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Asymptotically isometric and isometric copies of ℓ_1 in some Banach function lattices

Dedicated to Professor Julian Musielak on his 85th birthday

Abstract. We identify the class of Calderón-Lozanovskii spaces that do not contain an asymptotically isometric copy of ℓ_1, and consequently we obtain the corresponding characterizations in the classes of Orlicz-Lorentz and Orlicz spaces equipped with the Luxemburg norm. We also give a complete description of order continuous Orlicz-Lorentz spaces which contain (order) isometric copies of $\ell_1^{(n)}$ for each integer $n \geq 2$. As an application we provide necessary and sufficient conditions for order continuous Orlicz-Lorentz spaces to contain an (order) isometric copy of ℓ_1. In particular we give criteria in Orlicz and Lorentz spaces for (order) isometric containment of $\ell_1^{(n)}$ and ℓ_1. The results are applied to obtain the description of universal Orlicz-Lorentz spaces for all two-dimensional normed spaces.

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A Banach space X contains an asymptotically isometric copy of ℓ_1 if for every null sequence $(\varepsilon_n)_{n=1}^{\infty}$ in $(0,1)$, there exists a sequence $(x_n)_{n=1}^{\infty}$ of norm one elements in X such that

$$\sum_n (1-\varepsilon_n)\,|t_n| \leq \Big\| \sum_n t_n x_n \Big\|_X \leq \sum_n |t_n|$$

for all finite sequences (t_n) of scalars.

Dowling and Lennard [7] used the notion of an asymptotically isometric copy of ℓ_1 to show that every reflexive subspace of $L_1(0,1)$ fails the fixed point property for nonexpansive self-maps on closed bounded convex sets. Further Dowling, Lennard and Turrett [6] showed that if a Banach space contains asymptotically isometric

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copies of ℓ_1, then its dual space fails the fixed point property. In [5], Dilworth, Girardi and Hagler proved that a Banach space X contains asymptotically isometric copies of ℓ_1 if and only if $L_1(0,1)$ is linearly isometric to a subspace of the dual space X^* of X. This result shows a direct link between the papers [7] and [6].

Motivated by these results we consider the question when some important Banach lattices on measure spaces contain asymptotically isometric copies of ℓ_1. Our main aim is to identify a class of Calderón-Lozanovskii spaces that do not contain an asymptotically isometric copy of ℓ_1 and show applications to Orlicz-Lorentz and Orlicz spaces. We give an answer to the question posed in [6] showing that a separable Orlicz space L_φ equipped with the Luxemburg norm on a non-atomic measure space generated by an Orlicz N-function φ (at zero) does not contain asymptotically isometric copies of ℓ_1. It has been shown in [6] that in the case of Orlicz norm, the separable Orlicz space L_φ may contain an asymptotically isometric copy of ℓ_1. This occurs when φ satisfies the appropriate condition Δ_2 and its conjugate does not.

In the second part of the paper we give a complete description of order continuous Orlicz-Lorentz spaces which contain (order) isometric copies of $\ell_1^{(n)}$ for each integer $n \geq 2$. On the basis of this result we provide a necessary and sufficient condition for the order continuous Orlicz-Lorentz spaces to contain (order) isometric copies of ℓ_1. We complete this characterization by an equivalent condition for those spaces containing isometrically the space $L_1(A)$ for some measurable set A.

In the third part we obtain the corresponding results for Orlicz and Lorentz spaces. They recover and expand the characterizations received earlier in Lorentz and Orlicz spaces by Briskin and Semenov [2], and Wójtowicz [19], respectively. We conclude with the description of universal Orlicz-Lorentz spaces for all two-dimensional normed spaces.

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of real, non-negative real and natural numbers, respectively. Let $(\Omega, \mathcal{S}, \mu)$ be a σ-finite measure space, and L^0 be the set of all real valued μ-measurable functions. We say that $(X, \|\cdot\|_X)$ is a *Banach function lattice* (in short *Banach lattice*) on $(\Omega, \mathcal{S}, \mu)$ if X is an ideal in L^0 and whenever $x, y \in X$ and $|x| \leq |y|$ a.e., then $\|x\|_X \leq \|y\|_X$. Given a measurable set $A \subset \Omega$, $X(A)$ denotes the space of all elements in X restricted to A, i.e., $X(A) = \{x\chi_A \colon x \in X\}$. By $L_1 = L_1(\Omega)$ and $L_\infty = L_\infty(\Omega)$ we denote the spaces of integrable and μ-essentially bounded, real valued functions on Ω, respectively. They are equipped with the standard norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. An element $x \in X$ is called *order continuous* if for every $0 \leq x_n \leq |x|$ such that $x_n \downarrow 0$ a.e. it follows that $\|x_n\|_X \to 0$. By X_a denote the set of all order continuous elements of X. We say that X satisfies the *Fatou property* whenever for any $0 \leq x_n \in X$ and $x \in L^0$ such that $x_n \uparrow x$ a.e. and $\sup_n \|x_n\|_X < \infty$ we have that $x \in X$ and $\|x\|_X = \lim_n \|x_n\|_X$. By X^+ we denote the cone of non-negative elements in X.

Given two Banach lattices X, Y we will write $X = Y$ whenever the sets coincide and the norms are equivalent. Two expressions U, V are equivalent if for some constants $a, b > 0$ we have $aU \leq V \leq bU$. In this case we write $U \approx V$. The symbol $X \simeq Y$ means that X and Y are isometrically isomorphic. Banach lattices X and Y over $(\Omega, \mathcal{S}, \mu)$ are locally equivalent whenever $X(A) = Y(A)$ for any $A \in \mathcal{S}$ with $\mu(A) < \infty$.

The Köthe dual space X' of X is a collection of all elements $y \in L^0$ such that

$$\|y\|_{X'} = \sup\Big\{ \int_\Omega |xy|\, d\mu\colon\ \|x\|_X \le 1\Big\} < \infty.$$

The space X' equipped with the norm $\|\cdot\|_{X'}$ is a Banach function lattice with the Fatou property. It is well known that $(X_a)^* \simeq X'$ (see [2, 12, 14]).

A Banach lattice X is said to be *strictly monotone* if for any $x, y \in X$, we have $\|x\| < \|y\|$ whenever $0 \le x \le y$ and $x \ne y$.

Given $x \in L^0$, its *distribution function* is defined by $d_x(\lambda) = \mu\{t \in \Omega\colon |x|(t) > \lambda\}$, $\lambda \ge 0$, and its *decreasing rearrangement* by $x^*(t) = \inf\{s > 0\colon\ d_x(s) \le t\}$, for all $t \ge 0$. A Banach lattice X is called a *rearrangement invariant Banach space* (in short *r.i. Banach space*) if $\|x\| = \|y\|$ whenever $d_x = d_y$ and $x \in X$. The *fundamental function* of r.i. space X is defined by $\phi_X(t) = \|\chi_A\|_X$, where $\mu(A) = t$ and $0 \le t < \mu(\Omega)$.

1. Calderón-Lozanovskii Spaces. The class $\mathcal{U}$ consists of all functions $\Psi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ that are positively homogeneous (i.e., $\Psi(\lambda s, \lambda t) = \lambda\Psi(s,t)$ for every $s, t, \lambda \ge 0$) and concave. Recall that Ψ is concave whenever $\Psi(\alpha s_1 + \beta s_2, \alpha t_1 + \beta t_2) \ge \alpha\Psi(s_1, t_1) + \beta\Psi(s_2, t_2)$ for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and s_i, $t_i \ge 0$, for $i = 1, 2$. Note that any function $\Psi \in \mathcal{U}$ is continuous on $(0,\infty) \times (0,\infty)$.

