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Jerzy Grzybowski, Diethard Pallaschke, Ryszard Urbański

Algebraic Separation and Shadowing of Arbitrary Sets

Dedicated to Professor Julian Musielak on his 85^{th} *birthday*

Abstract. In this paper we consider a generalization of the separation technique proposed in [10, 4, 7] for the separation of finitely many compact convex sets $A_i,\ i \in I$ by another compact convex set S in a locally convex vector space to arbitrary sets in real vector spaces. Then we investigate the notation of shadowing set which is a generalization of the notion of separating set and construct separating sets by means of a generalized Demyanov-difference in locally convex vector spaces.

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1. Introduction. Separation of sets has been for long time an interesting research area for mathematicians. Basic concepts of classification theory are linear separability of sets, separation margin and kernel transformations. They have provided the theoretical background in constructing powerful classification tools for data classification.

A different approach to the separation of two sets was proposed by J. Grzybowski, D. Pallaschke and R. Urbański [7] and A. Astorino and M. Gaudioso [1] and M. Gaudioso, E. Gorgone et al. [4] which leads to a non-smooth optimization problem. It is based on the method of separating two compact convex sets by an other one. In this paper we generalize these results to the case of finitely many nonempty sets, which are not necessarily convex

The paper is organized as follows. We begin with a survey on basic properties of the family of bounded closed convex sets in a topological vector space. Then we prove a separation theorem for arbitrary sets and for closed bounded convex sets and discuss the connection between the different separation theorems for arbitrary and for bounded closed convex sets. Finally we use a generalization of the Demyanov-difference in locally convex vector spaces to construct separating sets.

2. The Semigroup of Closed Bounded Convex Sets. For a Hausdorff topological vector space (X, τ) let us denote by $\mathcal{A}(X)$ the set of all nonempty subsets of X, by $\mathcal{B}^*(X)$ the set of all nonempty bounded subsets of X, by $\mathcal{C}(X)$ the set of all nonempty closed convex subsets of X, by $\mathcal{B}(X) = \mathcal{B}^*(X) \cap \mathcal{C}(X)$ the set of all bounded closed convex sets of X and by $\mathcal{K}(X)$ the set of all nonempty compact convex subsets of X. (Note, that we consider only vector spaces over the reals). Recall that for $A, B \in \mathcal{A}(X)$ the *algebraic sum* is defined by $A + B = \{x = a + b \mid a \in A \text{ and } b \in B\}$ and for $\lambda \in \mathbb{R}$ and $A \in \mathcal{A}(X)$ the *multiplication* is defined by $\lambda A = \{x = \lambda a \mid a \in A\}$.

The *Minkowski sum* for $A, B \in \mathcal{A}(X)$ is defined by

$$A \dotplus B = \mathrm{cl}(\{x = a + b \mid a \in A \text{ and } b \in B\}),$$

where $\mathrm{cl}(A) = \bar{A}$ denotes the closure of $A \subset X$ with respect to τ. For compact convex sets, the Minkowski sum coincides with the algebraic sum, i,e., for $A, B \in \mathcal{K}(X)$ we have $A \dotplus B = A + B$. In quasidifferential calculus of Demyanov and Rubinov [3] pairs of bounded closed convex sets are considered. More precisely: For a Hausdorff topological vector space X two pairs $(A, B), (C, D) \in \mathcal{B}^2(X) = \mathcal{B}(X) \times \mathcal{B}(X)$ are called *equivalent* if $B \dotplus C = A \dotplus D$ holds and $[A, B]$ denotes the equivalence class represented by the pair $(A, B) \in \mathcal{B}^2(X)$. An ordering among equivalence classes is given by $[A, B] \leq [C, D]$ if and only if $A \dotplus D \subset B \dotplus C$. This is the ordering on the Minkowski-Rådström-Hörmander space and is independent of the choice of the representatives.

For $A \in \mathcal{B}(X)$ we denote by $\operatorname{ext} A$ the set of its extreme points and by $\exp A$ the set of its exposed points (see [10]). Next, for $A, B \in \mathcal{A}(X)$ we define: $A \vee B = \mathrm{cl}\,\mathrm{conv}(A \cup B)$, where $\mathrm{conv}(A \cup B)$ denotes the convex hull of $A \cup B$. We will use the abbreviation $A \dotplus B \vee C$ for $A \dotplus (B \vee C)$ and $C + d$ instead of $C + \{d\}$ for all bounded closed convex sets $A, B, C \in \mathcal{A}(X)$ and a point $d \in X$.

