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On a certain case of asymptotic stability of the solution $Y = 0$ of a system of ordinary differential equations $Y' = F(x, Y)$

In this paper we shall give sufficient conditions for the solution $Y = 0$ of the system of ordinary differential equations

$$Y' = F(x, Y), \tag{1}$$

where $Y = [y_1, y_2, \ldots, y_n]$, $F(x, Y) = [f_1(x, Y), \ldots, f_n(x, Y)]$, to be asymptotically and uniformly stable with respect to the initial conditions. The theorems which are given in this paper are generalizations of theorems of paper [2], where the asymptotic stability of the solution $y = y_0$ of the differential equation

$$y' = f(x, y)$$

was investigated.

We shall use the definition of uniform stability given in paper [1].

We start with two lemmas:

LEMMA 1. *Assume that*

1° *functions* $f_i(x, Y)$ $(i = 1, 2, \ldots, n)$ *appearing on the right-hand side system* (1) *are defined and continuous in the set*

$$D = \Delta_a \times D_a,$$

where $\Delta_a = \langle a, +\infty)$,

$$D_a = \{Y = [y_1, \ldots, y_n]: \|Y\| < a\}$$

$\Big(D_a \subset \boldsymbol{R}^n, \|Y\| =: \big(\sum_{i=1}^{n} y_i^2\big)^{1/2}\Big)$, $a > 0$, *in particular* $a = +\infty$ *is possible,*

2° $Y \cdot F(x, Y) = \sum_{i=1}^{n} y_i f_i(x, Y) \leqslant 0$ *for each* $(x, Y) \in D$,

3° $F(x, 0) = 0$ *for* $x \in \Delta_a$.

Under these assumptions the solution $Y(x) = 0$, $x \in \Delta_a$, *of system* (1) *is Liapunov stable.*

Proof. Let $(\bar{x}, \bar{Y})$ be an arbitrary point of D. It follows from Assumption 2° that each solution of system (1) issuing from the point $(\bar{x}, \bar{Y})$ is defined in $\langle\bar{x}, +\infty)$. We shall demonstrate that for each $\varepsilon > 0$ and $x_0 \in \Delta_a$ there exists a $\delta > 0$ such that each solution $Y = \Phi(x)$ of system (1) satisfying the condition $\|\Phi(x_0)\| < \delta$ satisfies also the inequality $\|\Phi(x)\| \leqslant \varepsilon$ for $x \in \langle x_0, +\infty)$.

It follows from Assumption 2° that the function $\varrho(x) = \|\Phi(x)\|$ is non-increasing. For a fixed $\varepsilon > 0$ and $x_0 \in \Delta_a$ we take $0 < \delta \leqslant \varepsilon$ and we obtain from the initial inequality $\|\Phi(x_0)\| < \delta$ the inequality $\|\Phi(x)\| < \delta$ for $x \in \langle x_0, +\infty)$ and hence also $\|\Phi(x)\| < \varepsilon$ for $x \in \langle x_0, +\infty)$ what completes the proof of Lemma 1.

LEMMA 2. *If*

1° *the function* $u(x)$ *is defined in* Δ_{x_0} *and there exists* $u'(x)$ *for* $x \in \Delta_{x_0}$,

2° $\limsup\limits_{x\to\infty} u'(x) = \delta,\ \delta < 0,$

then

$$\lim_{x\to\infty} u(x) = -\infty.$$

Proof. It follows from Assumption 2° that for each fixed k satisfying the condition $0 < k < |\delta|$ there exists an $A > 0$ such that

$$u'(x) < -k < 0 \quad \text{for } x > x_0 + A. \tag{2}$$

Let $\{a_n\}$ be a sequence such that $a_n > x_0 + A$ and $a_n - a_{n-1} = 1$ for $n = 1, 2, \ldots$ Then $u(a_n) - u(a_{n-1}) = u'(\lambda_n)$, where $a_{n-1} < \lambda_n < a_n$. From (2) we obtain $u(a_n) < u(a_0) - nk$ for $n = 1, 2, \ldots$ what completes the proof of Lemma 2.

Basing on these two lemmas, we come to the proof of the following theorem:

THEOREM 1. *Assume that*

1° *the functions* $f_i(x, Y)$ $(i = 1, 2, \ldots, n)$ *are defined and continuous in the set* D *defined in Lemma* 1,

2° *exactly one solution of system* (1) *passes through every point* $(x, Y) \in D$,

3° $Y \cdot F(x, Y) \leqslant 0$ *for any* $(x, Y) \in D$,

4° $F(x, 0) = 0$ *for* $x \in \Delta_a$,

5° $\limsup\limits_{\substack{x\to\infty \\ Y\to\bar{Y}}} Y \cdot F(x, Y) = \delta,\ \delta < 0$ *for any* $\bar{Y} \in D_a$, $\|\bar{Y}\| > 0$.