Given $\Psi \in \mathcal{U}$ and a couple of Banach lattices (X_0, X_1) on $(\Omega, \mathcal{S}, \mu)$, the Calderón-Lozanovskii space is defined as follows

$$\Psi(X_0, X_1) = \big\{x \in L^0\colon\ |x| = \Psi(x_0, x_1) \text{ for some } \ x_i \in X_i^+, i = 0, 1\big\}.$$

For any $1 \le p < \infty$, define a norm on the space $\Psi(X_0, X_1)$, as

$$\|x\|_{\Psi_p(X_0,X_1)} = \inf\big\{(\|x_0\|_{X_0}^p + \|x_1\|_{X_1}^p)^{1/p}\colon\ |x| = \Psi(x_0, x_1)\big\},$$

and for $p = \infty$ as

$$\|x\|_{\Psi_\infty(X_0,X_1)} = \inf\{\max(\|x_0\|_{X_0}, \|x_1\|_{X_1})\colon\ |x| = \Psi(x_0, x_1)\}.$$

Under each norm $\|\cdot\|_{\Psi_p(X_0,X_1)}$, the space $\Psi(X_0, X_1)$ is a Banach lattice [18]. Notice that all norms $\|\cdot\|_{\Psi_p(X_0,X_1)}$, $1 \le p \le \infty$, are equivalent on $\Psi(X_0, X_1)$. Denote by $\Psi_p(X_0, X_1)$ the space $\Psi(X_0, X_1)$ equipped with the norm $\|\cdot\|_{\Psi_p(X_0,X_1)}$.

We say that Ψ satisfies the left (resp., right) Δ_2 condition whenever there exists $C > 0$ such that $\Psi(2s, 2t) \le \Psi(Cs, t)$ (resp., $\Psi(2s, 2t) \le \Psi(s, Ct)$) for all $s, t \ge 0$.

Given $\Psi \in \mathcal{U}$, let

$$\widehat{\Psi}(s,t) = \inf_{u,v>0} \frac{us + vt}{\Psi(u,v)}.$$

It is well known that $\widehat{\widehat{\Psi}} = \Psi$. Moreover, for any $1 \le p \le \infty$,

$$(\Psi(X_0, X_1), \|\cdot\|_{\Psi_p(X_0,X_1)})' = (\widehat{\Psi}(X_0', X_1'), \|\cdot\|_{\Psi_{p'}(X_0',X_1')}),$$

where $1/p + 1/p' = 1$. In other words, $\Psi_p(X_0, X_1)' = \widehat{\Psi}_{p'}(X_0', X_1')$ (see [18]).

We need a lemma which is already implicit in [18]. It is however showed only for a subclass of functions in $\mathcal{U}$ with an advanced proof which uses a deep Köthe duality theorem of Lozanovskii. Here we include a simple proof based on a well-known theorem of Kolmos [13].

LEMMA 1.1 *Assume $\Psi \in \mathcal{U}$, $1 \le p \le \infty$ and (X_0, X_1) is a couple of Banach lattices over $(\Omega, \mathcal{S}, \mu)$ satisfying the Fatou property. Then $\Psi_p(X_0, X_1)$ is a Banach lattice with the Fatou property and for every $x \in \Psi(X_0, X_1)$ there exists an optimal factorization, i.e., there are $x_0 \in X_0^+$ and $x_1 \in X_1^+$ with*

$$|x| = \Psi(x_0, x_1) \quad \textit{and} \quad \|x\|_{\Psi_p(X_0,X_1)} = \|(x_0, x_1)\|_{X_0 \oplus_p X_1},$$

where $\|(x_0, x_1)\|_{X_0 \oplus_p X_1} = (\|x_0\|_{X_0}^p + \|x_1\|_{X_1}^p)^{1/p}$ for $1 \le p \le \infty$ (with the usual interpretation for $p = \infty$.)

PROOF Since for any Banach lattice $X \hookrightarrow X''$, there exists a positive measurable function w such that $X \hookrightarrow L_1(\nu)$ with $d\nu = w d\mu$. The inclusion $X \hookrightarrow X''$ follows immediately from the definition of the Köthe dual space. Now for any $0 < w \in X'$ with $\|w\|_{X'} \le 1$ we have that for any $x \in X$, $\int_\Omega xw\, d\mu \le \|x\|_X$ and so $X \hookrightarrow L_1(\nu)$.

Assume that $0 \le x_n \uparrow |x|$ a.e. and $\sup_{n \ge 1} \|x_n\|_{\Psi_p(X_0,X_1)} < \infty$. Then for all n there exist $u_n \in X_0$, $v_n \in X_1$ such that

$$x_n = \Psi(u_n, v_n), \quad \|(u_n, v_n)\|_{X_0 \oplus_p X_1} < \|x_n\|_{\Psi_p(X_0,X_1)} + 2^{-n}.$$

Hence (u_n) and (v_n) are bounded sequences in X_0 and X_1 respectively, and so in $L_1(\nu)$. It follows by the theorem of Kolmos [13] that by passing to a subsequence we may assume that

$$\frac{1}{n}\sum_{k=1}^n u_k \to u \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^n v_k \to v \quad \textit{a.e.}.$$

By concavity of Ψ we have

$$\sum_{k=1}^n \frac{1}{n}\Psi(u_k, v_k) \le \Psi\Big(\frac{1}{n}\sum_{k=1}^n u_k,\ \frac{1}{n}\sum_{k=1}^n v_k\Big) \quad \textit{a.e.},$$

and so for any $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=1}^n x_k \le \Psi\Big(\frac{1}{n}\sum_{k=1}^n u_k,\ \frac{1}{n}\sum_{k=1}^n v_k\Big) \quad \textit{a.e.}.$$

Then taking limits on both sides we obtain $|x| \le \Psi(u, v)$ a.e.. Clearly this implies that for some $0 \le x_0 \le u$ and $0 \le x_1 \le v$ we have $|x| = \Psi(x_0, x_1)$ a.e., and thus $x \in \Psi(X_0, X_1)$. Now combining the Fatou property of X_0 and X_1 with

$\|(u_n, v_n)\|_{X_0 \oplus_p X_1} < \|x_n\|_{\Psi_p(X_0,X_1)} + 1/n$, we obtain (for $1 \le p < \infty$)

$$\begin{aligned}
\|x\|_{\Psi_p(X_0,X_1)} &\le \|(x_0, x_1)\|_{X_0 \oplus_p X_1} \le \|(u, v)\|_{X_0 \oplus_p X_1} \\
&\le \Big(\liminf_{n\to\infty} \Big\| \frac{1}{n} \sum_{k=1}^{n} u_k \Big\|_{X_0}^{p} + \liminf_{n\to\infty} \Big\| \frac{1}{n} \sum_{k=1}^{n} v_k \Big\|_{X_1}^{p} \Big)^{1/p} \\
&\le \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|(u_k, v_k)\|_{X_0 \oplus_p X_1} \le \lim_{n\to\infty} \|x_n\|_{\Psi_p(X_0,X_1)} \\
&\le \|x\|_{\Psi_p(X_0,X_1)}.
\end{aligned}$$

Hence $\|x\|_{\Psi_p(X_0,X_1)} = \|(x_0, x_1)\|_{X_0 \oplus_p X_1}$ and thus (x_0, x_1) is an optimal factorization of x. Moreover $\|x_n\|_{\Psi_p(X_0,X_1)} \to \|x\|_{\Psi_p(X_0,X_1)}$. The proof is similar for $p = \infty$. ■

We apply the above factorization lemma to study the strict monotonicity of the Calderón-Lozanovskii spaces.