A distributivity relation between the Minkowski sum and the maximum operation is expressed by the *Pinsker Formula* (see [12]) which is stated in a more general form in [10] as:

PROPOSITION 2.1 *Let (X, τ) be a Hausdorff topological vector space, $A, B, C \in \mathcal{A}(X)$ and C be a convex set. Then*

$$(A \dotplus C) \vee (B \dotplus C) = C \dotplus (A \vee B).$$

The Minkowski-Rådström-Hörmander Theorem on the order cancellation property for bounded closed convex subsets in Hausdorff topological vector spaces states that for $A, B, C \in \mathcal{B}(X)$ the inclusion $A \dot{+} B \subset B \dot{+} C$ implies $A \subset C$. A generalization which is due to R. Urbański [14] (see also [10]) states:

THEOREM 2.2 *Let X be a Hausdorff topological vector space. Then for any $A \in \mathcal{A}(X)$, $B \in \mathcal{B}^*(X)$ and $C \in \mathcal{C}(X)$ the inclusion*

$$A + B \subset C \dot{+} B \quad \textit{implies} \quad A \subset C. \qquad \textbf{(olc)}$$

This implies that $\mathcal{B}(X)$ endowed with the Minkowski sum "$\dot{+}$" and the ordering induced by inclusion is a commutative ordered *semigroup* (i.e. a ordered set endowed with a group operation, without having inverse elements), which satisfies the order cancellation law and contains $\mathcal{K}(X)$ as a sub-semigroup.

3. Separation and Shadowing of Arbitrary Sets. The separation of two bounded closed convex sets by an other bounded closed set is extensively explained in [10]. A set S *separates* two sets A and B if $[a, b] \cap S \neq \emptyset$ for every $a \in A$ and $b \in B$ where $[a, b]$ is a closed interval. We say that a set S *strictly separates* A and B if $(a, b) \cap S \neq \emptyset$ for every $a \in A$, $b \in B$ where (a, b) is an open interval. In this section we discuss a possible generalization to the case of many nonempty sets in a real vector space which are not necessary convex.

Although a separation concept for two sets is intuitively clear, this is not so obvious for the separation of many sets. The following possible separation concept of convex sets has been recently formulated by J-E. Martinez-Legaz and A. Martinón in [9]:

"A subset S *separates* (*pairwise*) a finite family $\{A_i\}_{i \in I}$ of nonempty subsets if S separates every pair of sets A_i and A_j where $i, j \in I$ with $i \neq j$."

If a convex subset S separates pairwise a finite family $\{A_i\}_{i \in I}$ of nonempty subsets then the family $\{S \vee A_i\}_{i \in I}$ is pairwise convex, i.e. if the set $(S \vee A_i) \cup (S \vee A_j)$ is convex for all $i, j \in I$ ([11], Corollary 2.2).

Another possible generalization of separation is the notion of shadowing.

DEFINITION 3.1 Let X be a real vector space, a set of indices I is finite and $S, A_i \in \mathcal{A}(X)$, $i \in I$. We say that the set S *shadows* the family $\{A_i\}_{i \in I}$ if $S \cap \bigvee_{i \in I} a_i \neq \emptyset$ for every collection $a_i \in A_i$, $i \in I$.

Let us note that $\bigvee_{i \in I} a_i$ is the convex hull of the finite set $\{a_i\}_{i \in I}$. By relint$A$ we denote a relative interior of the set A.

DEFINITION 3.2 We say that the set S *strictly shadows* the family $\{A_i\}_{i \in I}$ if it shadows this family and $S \cap \operatorname{relint}(\bigvee_{i \in I} a_i) \neq \emptyset$ for every collection $a_i \in A_i$, $i \in I$.

The set S strictly shadows the family $\{A_i\}_{i \in I}$ if and only if for every collection $a_i \in A$, $i \in I$ there exist real numbers $\alpha_i > 0$ with $\sum_{i \in I} \alpha_i = 1$ and $\sum_{i \in I} \alpha_i a_i \in S$.

Let us remark, that for two sets A_1, A_2 these are exactly the definitions of separation and strict separation given in [10], Defintions 4.5.1 and 4.6.1.

