of class $\mathcal{A}^1(\mathbb{D})$ nowhere vanishing on $\mathbb{D}$ and let $f\in\mathcal{A}^n_G(\mathbb{D})$. Suppose that the set $h_0(f)$ of zeros of the function f is at most countable. If the inner-outer factorization of f is of the form $f=U_f g$ and $h_j(f) = \{z\in\mathbb{T}\setminus h_0(G): f(z)=f'(z)=\dots=f^{(j)}(z)=0,\ 0\leqslant j\}$ for $1\leqslant j\leqslant n$, then the closed ideal I_f generated in $\mathcal{A}^n_G(\mathbb{D})$ by f coincides with $I(U_f;h_0(f),\dots,h_n(f))$.*

In particular, if f is an outer function and $h_0(f)\subset h_0(G)$ then all functions f^k, $k\in\mathbb{N}$ generate the same ideal $I(1;h_0(f),\varnothing,\dots,\varnothing)$.

Theorem 3.2 applied to the case of $G\in\mathcal{A}^n(\mathbb{D})$ such that $h_0(G)$ is numerable leads to theorems which are analogues of the results from the paper [GW]. This is the principal subject of the paper [MW].

Nevertheless, there is a great number of open problems concerning the structure of algebras $\mathfrak{B}^n_G(\mathbb{D})$ and $\mathcal{A}^n_G(\mathbb{D})$.

In the case considered in the present paper it is necessary to investigate if for several G there appear ideals of $\mathcal{A}^n_G(\mathbb{D})$ which are not standard. As shows the history of the Benett-Gilbert conjecture about the ideals of the algebra $A^+(\mathbb{D})$ of analytic functions with absolutely convergent Taylor series, the construction of counterexamples in this area can be sometimes very difficult. The conjecture that all closed ideals in $A^+(\mathbb{D})$ are standard was solved in negative after 22 years by J. Esterle (see [E]).

The relation between the algebras $\mathfrak{B}^n_G(\mathbb{D})$ and $\mathcal{A}^n_G(\mathbb{D})$ is one of the principal problems. If there exist cases when $\mathfrak{B}^n_G(\mathbb{D}) \neq \mathcal{A}^n_G(\mathbb{D})$ the description of ideals of the algebra $\mathfrak{B}^n_G(\mathbb{D})$ is an interesting challenge. The definition of $\mathfrak{B}^n_G(\mathbb{D})$ suggests that the list of the standard ideals in this algebra should include in particular the ideals of the form

$$I_{w,k} = \{f \in \mathfrak{B}^n_G(\mathbb{D}) : \lim_{z\to w} G^j(z)f^{(j)}(z) = 0,\ 0 \leqslant j \leqslant k\}$$

for $w \in h_0(G)$ and $1 \leqslant k \leqslant n$.

However, Corollary 2.3 proved in [GW] implies that for smooth G and for w which is an isolated point in $h_0(G)$ this ideal coincides with $I(1, \{w\}, \varnothing, \dots, \varnothing)$.

Only for w which are accumulation points of $h_0(G)$ this ideals can be essentially bigger.

The list of problems not undertook in this paper can be extended. It is just the first approach to a very extensive subject.

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