Theorem 1.2 *Let X_i, $i = 0, 1$, be Banach lattices over $(\Omega, \mathcal{S}, \mu)$ satisfying the Fatou property. If $\Psi \in \mathcal{U}$ is such that $\Psi(0, 1) = 0$ (resp., $\Psi(1, 0) = 0$) and X_0 (resp., X_1) is strictly monotone, then the space $\Psi_p(X_0, X_1)$ is strictly monotone for every $1 \le p < \infty$.*

Proof Assume that X_0 is strictly monotone and $\Psi(0, 1) = 0$. Let $0 \le x, y \in \Psi_p(X_0, X_1)$, $x \le y$ and $x \ne y$ on a set A with $\mu(A) > 0$. By Lemma 1.1, there exist $y_j \in X_j^+$, $j = 0, 1$, such that $y = \Psi(y_0, y_1)$ and

$$\|y\|_{\Psi_p(X_0,X_1)} = (\|y_0\|_{X_0}^p + \|y_1\|_{X_1}^p)^{1/p}.$$

Letting $v = x/\Psi(y_0, y_1)$ on supp y, and 0 outside, we have that $0 \le v \le 1$ and $v < 1$ on A. Defining $x_j = v y_j$ for $j = 0, 1$ we get $0 \le x_j \le y_j$. In addition by the assumption $\Psi(0, 1) = 0$, if $y(t) > 0$ then $y_0(t) > 0$. Hence if $t \in A$ then $0 \le x(t)/y(t) < 1$, $y_0(t) > 0$ and so $x_0(t) = (x(t)/y(t))y_0(t) < y_0(t)$. We also have $x = \Psi(x_0, x_1)$. Consequently, in view of strict monotonicity of X_0,

$$\|x\|_{\Psi_p(X_0,X_1)} \le (\|x_0\|_{X_0}^p + \|x_1\|_{X_1}^p)^{1/p} < (\|y_0\|_{X_0} + \|y_1\|_{X_1}^p)^{1/p} = \|y\|_{\Psi_p(X_0,X_1)},$$

which shows strict monotonicity of $\Psi_p(X_0, X_1)$. ■

Remark 1.3 In Theorem 1.2, the assumption $\Psi(0, 1) = 0$ or $\Psi(1, 0) = 0$ is necessary in general. To see this consider $\Psi(s, t) = s + t$, and $X_0 = L_1$ and $X_1 = L_\infty$ on $\mathbb{R}_+$ where $\|x\|_{L_1+L_\infty} = \inf\{\|x_0\|_1 + \|x_1\|_\infty : x = x_0 + x_1,\, x_0 \in L_1, x_1 \in L_\infty\}$. Then $\Psi(0, 1) = \Psi(1, 0) = 1$ and $\Psi_1(L_1, L_\infty) = L_1 + L_\infty$ with equality of norms. By the well known formula $\|x\|_{L_1+L_\infty} = \int_0^1 x^*$ [2, 12], we conclude that $L_1 + L_\infty$ is not strictly monotone although L_1 is strictly monotone.

The following lemma of independent interest shows that under some geometrical assumptions on Banach lattices, an optimal factorization presented in Lemma 1.1

is unique. Before presenting the result we need further notation. For any $\Psi \in \mathcal{U}$ we define $\overline{\Psi}$ on $(0,\infty)\times(0,\infty)$ by

$$\overline{\Psi}(s,t) = \sup\Big\{\frac{\Psi(us,vt)}{\Psi(u,v)} :\ u,v>0\Big\}, \qquad s,t>0.$$

Note that if $\Psi \in \mathcal{U}$ and (X_0,X_1) is a couple of Banach lattices and $|x| = \Psi(x_0,x_1)$ a.e. with $0 \neq x_j \in X_j^+$, then we have $|x| = \Psi(\|x_0\|_{X_0}u_0, \|x\|_{X_1}u_1)$ with $u_j = x_j/\|x_j\|_{X_j}$ for $j=0,1$, which implies $|x| \le \overline{\Psi}\big(\|x_0\|_{X_0},\|x_1\|_{X_1}\big)\Psi(u_0,u_1)$ a.e. and so

$$\|x\|_{\Psi_\infty(X_0,X_1)} \le \overline{\Psi}\big(\|x_0\|_{X_0},\|x_1\|_{X_1}\big). \tag{1}$$

In the proof of lemma below we use the following fact (which follows from [12, Lemma 1.3, Chapter II, p. 55]): if $\Psi \in \mathcal{U}$ and $\overline{\Psi}(s_0,1)=1$ (resp., $\overline{\Psi}(1,s_0)=1$) for some $0<s_0<1$, then $\overline{\Psi}(s,1)=1$ (resp., $\overline{\Psi}(1,s)=1$) for every $0<s<1$.

Lemma 1.4 *Let (X_0,X_1) be a couple of Banach lattices over $(\Omega,\mathcal{S},\mu)$ satisfying the Fatou property. If $\Psi \in \mathcal{U}$ is such that $\Psi(s,t)=0$ if and only if $s=t=0$, then the following holds.*

(i) *If Ψ is strictly concave and there exits $0<s<1$ such that $\overline{\Psi}(s,1)<1$ (resp., $\overline{\Psi}(1,s)<1$) and X_0 is strictly monotone (resp., X_1 is strictly monotone), then for every $x \in \Psi_\infty(X_0,X_1)$ there is an unique optimal factorization of x shown in Lemma 1.1 with $p=\infty$.*

(ii) *If Ψ is strictly concave and X_0 or X_1 is strictly monotone, then for each $1 \le p < \infty$ and for each $x \in \Psi(X_0,X_1)$ there is an unique optimal factorization of x shown in Lemma 1.1.*

Proof (i) Let first $p=\infty$. Assume that X_0 is strictly monotone and $\overline{\Psi}(s,1)<1$ for some $0<s<1$. Fix $x \in \Psi(X_0,X_1)$ with $\|x\|_{\Psi_p(X_0,X_1)}=1$. Suppose that $|x| = \Psi(x_0,x_1)$ with $x_0 \in X_0^+$, $x_1 \in X_1^+$ and $|x| = \Psi(y_0,y_1)$ with $y_0 \in X_0^+$, $y_1 \in X_1^+$ and

$$1 = \|x\|_{\Psi_\infty(X_0,X_1)} = \|(y_0,y_1)\|_{X_0\oplus_\infty X_1} = \|(x_0,x_1)\|_{X_0\oplus_\infty X_1}.$$

By concavity of Ψ we obtain

$$|x| \le y := \Psi\Big(\tfrac{x_0+y_0}{2},\tfrac{x_1+y_1}{2}\Big).$$

Letting $v = x/y$ on supp y and 0 outside, we have that $0 \le v \le 1$. Defining $z_j = v(x_j+y_j)/2$ for $j=0,1$ we have $|x| = \Psi(z_0,z_1)$ with $z_0 \in X_0^+$, $z_1 \in X_1^+$ and so it follows by (1),

$$1 = \|x\|_{\Psi_\infty(X_0,X_1)} \le \overline{\Psi}(\|z_0\|_{X_0},\|z_1\|_{X_1}). \tag{2}$$

We claim that $v = 1$ a.e. on supp y. If we have $v < 1$ on a measurable subset $A \subset$ supp y with $\mu(A) > 0$, then the strict monotonicity of X_0 implies that $s_0 := \|z_0\|_{X_0} < \|(x_0 + y_0)/2\|_{X_0} \leq 1$. Thus by (2),

$$1 = \|x\|_{\Psi_\infty(X_0,X_1)} \leq \overline{\Psi}(\|z_0\|_{X_0}, \|z_1\|_{X_1}) \leq \overline{\Psi}(s_0, 1) \leq 1,$$

and so $\overline{\Psi}(s_0, 1) = 1$. Hence by the fact mentioned before theorem (see Lemma 1.3 in [12]), $\Psi(s, 1) = 1$ for all $0 < s < 1$, which contradicts our assumption.

In consequence we conclude that on supp y we have

$$\Psi(x_0, x_1) = \Psi\Big(\tfrac{x_0+y_0}{2}, \tfrac{x_1+y_1}{2}\Big).$$

Therefore and by our hypotheses that $\Psi(s,t) = 0$ implies $s = t = 0$, and strict concavity of the function Ψ, we have that $x_0 = y_0$, $x_1 = y_1$.

(ii) In the case $1 \leq p < \infty$, we may assume without loss of generality that X_0 is strictly monotone. Similarly as above let $\|x\|_{\Psi_p(X_0,X_1)} = 1$, and let x have two optimal factorizations

$$1 = \|x\|_{\Psi_p(X_0,X_1)} = \|(y_0, y_1)\|_{X_0 \oplus_p X_1} = \|(x_0, x_1)\|_{X_0 \oplus_p X_1}.$$

Then $|x| = \Psi(z_0, z_1)$ with $z_j = (x_j + y_j)/2 \in X_j^+$ for $j = 0, 1$ and so by strict monotonicity of X_0 we get

$$\begin{aligned} 1 = \|x\|_{\Psi_p(X_0,X_1)} \leq \|(z_0, z_1)\|_{X_0 \oplus_p X_1} &< \frac{1}{2}\Big(\|(x_0, x_1)\|_{X_0 \oplus_p X_1} + \|(y_0, y_1)\|_{X_0 \oplus_p X_1}\Big) \\ &= \frac{1}{2}\Big(\|x\|_{\Psi_p(X_0,X_1)} + \|x\|_{\Psi_p(X_0,X_1)}\Big) = 1, \end{aligned}$$

which is a contradiction. Thus $v = x/y = 1$ a.e. on supp y, and we finish the proof as in case (i). ■

In what follows we will use the following fact that for any couple (X_0, X_1) of r.i. spaces, we have $\phi_{\Psi(X_0,X_1)} \approx \Psi(\phi_{X_0}, \phi_{X_1})$ (see [16]).