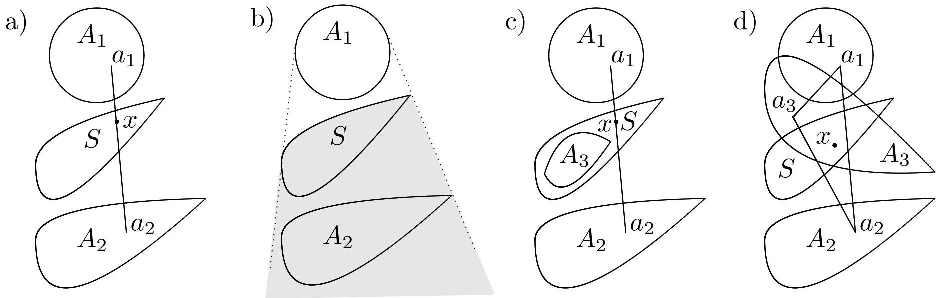

Figure 1. a), b) Separation of two sets. c) Pairwise separation. d) Shadowing of three sets.

Remark 3.3 Let us add that the notion of shadowing, which generalizes separation, comes from a physical interpretation of the separation by sets as follows: if the sets A, B, S are considered as celestial and A shines, then S separates A and B if and only if B lies in the shadow of S. (see Fig. 1.b))

In [6] the concept of shadowing is generalized to commutative semigroups with cancellation property and the following equivalence is proved in Corollary 5.3:

Proposition 3.4 *Let X be a topological vector space, I a finite index set and $S, A_i \in \mathcal{B}(X),\ i \in I$. Then S shadows the sets $A_i,\ i \in I$ if and only if* $\inf_{i\in I}[A_i, 0] \leq [S, 0]$ *in the sense of the ordering among equivalence classes in the Minkowski-Rådström-Hörmander space.*

Now we prove that the concept of separation of sets satisfies the fundamental principle on set-separation of J-E. Martinez-Legaz, and A. Martinón [9]:

Proposition 3.5 *Let X be a real vector space, I a finite index set, $S, A_i \in \mathcal{A}(X),\ i \in I$ and S be convex. If S [strictly] separates pairwise the family $\{A_i\}_{i\in I}$ then S [strictly] shadows the family $\{A_i\}_{i\in I}$.*

Proof The case of separation is obvious. Assume strict separation. Let $S, A_i \subset X,\ i \in I$ be given, where I consists of k elements. The set S strictly separates all pairs of sets A_i and A_j with $i, j \in I$ and $i \neq j$. Then for every collection $\{a_i\}_{i\in I}$ with $a_i \in A_i$ there exist real numbers $\alpha_{ij} > 0$ with $z_{ij} = \alpha_{ij} a_i + (1 - \alpha_{ij})\, a_j \in S$ for $i \neq j$. Put $\sigma = \frac{1}{k(k-1)}$ then the convex combination $\sigma \sum_{\substack{i,j\in I\\ i\neq j}} z_{ij} \in S$ has only nonzero coefficients, i.e., S strictly shadows the sets $A_i,\ i \in I$. ■

3.1. Algebraic separation law for arbitrary sets. The following theorem states the algebraic separation law for arbitrary sets.

THEOREM 3.6 *Let X be a real vector space, I a finite index set and $S, A_i \subset X,\ i \in I$. Then the following statements are equivalent*:

(S1) *S shadows the family $\{A_i\}_{i\in I}$.*
(S2) *For any convex set $C \subset X$ we have*

$$\bigcap_{i\in I}(C + A_i) \subset C + S.$$

(S3) *For any $\{b_i\}_{i\in I} \in X^I$ we have*

$$\bigcap_{i\in I}(A_i + \bigvee_{k\in I} b_k) \subset S + \bigvee_{i\in I} b_i.$$

PROOF (S1) $\Rightarrow$ (S2) Let $x \in \bigcap_{i\in I}(C + A_i)$. Then $x = c_i + a_i, i \in I$ for some $c_i \in C, a_i \in A_i$. By (S1) there exist $y \in S \cap \bigvee_{i\in I} a_i$. We have $y = \sum_{i\in I} \lambda_i a_i$ for some $\lambda_i \geqslant 0$ such that $\sum_{i\in I} \lambda_i = 1$. Hence $x = y + \sum_{i\in I} \lambda_i c_i \in S + C$.

(S2) $\Rightarrow$ (S3) Obvious. (S3) $\Rightarrow$ (S1) Let $\{a_i\}_{i\in I} \in \prod_{i\in I} A_i$. Denote $b_i = -a_i$. Notice that $0 \in \bigcap_{i\in I}(A_i + \bigvee_{k\in I} b_k) \subset S + \bigvee_{i\in I} b_i$. There exist $x \in S$ and $\lambda_i \geqslant 0, i \in I$ such that $\sum_{i\in I} \lambda_i = 1$ and $0 = x + \sum_{i\in I} \lambda_i b_i$. Hence $x = \sum_{i\in I} \lambda_i a_i \in S \cap \bigvee_{i\in I} a_i$.■

The following two corollaries are an immediate consequence:

COROLLARY 3.7 *The following conditions are equivalent*:
(a) *$\bigcap_{i\in I} A_i$ shadows the family $\{A_i\}_{i\in I}$.*
(b) *For any convex set $C \subset X$ we have*

$$\bigcap_{i\in I}(C + A_i) = C + \bigcap_{i\in I} A_i$$

(Translation property of intersection with respect to the family of convex sets).
(c) *For any $\{b_i\}_{i\in I} \in X^I$ we have*

$$\bigcap_{i\in I}(A_i + \bigvee_{k\in I} b_k) = (\bigcap_{i\in I} A_i) + \bigvee_{i\in I} b_i.$$

COROLLARY 3.8 *The following conditions are equivalent:*
(a) *$A \cap B$ separates A and B.*
(b) *For any convex set $C \subset X$ we have $(C + A) \cap (C + B) = C + A \cap B$* (Translation property of intersection with respect to the family of convex sets).
(c) *For any $a \in A, b \in B$ we have $(A + a \vee b) \cap (B + a \vee b) = A \cap B + a \vee b$.*

Second corollary is a generalization of Lemma 3.4 in [15].

3.2. Algebraic separation law for bounded closed convex sets. We write in the following: $\displaystyle\sum_{i=1}^{k} A_i \;=\; A_1 \dotplus A_2 \dotplus \cdots \dotplus A_k$.

For the weaker concept of separation we have the following algebraic characterization:

THEOREM 3.9 *Let X be a topological vector space, I a finite index set and $S, A_i \in \mathcal{B}(X)$, $i \in I$. Then S shadows the family $\{A_i\}_{i\in I}$ if and only if*

$$\sum_{i\in I} A_i \subset \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) \dotplus S.$$

PROOF *Necessity:* Let $a_i \in A_i$, $i \in I$ be given. Then there exist $\alpha_i \geq 0$, $\sum\limits_{i\in I} \alpha_i = 1$ such that $\sum\limits_{i\in I} \alpha_i a_i \in S$. Therefore,

$$\begin{aligned} \sum_{i\in I} a_i &= \sum_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} \alpha_k \Big) a_i + \sum_{i\in I} \alpha_i a_i \\ &= \sum_{i\in I} \alpha_i \Big(\sum_{k\in I\setminus\{i\}} a_k \Big) + \sum_{i\in I} \alpha_i a_i \in \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) \dotplus S, \end{aligned}$$

which proves the necessity.

Sufficiency: Now fix any $a_i \in A_i$, $i \in I$. Then it follows from the assumption

$$\sum_{i\in I} A_i \subset \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) \dotplus S$$

that for every $i \in I$

$$a_i + \sum_{k\in I\setminus\{i\}} A_k \subset \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) \dotplus S,$$

which means:

$$\sum_{k\in I\setminus\{i\}} A_k \subset \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) \dotplus (S - a_i),\ i \in I.$$

From the Pinsker rule we get:

$$\begin{aligned} \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) &\subset \bigvee_{i\in I} \Big[\bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) \dotplus (S - a_i) \Big] \\ &= \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k \Big) \dotplus \bigvee_{i\in I} (S - a_i) \end{aligned}$$

and from the order cancellation law we get $0 \in \bigvee\limits_{i\in I} (S - a_i)$. Again from the Pinsker rule follows $0 \in \bigvee\limits_{i\in I} (S - a_i) = S \dotplus \bigvee\limits_{i\in I} \{-a_i\}$, which implies $S \cap \bigvee_{i\in I} a_i \neq \emptyset$. ∎

4. The relation between the different separation theorems. The following implication holds:

Proposition 4.1 *Let X be a topological vector space, I a finite index set and $S, A_i \in \mathcal{B}(X)$, $i \in I$. If S shadows the sets A_i, $i \in I$ then condition* (S2) *of* Theorem 3.6 *implies the condition of* Theorem 3.9.