Theorem 1.5 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic separable measure space and (X_0, X_1) be a couple of r.i. spaces on $(\Omega, \mathcal{S}, \mu)$. Assume that* supp $[\Psi_p(X_0, X_1)]_a = \Omega$. *If X_0' (resp., X_1') is strictly monotone, $\widehat{\Psi}(X_0', X_1')$ is not locally equivalent to L_1 and $\lim_{t\to 0} \frac{t}{\Psi(1,t)} = 0$ (resp., $\lim_{t\to 0} \frac{t}{\Psi(t,1)} = 0$), then for every $1 < p \leq \infty$, the Caledrón-Lozanovskii space $[\Psi_p(X_0, X_1)]_a$ does not contain an asymptotically isometric copy of ℓ_1.*

Proof If $[\Psi_p(X_0, X_1)]_a$ contains an asymptotically isometric copy of ℓ_1, then by [5] its dual $([\Psi_p(X_0, X_1)]_a)^*$ contains an isometric copy of $L_1(0,1)$. But

$$([\Psi_p(X_0, X_1)]_a)^* \simeq (\Psi_p(X_0, X_1))' = \widehat{\Psi}_{p'}(X_0', X_1').$$

The spaces X_i', $i = 0, 1$, satisfy the Fatou property. Moreover by concavity of Ψ,

$$\widehat{\Psi}(0, 1) = \inf_{u,v>0} \frac{u\frac{v}{u}}{u\Psi(1, \frac{v}{u})} = \inf_{t>0} \frac{t}{\Psi(1,t)} = \lim_{t\to 0} \frac{t}{\Psi(1,t)},$$

and similarly $\widehat{\Psi}(1,0) = \lim_{t\to 0} \frac{t}{\Psi(t,1)}$. Now by Theorem 1.2 and our assumptions, $\widehat{\Psi}_{p'}(X_0', X_1')$ is strictly monotone. Notice that $1 \le p' < \infty$. Thus by [19], $\widehat{\Psi}_{p'}(X_0', X_1')$ contains also an order isometric copy of $L_1(0,1)$. Since μ is separable, it follows by the Caratheodory theorem [9] that $L_1(0,1)$ is order isometric to $L_1(A)$ for every $A \in \mathcal{S}$ with $\mu(A) < \infty$. Thus $L_1(A)$ is order isometrically embedded into $\widehat{\Psi}_{p'}(X_0', X_1')$. Now by Corollary 9 in [1] we get $L_1(A) \subset \widehat{\Psi}_{p'}(X_0', X_1')(A)$ for every $A \in \mathcal{S}$ with $\mu(A) < \infty$. Recall now that for any r.i. space X and any $A \in \mathcal{S}$ with $\mu(A) < \infty$, we have that $X(A) \subset L_1(A)$ [2, 12]. Thus

$$\widehat{\Psi}_{p'}(X_0', X_1')(A) \subset L_1(A)$$

by the fact that any Calderón-Lozanovskii space is r.i. whenever both X_0 and X_1 are r.i. spaces. Hence $L_1(A) = \widehat{\Psi}(X_0', X_1')(A)$ up to equivalence of norms, which contradicts the assumption that $\widehat{\Psi}(X_0', X_1')$ is not locally equivalent to L_1, and concludes the proof. ∎

The next result is a corollary of the previous one in view of the fact that if Ψ satisfies the left (resp., right) Δ_2 condition and X_0 (resp., X_1) is order continuous, then $\Psi(X_0, X_1)$ is also order continuous (see [18]). In this case $\Psi_p(X_0, X_1) = [\Psi_p(X_0, X_1)]_a$ and then $(\Psi_p(X_0, X_1))^* \simeq (\Psi_p(X_0, X_1))'$.

Corollary 1.6 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic separable measure space and let (X_0, X_1) be a couple of r.i. spaces such that X_0 (resp., X_1) is order continuous. Let Ψ satisfy the left (resp., right) Δ_2 condition and $\lim_{t\to 0} \frac{t}{\Psi(1,t)} = 0$ (resp., $\lim_{t\to 0} \frac{t}{\Psi(t,1)} = 0$). If X_0' (resp., X_1') is strictly monotone and $\widehat{\Psi}(X_0', X_1')$ is not locally equivalent to L_1, then for every $1 < p \le \infty$, the Caledrón-Lozanovskii space $\Psi_p(X_0, X_1)$ does not contain an asymptotically isometric copy of ℓ_1.*

2. Orlicz-Lorentz spaces. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an Orlicz function, that is φ is convex, $\varphi(0) = 0$ and φ is positive on $(0, \infty)$. We say that the Orlicz function φ is N-function at 0 whenever $\lim_{s\to 0} \varphi(s)/s = 0$ [17]. Let further $\gamma = \mu(\Omega)$. We assume that $\psi \in \mathcal{P}$, that is $\psi\colon [0, \gamma) \to \mathbb{R}_+$ is concave, $\psi(0) = 0$ and ψ is not trivially equal to zero. Denote $\psi(0+) = \lim_{t\to 0+} \psi(t)$ and $\psi(\infty) = \lim_{t\to\infty} \psi(t)$ if $\gamma = \infty$. Let ρ be the following modular on L^0,

$$\rho(x) = \int_0^\gamma \varphi(x^*) d\psi = \|\varphi(x^*)\|_\infty \psi(0+) + \int_0^\gamma \varphi(x^*(s)) \psi'(s)\, ds,$$

where ψ' denotes the derivative of ψ. Since ψ is concave its derivative exists except a countable set. In fact we have $\psi(t) = \psi(0+) + \int_0^t \psi'$, $t \in \mathbb{R}_+$, where ψ' is decreasing. If there is no confusion we often write $\int_0^\gamma \varphi(x^*)\psi'$ instead of $\int_0^\gamma \varphi(x^*(s))\psi'(s)\, ds$. The *Orlicz-Lorentz space* $\Lambda_{\varphi,\psi}$ consists of all $x \in L^0$ such that there exists $\lambda > 0$ with $\rho(x/\lambda) < \infty$. The space $\Lambda_{\varphi,\psi}$ equipped with the norm

$$\|x\|_{\varphi,\psi} = \inf\{\lambda > 0\colon\ \rho(x/\lambda) \le 1\}$$

is a r.i. space satisfying the Fatou property.

In the context of Orlicz-Lorentz spaces we will always say that φ satisfies the *appropriate* Δ_2*-condition* whenever (i) φ satisfies condition Δ_2 for all arguments, that is there exists $K > 0$ such that $\varphi(2t) \leq K\varphi(t)$ for all $t \in \mathbb{R}_+$, in the case when μ is non-atomic, $\mu(\Omega) = \infty$ and $\psi(\infty) = \infty$; (ii) φ satisfies condition Δ_2^∞ for large arguments, that is there are $K > 0$ and $t_0 \geq 0$ such that $\varphi(2t) \leq K\varphi(t)$ for all $t \geq t_0$, in the case when μ is non-atomic, $\mu(\Omega) < \infty$, or $\mu(\Omega) = \infty$ and $\psi(\infty) < \infty$.

We easily observe that $(\Lambda_{\varphi,\psi})_a \neq \{0\}$ if and only if $\psi(0+) = 0$. It is also well known that if $\psi(0+) = 0$ then $(\Lambda_{\varphi,\psi})_a$ coincides with the collection of $x \in L^0$ such that $\rho(kx) < \infty$ for every $k \in \mathbb{R}$. Moreover $(\Lambda_{\varphi,\psi})_a = \Lambda_{\varphi,\psi}$ if and only if φ satisfies the appropriate Δ_2-condition and $\psi(0+) = 0$, and $\psi(\infty) = \infty$ in the case of $\mu(\Omega) = \infty$ (see [11]). In the sequel we will use the fact that for any $x \in (\Lambda_{\varphi,\psi})_a$ we have that $\|x\|_{\varphi,\psi} = 1$ if and only if $\rho(x) = 1$ (see [4]). It is also clear that if $\psi(\infty) = \infty$ in the case of $\mu(\Omega) = \infty$, then for any $x \in \Lambda_{\varphi,\psi}$ it holds $\lim_{t\to\infty} x^*(t) = 0$ which is equivalent to $d_x(\lambda) < \infty$ for every $\lambda > 0$.

Recall that if $\varphi(t) = t$, $t \in \mathbb{R}_+$, then $\Lambda_{\varphi,\psi}$ is a *Lorentz space* denoted by Λ_ψ. Given $\psi \in \mathcal{P}$, the *Marcinkiewicz space* M_ψ is the set of all $x \in L^0$ such that

$$\|x\|_{M_\psi} = \sup_{0<t<\gamma} \frac{\int_0^t x^*}{\psi(t)} < \infty.$$

The Marcinkiewicz space is an r.i. space with its fundamental function $\phi_{M_\psi}(t) = t/\psi(t)$, $t \in (0,\gamma)$, and such that $(\Lambda_\psi)' = M_\psi$ (see [12, Theorem 5.2, Chapter II, p. 112]).

Now letting $\Psi(s,t) = t\varphi^{-1}(s/t)$ for $s,t > 0$, and $\Psi(s,t) = 0$ for $t = 0$ and any $s \geq 0$, we have $\Lambda_{\varphi,\psi} = \Psi_\infty(\Lambda_\psi, L_\infty)$ with equality of norms (see [16]). This implies that $(\Lambda_{\varphi,\psi})' = \widehat{\Psi}_1((\Lambda_\psi)', (L_\infty)') = \widehat{\Psi}_1(M_\psi, L_1)$ with equality of norms. We also recall that in this case $\widehat{\Psi}(s,t) = t\varphi_*^{-1}(s/t)$ for $s,t > 0$, where $\varphi_*(s) = \sup_{t\geq 0}\{st - \varphi(t)\}$ (see [15]).

Corollary 2.1 *For any Orlicz function φ which is N-function at 0, and $\psi \in \mathcal{P}$, the Köthe dual $(\Lambda_{\varphi,\psi})'$ of Orlicz-Lorentz space is strictly monotone.*

Proof By the assumption that φ is N-function at 0, we have

$$\widehat{\Psi}(1,0) = \inf_{s>0} \frac{s}{\Psi(s,1)} = \lim_{s\to 0} \frac{s}{\Psi(s,1)} = \lim_{s\to 0} \frac{s}{\varphi^{-1}(s)} = \lim_{t\to 0} \frac{\varphi(t)}{t} = 0.$$

Since L_1 is strictly monotone, by Theorem 1.2 the space $(\Lambda_{\varphi,\psi})' = \widehat{\Psi}_1(M_\psi, L_1)$ is strictly monotone. ∎

Corollary 2.2 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic separable measure space. Let φ be N-function at 0 and let $\psi \in \mathcal{P}$ be such that $\psi(0+) = 0$. Then the order continuous part $(\Lambda_{\varphi,\psi})_a$ of Orlicz-Lorentz space does not contain an asymptotically isometric copy of ℓ_1.*

PROOF We shall apply Theorem 1.5 to $\Lambda_{\varphi,\psi} = \Psi_\infty(\Lambda_\psi, L_\infty)$ with $\Psi(s,t) = t\varphi^{-1}(s/t)$ for $s,t > 0$ and $\Psi(s,t) = 0$ for $t = 0$ and any $s \geq 0$. We have that $(\Lambda_\psi)' = M_\psi$, $(L_\infty)' = L_1$ and clearly L_1 is strictly monotone. Moreover $\widehat{\Psi}(1,0) = \lim_{t\to 0} \frac{\varphi(t)}{t} = 0$, by the assumption that φ is N-function at 0.

Recall that for a r.i. space X we have that $\lim_{t\to 0} \phi_X(t) = 0$ if and only if X_a is not trivial which is equivalent to supp $X_a = \Omega$. Therefore we have that supp $(\Lambda_{\varphi,w})_a = \Omega$ if and only if $\lim_{t\to 0+} \phi_{\Lambda_{\varphi,\psi}}(t) = \lim_{t\to 0+} 1/\varphi^{-1}(1/\psi(t)) = 0$. The latter is true by the assumption $\psi(0+) = 0$.

Observe also that $\widehat{\Psi}(M_\psi, L_1)$ is not locally equivalent to L_1. Indeed, we will show that $\phi_{\widehat{\Psi}(M_\psi,L_1)}$ is not equivalent to ϕ_{L_1} in a neighborhood of zero, which yields that $\widehat{\Psi}(M_\psi, L_1)(A) \neq L_1(A)$ for some $A \in \mathcal{S}$ with $\mu(A) < \infty$. For any $0 < t < \mu(\Omega)$,

$$\phi_{\widehat{\Psi}(M_\psi,L_1)}(t) \approx \widehat{\Psi}(\phi_{M_\psi}(t), \phi_{L_1}(t)) = \widehat{\Psi}\Big(\frac{t}{\psi(t)}, t\Big) = t\widehat{\Psi}\Big(\frac{1}{\psi(t)}, 1\Big).$$

In our case $\widehat{\Psi}(s,t) = t\varphi_*^{-1}(s/t)$, and thus

$$\phi_{\widehat{\Psi}(M_\psi,L_1)}(t) \approx t\widehat{\Psi}\left(\frac{1}{\psi(t)}, 1\right) = t\varphi_*^{-1}\left(\frac{1}{\psi(t)}\right).$$

Supposing now that the latter fundamental function is equivalent to the fundamental function $\phi_{L_1}(t) = t$, we get that $1/\psi(t)$ is equivalent to a constant, which contradicts the assumption that $\psi(0+) = 0$. Thus $\widehat{\Psi}(M_\psi, L_1)$ is not locally equivalent to L_1, and application of Theorem 1.5 completes the proof. ∎

The next result is a direct corollary of the previous one, since $\Lambda_{\varphi,\psi} = (\Lambda_{\varphi,\psi})_a$ whenever φ satisfies the appropriate Δ_2-condition and $\psi(0+) = 0$ and $\psi(\infty) = \infty$ in the case of $\mu(\Omega) = \infty$ (for details we refer to [11]).

COROLLARY 2.3 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic separable measure space. We assume that φ is an Orlicz function satisfying the appropriate Δ_2-condition, and φ is N-function at 0. Moreover let $\psi \in \mathcal{P}$ be such that $\psi(0+) = 0$ and $\psi(\infty) = \infty$ in the case of $\mu(\Omega) = \infty$. Then the Orlicz-Lorentz space $\Lambda_{\varphi,\psi}$ does not contain an asymptotically isometric copy of ℓ_1.*

Letting $\psi(t) = t$, the Orlicz-Lorentz space $\Lambda_{\varphi,\psi}$ is the Orlicz space L_φ equipped with the Luxembourg norm. In [6] the authors showed that if φ_* does not satisfy the appropriate Δ_2-condition and φ does, then the Orlicz space equipped with the Orlicz norm contains an asymptotically isometric copy of ℓ_1. The corollary below states that whenever φ satisfies Δ_2 independently of the behavior of φ_*, L_φ equipped with the Luxemburg norm never contains an asymptotically isometric copy of ℓ_1. This answers the question posed in [6].

COROLLARY 2.4 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic separable measure space. If φ satisfies the appropriate Δ_2-condition and φ is N-function at 0, then the Orlicz spaces $(L_\varphi, \|\cdot\|_\varphi)$ over $(\Omega, \mathcal{S}, \mu)$ does not contain an asymptotically isometric copy of ℓ_1.*

In what follows we will study subspaces of Orlicz-Lorentz spaces isometrically isomorphic to $\ell_1^{(n)}$ for $n \in \mathbb{N}$ and $n \geq 2$, where as usual by $\ell_1^{(n)}$ we denote the n dimensional real vector space equipped with ℓ_1-norm. It is well known that $\ell_1^{(n)}$ is isometrically embedded (resp., embedded order isometrically) into a Banach space (resp., Banach lattice) X if and only if there exist $x_i \in X$, $i = 1, \dots, n$ such that each $\|x_i\|_X = 1$ and $\|\sum_{i=1}^n \theta_i x_i\|_X = n$ for each combination of signs $\theta_i = \pm 1$ (resp., $\|x_i\|_X = 1$ with $|x_i| \wedge |x_j| = 0$ for $i \neq j$, and $\|\sum_{i=1}^n x_i\|_X = n$).