Proof Let us assume that the bounded closed convex set $S \in \mathcal{B}(X)$ separates the sets $A_i \in \mathcal{B}(X)$, $i \in I$. Now choose for the convex set $C \subset X$ in condition (S2) of Theorem 3.6 the set

$$C = \bigvee_{i \in I} \Big(\sum_{k \in I \setminus \{i\}} A_k \Big).$$

Now we observe that for every $i \in I$ we have

$$A_i + \sum_{k \in I \setminus \{i\}} A_k \subset \bigcap_{i \in I} \left(A_i + \bigvee_{i \in I} \sum_{k \in I \setminus \{i\}} A_k \right).$$

Using condition (S2) this implies:

$$\sum_{k \in I} A_k \subset \mathrm{cl} \left[\bigcap_{i \in I} \left(A_i + \bigvee_{i \in I} \Big(\sum_{k \in I \setminus \{i\}} A_k \Big) \right) \right] \subset \bigvee_{i \in I} \Big(\sum_{k \in I \setminus \{i\}} A_k \Big) \dot{+} S$$

which completes the proof. ■

5. The Demyanov-Difference. Demyanov original subtraction $A \stackrel{..}{-} B$ (see [13]) of compact convex subsets in finite dimensional space is defined with the help of the Clarke subdifferential (see [2]) of the difference of support functions, i.e.

$$A \stackrel{..}{-} B = \partial_{\mathrm{cl}}(p_A - p_B)\Big|_0,$$

where p_A and p_B are the support functions of A and B, i.e. $p_A(x) = \max_{a \in A} \langle a, x \rangle$

This can be equivalently formulated by

$$A \stackrel{..}{-} B = \overline{\mathrm{conv}}\{a - b | a \in A, b \in B, a + b \in \exp(A + B)\},$$

where $\exp(A+B)$ are the exposed points of $A+B$. For the proof see [13] and note that every exposed point of $A+B$ is the unique sum of an exposed point of A with an exposed point of B.

To extend the definition of the difference $A \stackrel{..}{-} B$ to locally convex vector spaces, the set of exposed points will be replaced by the set of extremal points.

Definition 5.1 Let (X,τ) be a locally convex vector space and $\mathcal{K}(X)$ the family of all nonempty compact convex subsets of X. Then for $A,B \in \mathcal{K}(X)$, the set

$$A \dot{\overline{-}} B = \overline{\mathrm{conv}}\{a-b | a \in A, b \in B, a+b \in \mathrm{ext}(A+B)\} \in \mathcal{K}(X)$$

is called the *Demyanov Difference* of A and B.

This is a canonical generalization of the above definition, because for $A,B \in \mathcal{K}(X)$ every extremal point $z \in \mathrm{ext}(A+B)$ has a unique decomposition $z = x+y$ into the sum of two extreme points $x \in \mathrm{ext}\,A$ and $y \in \mathrm{ext}\,B$ (see [8], Proposition 1).

Since in the finite dimensional case the exposed points are dense in the set of extreme points of a compact convex set, this definition coincides with the original definition of the Demyanov difference in finite dimensional spaces.

The Demyanov difference in finite dimensional spaces possesses many important properties. Some of them hold also for its generalization (see [5]):

Proposition 5.2 *Let X be a locally convex vector space and $A,B,C \in \mathcal{K}(X)$. The Demyanov-Difference has the following properties:*

(D1) *If $A = B + C$, then $C = A \dot{\overline{-}} B$.*
(D2) $(A \dot{\overline{-}} B) + B \supset A$.
(D3) *If $B \subset A$, then $0 \in A \dot{\overline{-}} B$.*
(D4) $(A \dot{\overline{-}} B) = -(B \dot{\overline{-}} A)$.
(D5) $A \dot{\overline{-}} C \subset (A \dot{\overline{-}} B) + (B \dot{\overline{-}} C)$.

From property (D2) of the above proposition follows immediately:

Theorem 5.3 *Let X be a locally convex vector space, I a finite index set and $S, A_i \in \mathcal{K}(X),\ i \in I$. Then the Demyanov-Difference*

$$S \;=\; \Big(\sum_{i\in I} A_i\Big) \dot{\overline{-}} \bigvee_{i\in I} \Big(\sum_{k\in I\setminus\{i\}} A_k\Big)$$

shadows the family $\{A_i\}_{i\in I}$.