We will need further the following technical lemma which is a consequence of a more general result due to Wójtowicz [19]: If a strictly monotone Banach lattice X contains an isometric copy of $L_1(\nu)$, then X contains a lattice-isometric copy of $L_1(\nu)$. Below we include a simple, direct proof of the local variant of the mentioned result.

Lemma 2.5 *Let E be a strictly monotone Banach lattice on a measure space $(\Omega, \mathcal{S}, \mu)$ and let $n \geq 2$. Then E contains $\ell_1^{(n)}$ order isometrically whenever E contains $\ell_1^{(n)}$ isometrically.*

Proof Assume that $\ell_1^{(2)}$ is isometrically embedded in E. Thus there exist x, y in the unit sphere of E such that *span*$\{x, y\}$ is isometric to $\ell_1^{(2)}$, that is $\|x + y\|_E = \|x - y\|_E = 2$. We can always assume that $x(t) > 0$ for a.e. $t \in \operatorname{supp} x$. In fact $2 = \|x - y\|_E = \||x| - (\operatorname{sign} x) y\|_E = \|x + y\|_E$.

Now we shall prove that $|x| \wedge |y| = 0$. Then *span*$\{|x|, |y|\}$ is isometric to $\ell_1^{(2)}$ and so $\ell_1^{(2)}$ is an order isometric copy in E. Assume for a contrary without loss of generality that

$$\mu\{t \in \operatorname{supp} x \colon\ y(t) > 0\} > 0.$$

There exist $\varepsilon > 0$ and $A \in \mathcal{S}$ with $\mu(A) > 0$ such that for all $t \in A$, we have that $x(t) > \varepsilon$ and $y(t) > \varepsilon$. Define

$$x_1(t) = \begin{cases} x(t), & t \notin A; \\ x(t) - \varepsilon, & t \in A, \end{cases} \qquad y_1(t) = \begin{cases} y(t), & t \notin A; \\ y(t) - \varepsilon, & t \in A. \end{cases}$$

Then $x_1(t) - y_1(t) = x(t) - y(t)$ a.e. and $x_1(t) = x(t) - \varepsilon < x(t)$ for $t \in A$. Since E is strictly monotone, $\|x_1\|_E < \|x\|_E = 1$. Thus

$$2 = \|x - y\|_E = \|x_1 - y_1\|_E \leq \|x_1\|_E + \|y_1\|_E < 2,$$

which is a contradiction and so $|x| \wedge |y| = 0$. The above proof gives that if a linear *span*$\{x_1, ..., x_n\}$ is isometrically isomorphic to $\ell_1^{(n)}$, then $|x_i| \wedge |x_j| = 0$ for each $1 \leq i, j \leq n$ with $i \neq j$ and this completes the proof. ∎

In what follows for an Orlicz function φ and a function $\psi \in \mathcal{P}$ we define,

$$t_\varphi = \sup\{t > 0 \colon\ \varphi(t) = kt \ \text{ for some } \ k > 0\},$$
$$t_\psi = \sup\{t \in (0, \gamma) \colon\ \psi(t) = lt \ \text{ for some } \ l > 0\}.$$

Theorem 2.6 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space. Assume $\psi' > 0$ on $(0, \gamma)$ and $\psi(0+) = 0$. Let $n \in \mathbb{N}$ and $n \geq 2$.*

(a) *If $0 < t_\psi < \infty$ and $t_\varphi t_\psi \geq \frac{n}{kl}$, or $t_\psi = \infty$ and $t_\varphi > 0$, then the space $(\Lambda_{\varphi,\psi})_a$ contains an order isometric copy of $\ell_1^{(n)}$.*

(b) *If the space $(\Lambda_{\varphi,\psi})_a$ contains an order isometric copy of $\ell_1^{(n)}$ then both $t_\varphi > 0$ and $t_\psi > 0$, and $t_\varphi t_\psi \geq \frac{n}{kl}$.*

Consequently, if $0 < t_\varphi \leq \infty$ and $0 < t_\psi < \infty$ then the space $(\Lambda_{\varphi,\psi})_a$ contains an order isometric copy of $\ell_1^{(n)}$ if and only if $t_\varphi t_\psi \geq \frac{n}{kl}$. In particular if $\gamma = t_\psi < \infty$ then $(\Lambda_{\varphi,\psi})_a$ contains an order isometric copy of $\ell_1^{(n)}$ if and only if $t_\varphi \geq \frac{n}{kl\gamma}$.

Proof (a) Fix $n \geq 2$ and assume that $0 < t_\psi < \infty$. Choose for all $1 \leq i \leq n$ the sets $A_i \in \mathcal{S}$ such that they are disjoint and $\mu(A_i) = t_\psi/n$. Then we find $c \in (0, t_\varphi]$ such that $ct_\psi = \frac{n}{kl}$. Now for the functions $c\chi_{A_i} \in (\Lambda_{\varphi,\psi})_a$, $1 \leq i \leq n$ we have

$$\rho(c\chi_{A_i}) = \varphi(c) \int_0^{t_\psi/n} \psi' = \frac{kclt_\psi}{n} = 1.$$

Moreover

$$\begin{aligned}\rho\left(\frac{1}{n}\sum_{i=1}^n c\chi_{A_i}\right) &= \int_0^\infty \varphi\left(\frac{1}{n}\sum_{i=1}^n c\chi_{(\frac{i-1}{n}t_\psi, \frac{i}{n}t_\psi)}\right)\psi' \\ &= \sum_{i=1}^n \int_{\frac{i-1}{n}t_\psi}^{\frac{i}{n}t_\psi} \varphi\left(\frac{c}{n}\right)\psi' = \frac{kclt_\psi}{n} = 1.\end{aligned}$$

Hence $\|c\chi_{A_i}\|_{\varphi,\psi} = 1$ for each $1 \leq i \leq n$ and $\left\|\frac{1}{n}\sum_{i=1}^n c\chi_{A_i}\right\|_{\varphi,\psi} = 1$, and so $(\Lambda_{\varphi,\psi})_a$ contains an order isometric copy of $\ell_1^{(n)}$.

Let now $t_\psi = \infty$ and $t_\varphi > 0$. Hence for $c \in (0, t_\varphi)$ we choose n disjoint measurable sets A_i such that $\rho(c\chi_{A_i}) = 1$ for each $1 \leq i \leq n$. The rest of the proof is similar to the one above.

(b) Assume now that $\ell_1^{(n)}$ is an order isometric copy of $(\Lambda_{\varphi,\psi})_a$. There exist $x_i \in (\Lambda_{\varphi,\psi})_a$, $1 \leq i \leq n$ such that $|x_i| \wedge |x_j| = 0$ for $i \neq j$, $\|x_i\|_{\varphi,\psi} = 1$ for each i, and $\|\sum_{i=1}^n x_i\|_{\varphi,\psi} = n$. Thus we get that $\rho(x_i) = \rho\left(\frac{1}{n}\sum_{i=1}^n x_i\right) = 1$ for each $1 \leq i \leq n$. Hence

$$\begin{aligned}1 = \rho\left(\frac{1}{n}\sum_{i=1}^n x_i\right) &= \int_0^\gamma \left(\sum_{i=1}^n \varphi\left(\frac{|x_i|}{n}\right)\right)^* \psi' \\ &\leq \int_0^\gamma \left(\sum_{i=1}^n \varphi\left(\frac{x_i^*}{n}\right)\right)\psi' \leq \frac{1}{n}\sum_{i=1}^n \int_0^\gamma \varphi(x_i^*)\psi' = 1.\end{aligned}$$