Corollary 5.4 *Let $A_1, A_2, ..., A_k \in \mathcal{K}(\mathbb{R}^n)$ be given. Then for the Demyanov-Difference holds*

$$\Big(\sum_{i=1}^{k} A_i\Big) \dot{\overline{-}} \bigvee_{i=1}^{k} \Big(\sum_{\substack{j=1\\ j\neq i}}^{k} A_j\Big) \;=\; \partial_{\mathrm{cl}} P\big|_0,$$

where $\partial_{\mathrm{cl}} P\big|_0$ is the Clarke subdifferential of $P = \min\{p_{A_1}, p_{A_1}, ..., p_{A_k}\}$, at $0 \in \mathbb{R}^n$, i.e. the minimum of the support functions of the sets A_i.

Proof This follows immediately from the definition of the Demyanov-Difference for the finite dimensional case (see [13]) and the formula

$$\Big(\sum_{i=1}^{k} p_{A_1}\Big) \;-\; \max\left\{ \sum_{\substack{j=1\\ j\neq i}}^{k} p_{A_1} \mid i \in \{1,..,k\} \right\} = \min\{p_{A_1}, p_{A_1}, ..., p_{A_k}\},$$

which completes the proof. ■

References

[1] A. Astorino and M. Gaudioso, (2002): *Polyhedral separability through successive LP,* Journ. of Optimization Theory **112**, 265-293.

[2] F. H. Clarke, (1983): *Optimization and Nonsmooth Analysis,* J. Wiley Pub. Comp., New York.

[3] V. F. Demyanov and A. M. Rubinov, (1986): *Quasidifferential calculus*, Optimization Software Inc., Publications Division, New York.

[4] M. Gaudioso, E. Gorgone and D. Pallaschke, (2011): *Separation of convex sets by Clarke subdifferential,* Optimization **59**, 1199-1210.

[5] J. Grzybowski, D. Pallaschke and R. Urbański, (2012): *Demyanov Difference in infinite dimensional Spaces,* submitted to the Proceedings of CNSA, St. Petersburg.

[6] J. Grzybowski, D. Pallaschke and R. Urbański, (2010): *Reduction of finite exhausters,* Journ. of Global Optimization **46**, 589-601.

[7] J. Grzybowski, D. Pallaschke and R. Urbański, (2005): *A pre-classification and the separation law for closed bounded convex sets,* Optimization Methods and Software **20**, 219-229.

[8] T. Husain and I. Tweddle, (1970): *On the extreme points of the sum of two compact convex sets,* Math. Ann. **188**, 113-122.

[9] J-E. Martinez-Legaz and A. Martinón, (2012): *On the infimum of a quasiconvex vector function over an intersection,* TOP **20**, 503-516.

[10] D. Pallaschke and R. Urbański, (2002): *Pairs of Compact Convex Sets —Fractional Arithmetic with Convex Sets,* Mathematics and its Applications, Vol. **548**, Kluwer Acad. Publ. Dordrecht.

[11] D. Pallaschke and R. Urbański, (2002): *On the separation and order law of cancellation for bounded sets,* Optimization, Vol. **51**(3), 487-496.

[12] A. G. Pinsker, (1966): *The space of convex sets of a locally convex space,* Trudy Leningrad Engineering-Economic Institute **63**, 13-17.

[13] A. M. Rubinov and I. S. Akhundov, (1992): *Differences of compact sets in the sense of Demyanov and its application to non-smooth-analysis,* Optimization **23** (1992), 179-189.

[14] R. Urbański, (1976): *A generalization of the Minkowski-Rådström-Hörmander theorem,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **24**, 709-715.

[15] R. Zang, Y. Ma and Y. Liu, (2013): *(F, K, b)-vex sets and some related properties,* International Journal of Computer Mathematics, (2013), 1-7.

Jerzy Grzybowski
Adam Mickiewicz University Poznań
Faculty of Mathematics and Computer Science, Umultowska 87, PL-61-614 Poznań, Poland
E-mail: jgrz@amu.edu.pl

Diethard Pallaschke
University of Karlsruhe (KIT)
Institute of Operations, Kaiserstr. 12, D-76128 Karlsruhe, Germany
E-mail: diethard.pallaschke@kit.edu

Ryszard Urbański
Adam Mickiewicz University Poznań
Faculty of Mathematics and Computer Science, Umultowska 87, PL-61-614 Poznań, Poland
E-mail: rich@amu.edu.pl

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