It follows that $\int_0^\gamma \varphi(\frac{x_i^*}{n})\psi' = \int_0^\gamma \frac{1}{n}\varphi(x_i^*)\psi'$ for each $1 \leq i \leq n$. Thus in view of the inequalities $\varphi(\frac{x_i^*}{n}) \leq \frac{1}{n}\varphi(x_i^*)$ and $\psi' > 0$ on $(0, \gamma)$ we get that for all $t \in (0, \gamma)$

and each $1 \le i \le n$ it holds $\varphi(\frac{x_i^*(t)}{n}) = \frac{1}{n}\varphi(x_i^*(t))$. Hence φ must be linear on the interval $(0, \max_{1\le i\le n} x_i^*(0+))$. Therefore

$$t_\varphi \ge \max_{1\le i\le n} x_i^*(0+). \tag{3}$$

It is well known [12, formula (5.4)] that if $\psi(0+) = 0$ then the following formula holds for any $f \in \Lambda_\psi$,

$$\int_0^\gamma f^*(t)\psi'(t)\,dt = \int_0^\infty \psi(d_f(t))\,dt.$$

This implies that for all $f \in \Lambda_{\varphi,\psi}$ we have

$$\begin{aligned}\rho(f) &= \int_0^\gamma \varphi(|f(t)|)^*\psi'(t)\,dt = \int_0^\infty \psi(d_{\varphi(|f|)}(t))\,dt \\ &= \int_0^\infty \psi(d_f(\varphi^{-1}(t))\,dt = \int_0^\infty \psi(d_f(t))\varphi'(t)\,dt.\end{aligned} \tag{4}$$

Now by the orthogonality $|x_i| \wedge |x_j| = 0$ for $i \ne j$, concavity of ψ and (4),

$$\begin{aligned}1 = \rho\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big) &= \int_0^\infty \psi\Big(\sum_{i=1}^n d_{\frac{x_i}{n}}(t)\Big)\varphi'(t)\,dt \\ &\le \sum_{i=1}^n \int_0^\infty \psi(d_{\frac{x_i}{n}}(t))\varphi'(t)\,dt \le \frac{1}{n}\sum_{i=1}^n \rho(x_i) = 1.\end{aligned}$$

Combining with the inequality $\psi\Big(\sum_{i=1}^n d_{x_i/n}(t)\Big) \le \sum_{i=1}^n \psi(d_{x_i/n}(t))$, it yields that for all $t \in (0,\infty)$ we have $\psi\Big(\sum_{i=1}^n d_{x_i/n}(t)\Big) = \sum_{i=1}^n \psi(d_{x_i/n}(t))$. Since $0 \le d_{x_i/n}(t) \le \mu(\text{supp } (x_i/n)) = \mu(\text{supp } x_i)$ for each $1 \le i \le n$, so ψ is linear on the interval $(0, \sum_{i=1}^n \mu(\text{supp } x_i))$. Thus

$$t_\psi \ge \sum_{i=1}^n \mu(\text{supp } x_i). \tag{5}$$

Notice that by (3) and (5) both t_φ and t_ψ must be positive. If $t_\varphi = \infty$ or $t_\psi = \infty$ then the inequality $t_\varphi t_\psi \ge \frac{n}{kl}$ is satisfied. Otherwise we have that for each $1 \le i \le n$, $x_i^*(0+), \mu(\text{supp } x_i) \in (0,\infty)$. Then by (3) and (5),

$$\begin{aligned}1 &= \rho\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big) \le \sum_{i=1}^n \frac{1}{n}\rho(x_i) = \frac{1}{n}\sum_{i=1}^n \int_0^{\mu(\text{supp } x_i)} \varphi(x_i^*)\psi' \\ &= \frac{1}{n}\sum_{i=1}^n \int_0^{\mu(\text{supp } x_i)} kx_i^* l \le \frac{1}{n}\sum_{i=1}^n klx_i^*(0+)\mu(\text{supp } x_i) \le \frac{1}{n}\sum_{i=1}^n klt_\varphi\mu(\text{supp } x_i) \\ &\le \frac{1}{n}klt_\varphi\Big(t_\psi - \sum_{i=2}^n \mu(\text{supp } x_i)\Big) + \frac{1}{n}\sum_{i=2}^n klt_\varphi\mu(\text{supp } x_i) = \frac{1}{n}klt_\varphi t_\psi.\end{aligned}$$

Hence $t_\varphi t_\psi \ge \frac{n}{kl}$ and the proof is complete.

The direct consequence of Theorem 2.6 is the next result.

COROLLARY 2.7 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space. Assume $\psi' > 0$ on $(0, \gamma)$ and $\psi(0+) = 0$. If for some $n \geq 2$, $\ell_1^{(n)}$ is order isometrically contained in $(\Lambda_{\varphi,\psi})_a$ then both φ and ψ are linear functions in some neighborhoods of zero.*

COROLLARY 2.8 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space. Assume $\psi' > 0$ on $(0, \gamma)$, $\psi(0+) = 0$ and $\psi(\infty) = \infty$ when $\gamma = \infty$. Let $0 < t_\varphi \leq \infty$ and $0 < t_\psi < \infty$. Then for each $n \geq 2$ the following conditions are equivalent.*

(i) *$\ell_1^{(n)}$ is isometrically embedded in $(\Lambda_{\varphi,\psi})_a$.*

(ii) *$\ell_1^{(n)}$ is order isometrically embedded in $(\Lambda_{\varphi,\psi})_a$.*

(iii) *$t_\varphi t_\psi \geq \frac{n}{kl}$.*

PROOF We first observe that under our assumptions on ψ the space $(\Lambda_{\varphi,\psi})_a$ is strictly monotone. Indeed let $x, y \in (\Lambda_{\varphi,\psi})_a$ be such that $0 \leq x \leq y$ a.e., $x \neq y$ with $\|y\|_{\varphi,\psi} = 1$. Since $\psi' > 0$ and $\psi(\infty) = \infty$ so $\lim_{t\to\infty} y^*(t) = 0$ in case when $\gamma = \infty$. Hence $x^* \leq y^*$ a.e. and $x^* \neq y^*$. This yields

$$\rho(x) < \int_0^\gamma \varphi(y^*)\psi' = \rho(y) = 1,$$

and so $\|x\|_{\varphi,\psi} < \|y\|_{\varphi,\psi} = 1$. To conclude the proof, we apply Lemma 2.5 and Theorem 2.6. ■

PROPOSITION 2.9 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space. If $t_\psi > 0$ then there exists $l > 0$ such that for any $B \in \mathcal{S}$ with $0 < \mu(B) \leq t_\psi$ we have $\Lambda_{\varphi,\psi}(B) = L_{l\varphi}(B)$ with equality of norms, where $L_{l\varphi}$ is the Orlicz space associated to the Orlicz function $l\varphi$.*

PROOF By the assumption $t_\psi > 0$ we have that $\psi(t) = lt$ for some $l > 0$ and all $t \in [0, t_\psi]$. Let $B \in \mathcal{S}$ be such that $0 < \mu(B) \leq t_\psi$. Based on formula (4) we conclude that for all $x \in \Lambda_{\varphi,\psi}$ and $\varepsilon > 0$,

$$\rho\Big(\frac{x\chi_B}{\varepsilon}\Big) = \int_0^\infty \psi\Big(d_{\varphi(\frac{|x|\chi_B}{\varepsilon})}(t)\Big)\,dt = \int_0^\infty l\,d_{\varphi(\frac{|x|\chi_B}{\varepsilon})}(t)\,dt = \int_B l\,\varphi\Big(\frac{|x(t)|}{\varepsilon}\Big)\,dt,$$

which implies that $\|x\chi_B\|_{\varphi,\psi} = \|x\chi_B\|_{L_{l\varphi}}$ and completes the proof. ■

COROLLARY 2.10 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space. Assume $\psi' > 0$ on $(0, \gamma)$ with $\gamma = \mu(\Omega)$, $\psi(0+) = 0$ and $\psi(\infty) = \infty$ when $\gamma = \infty$, and let $t_\psi < \infty$. Then the following conditions are equivalent.*

(i) *For some $A \in \mathcal{S}$ the space $L_1(A)$ is isometrically embedded into $(\Lambda_{\varphi,\psi})_a$.*

(ii) *For some $A \in \mathcal{S}$ the space $L_1(A)$ is order isometrically embedded into $(\Lambda_{\varphi,\psi})_a$.*

(iii) *ℓ_1 is isometrically embedded in $(\Lambda_{\varphi,\psi})_a$.*

(iv) *ℓ_1 is order isometrically embedded in $(\Lambda_{\varphi,\psi})_a$.*

(v) *φ is linear on $[0, \infty)$ and ψ is linear in a neighborhood of zero.*

In particular if $\gamma = \mu(\Omega) < \infty$ then the above five conditions are equivalent.

Proof The conditions (i) and (ii) as well as (iii) and (iv) are equivalent by strict monotonicity of $(\Lambda_{\varphi,\psi})_a$ and the above mentioned result from [19].

(iv) $\Rightarrow$ (v) If ℓ_1 is order isometrically embedded in $(\Lambda_{\varphi,\psi})_a$ then $(\Lambda_{\varphi,\psi})_a$ contains $\ell_1^{(n)}$ for each positive integer $n \geq 2$, and it follows from Theorem 2.6 (b) that $t_\varphi > 0$, $t_\psi > 0$ and $t_\varphi t_\psi \geq \frac{n}{kl}$ for all integer $n \geq 2$. Since $0 < t_\psi < \infty$, ψ is linear in a neighborhood of zero, and $t_\varphi = \infty$ and so φ is linear on $[0, \infty)$.

(v) $\Rightarrow$ (ii). By Proposition 2.9 we have that $\Lambda_{\varphi,\psi}(B) = L_{l\varphi}(B)$ with equality of norms for $B \in \mathcal{S}$ with $0 < \mu(B) \leq t_\psi$. Since φ is linear on $\mathbb{R}_+$, $\Lambda_{\varphi,\psi}(B)$ is order isometric to $L_1(B)$.

The implication (ii) $\Rightarrow$ (iv) is clear and this completes the proof. ■

3. Applications. In this section we shall present consequences of the results from the previous section, as well as related observations for Orlicz and Lorentz spaces. We recall that if $\psi(t) = t$ for all $t \in (0, \gamma)$ then the Orlicz-Lorentz space $\Lambda_{\varphi,\psi}$ becomes the Orlicz space L_φ equipped with the Luxemburg norm.

We start with the result which characterizes the containment of an isometric copy of $\ell_1^{(n)}$, $n \geq 2$, in the Orlicz space $(L_\varphi)_a$ over a non-atomic and finite measure space $(\Omega, \mathcal{S}, \mu)$. It extends and improves Theorem 2 in [19].

Corollary 3.1 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space with $\gamma = \mu(\Omega) < \infty$. Let $n \in \mathbb{N}$ and $n \geq 2$. Then the following conditions are equivalent.*

(i) *The order continuous Orlicz space $(L_\varphi)_a$ contains an order isometric copy of $\ell_1^{(n)}$.*

(ii) *The order continuous Orlicz space $(L_\varphi)_a$ contains an isometric copy of $\ell_1^{(n)}$.*

(iii) $t_\varphi \geq \frac{n}{k\gamma}$.

Proof The equivalence of (i) and (ii) is a result of Lemma 2.5 and the fact that $(L_\varphi)_a$ is strictly monotone. For the equivalence of (i) and (iii) we apply the last part of Theorem 2.6 for $l = 1$ and $t_\psi = \gamma$. ■

The conditions (iii)-(v) in the next corollary recover Theorem 2 for Orlicz space L_φ in [19] in the case of $\mu(\Omega) = \infty$. The corollary is a consequence of Lemma 2.5 and Theorem 2.6 applied to $t_\psi = \infty$. In this case $\Lambda_{\varphi,\psi} = L_{l\varphi}$ isometrically.

Corollary 3.2 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space with $\gamma = \mu(\Omega) = \infty$. Let $n \in \mathbb{N}$ and $n \geq 2$. Then the following conditions are equivalent.*

(i) *The order continuous Orlicz space $(L_\varphi)_a$ contains an order isometric copy of $\ell_1^{(n)}$.*

(ii) *The order continuous Orlicz space $(L_\varphi)_a$ contains an isometric copy of $\ell_1^{(n)}$.*

(iii) *The order continuous Orlicz space $(L_\varphi)_a$ contains an isometric copy of ℓ_1.*

(iv) *The order continuous Orlicz space $(L_\varphi)_a$ contains an order isometric copy of ℓ_1.*

(v) *φ is linear in a neighborhood of zero.*

The corollary below follows from Corollary 2.10. The equivalence of (iii)–(v) has been proved in Theorem 3(b) in [19].

Corollary 3.3 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space such that $\mu(\Omega) < \infty$. Then the following conditions are equivalent.*

(i) *For some $A \in \mathcal{S}$ the space $L_1(A)$ is isometrically embedded into $(L_\varphi)_a$.*

(ii) *For some $A \in \mathcal{S}$ the space $L_1(A)$ is order isometrically embedded into $(L_\varphi)_a$.*

(iii) *ℓ_1 is isometrically embedded in $(L_\varphi)_a$.*

(iv) *ℓ_1 is order isometrically embedded in $(L_\varphi)_a$.*

(v) *φ is linear on $[0, \infty)$ that is $L_\varphi = (L_\varphi)_a \simeq L_1$.*

Remark 3.4 If we assume in Theorem 2.6 and all above corollaries in this section that φ satisfies the appropriate condition Δ_2 then they remain true if the space $(\Lambda_{\varphi,\psi})_a$ or $(L_\varphi)_a$ is replaced by the whole space $\Lambda_{\varphi,\psi}$ or L_φ respectively.

If $\varphi(t) = t$, then the Orlicz-Lorentz space becomes the Lorentz space Λ_ψ. If in addition $\psi(0+) = 0$ and $\psi(\infty) = \infty$ when $\gamma = \infty$, then $(\Lambda_{\varphi,\psi})_a = \Lambda_\psi$. Our last result follows immediately from Corollary 2.10 and Theorem 2.6 and implies the description of Lorentz spaces Λ_ψ on $[0, 1]$ containing $\ell_1^{(2)}$ due to Briskin and Semenov in [3].

Corollary 3.5 *Let $(\Omega, \mathcal{S}, \mu)$ be a non-atomic measure space. If $n \in \mathbb{N}$, $n \geq 2$, and $\psi \in \mathcal{P}$ is such that $\psi' > 0$ on $(0, \gamma)$, $\psi(0+) = 0$ and $\psi(\infty) = \infty$ when $\mu(\Omega) = \infty$, then the following conditions are equivalent.*

(i) *For some $A \in \mathcal{S}$ the space $L_1(A)$ is isometrically embedded into Λ_ψ.*

(ii) *For some $A \in \mathcal{S}$ the space $L_1(A)$ is order isometrically embedded into Λ_ψ.*

(iii) ℓ_1 *is isometrically embedded in* Λ_ψ.

(iv) ℓ_1 *is order isometrically embedded in* Λ_ψ.

(v) $\ell_1^{(n)}$ *is isometrically embedded in* Λ_ψ.

(vi) $\ell_1^{(n)}$ *is order isometrically embedded in* Λ_ψ.

(vii) ψ *is linear in a neighborhood of zero.*

We conclude with the following result on the description of universal Orlicz-Lorentz (and so also Orlicz and Lorentz) spaces for all two-dimensional normed spaces. We recall that a Banach space U is called universal for all two-dimensional normed spaces if for each two-dimensional normed space X there is a subspace Y in U such that X is isometrically isomorphic to Y.

Theorem 3.6 *Let* $(\Omega, \mathcal{S}, \mu)$ *be a non-atomic separable measure space. Assume* $\psi' > 0$ *on* $(0, \gamma)$ *with* $\gamma = \mu(\Omega)$, $\psi(0+) = 0$ *and* $\psi(\infty) = \infty$ *when* $\gamma = \infty$, *and let* $t_\psi < \infty$. *Then the separable Orlicz-Lorentz space* $\Lambda_{\varphi,\psi}$ *is universal for all two-dimensional normed spaces if and only if* φ *is linear on* $[0, \infty)$ *and* ψ *is linear in a neighborhood of zero.*

Proof Ferguson [8] (independently, Herz [10]) proved that any two-dimensional normed space can be embedded isometrically into $L_1(0, 1)$. Consequently our hypothesis follows from Corollary 2.10 and the fact that $L_1(A)$ with $\mu(A) > 0$ is isometrically isomorphic to $L_1(0, 1)$. ∎

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