Under these assumptions the solution $Y = 0$ *of system* (1) *is asymptotically stable in* Δ_a *uniformly with respect to the initial conditions given on the set*

$$K_{x_0\beta} = \{(x, Y):\ x = x_0,\ \|Y\| < \beta,\ 0 < \beta < a\}$$

for fixed x_0, β *what means that*

1) *the integral* $Y = 0$ *is stable in* Δ_a *and that*

2) *for any* $\varepsilon > 0$ *and for any* $x_0, x_0 \geqslant a$ *there exists an* $A > 0$ *dependent on* ε *and on* x_0 *such that for an arbitrary solution* $Y = \Phi(x)$ *of* (1) *issuing from an arbitrary point of the set* $K_{x_0\beta}$, *the inequality* $\|\Phi(x)\| < \varepsilon$ *holds if only* $x > x_0 + A$

Proof. It follows from Assumptions 1°, 3°, 4° and Lemma 1 that the solution $Y = 0$ is Liapunov stable. Now we shall show that for any $x_0 \in \Delta_a$ the solution $Y = \Phi(x)$ $(\Phi(x) = [\varphi_1(x), \ldots, \varphi_n(x)])$ of system (1) satisfying the condition $\|\Phi(x_0)\| \leqslant \beta$ satisfies also the condition

$$\lim_{x\to\infty} \|\Phi(x)\| = 0.$$

For this purpose denote by u the function

$$u(x) = \|\Phi(x)\|^2 = \sum_{i=1}^{n} \varphi_i^2(x). \tag{3}$$

Then

$$u'(x) = 2\sum_{i=1}^{n} \varphi_i(x)\varphi_i'(x) = 2\sum_{i=1}^{n} \varphi_i(x) f_i\big(x, \Phi(x)\big) \tag{4}$$
$$= 2\Phi(x)\cdot F\big(x, \Phi(x)\big) \leqslant 0.$$

We shall demonstrate that $\lim_{x\to\infty} u(x) = 0$.

Assume that the last condition does not hold. Then $u(x) > 0$ and $u'(x) \leqslant 0$ for any $x \in \Delta_a$ implies that

$$\lim_{x\to\infty} u(x) = \gamma, \qquad \gamma > 0. \tag{5}$$

From (3), (4) and (5) we infer that there exists the limit

$$\limsup_{x\to\infty} \Phi(x) = Y_1 \neq 0$$

and therefrom and from Assumption 5° we infer that

$$\limsup_{x\to\infty} u'(x) = \limsup_{x\to\infty} 2\Phi(x)\cdot F\big(x, \Phi(x)\big) = 2\delta < 0.$$

It would follow from the above given conditions and from Lemma 2 that

$$\lim_{x\to\infty} u(x) = -\infty$$

what contradicts (5). Therefore the stability is indeed asymptotic.

To complete the proof we must show that this asymptotic stability is uniform with respect to the initial conditions given on a fixed set $K_{x_0\beta}$.

Assume on the contrary that for some $\varepsilon_0 > 0$ no such $A > 0$ can be chosen that if

$$\|\Phi(x_0)\| \leqslant \beta,$$

then

$$\|\Phi(x)\| < \varepsilon_0 \quad \text{for } x > x_0 + A$$

and for all solutions $Y = \Phi(x)$ of system (1) issuing from $K_{x_0\beta}$ with fixed x_0 and β.

This would mean that for each positive integer n there exists a point $(x_0, Y_n) \in K_{x_0\beta}$ such that the solution $Y = \Phi_n(x)$ issuing from this point satisfies the condition

$$(6) \qquad \|\Phi_n(x_0+n)\| > \varepsilon_0.$$

As $\overline{K}_{x_0\beta}$ is a compact set, therefore the sequence $\{(x_0, Y_n)\}$ contains the subsequence $\{(x_0, Y_{n_k})\}$ convergent to some point $(x_0, \hat{Y}) \in \overline{K}_{x_0\beta}$.

Denote by $Y = \hat{Y}(x)$ the solution of system (1) satisfying the initial condition

$$\hat{Y}(x_0) = \hat{Y}.$$

As the solution $Y = 0$ of system (1) is asymptoticaly stable,

$$\lim_{x\to\infty} \|\hat{Y}(x)\| = 0.$$

and hence there exists an $A_0 > 0$ such that

$$\|\hat{Y}(x_0 + A_0)\| < \varepsilon_0.$$

As the solution of (1) depends continuously on the initial conditions, there exists a neighbourhood $U(x_0, \hat{Y})$ of the point $(x_0, \hat{Y})$ such that all solutions of system (1) issuing from this neighbourhood satisfy the inequality

$$\|Y(x_0 + A_0)\| < \varepsilon_0.$$

As for a sufficiently large n_k the points (x_0, Y_{n_k}) belong to the neighbourhood $U(x_0, \hat{Y})$ therefore

$$\|\Phi_{n_k}(x_0 + A_0)\| < \varepsilon_0$$

for the sufficiently large n_k, what contradicts inequality (6).

Thus the stability is uniform with respect to the initial conditions given on a fixed set $\overline{K}_{x_0\beta}$.

Remark 1. In the case where $\delta = 0$ the stable solution $Y = 0$ of system (1) may not be asymptotically stable as the following example shows:

Consider the following system of differential equations

$$(7) \qquad \frac{dy_1}{dx} = -\frac{y_1}{x}, \qquad \frac{dy_2}{dx} = -\frac{y_1^2 y_2}{x}$$

in the set

$$D = \{(x, y_1, y_2): x \in \langle 1, +\infty), \; y_1^2 + y_2^2 < +\infty\}.$$

Functions occurring on the right-hand side of (7) satisfy assumptions 1°–4° of Theorem 1, but

$$\limsup_{\substack{x\to\infty\\(y_1,y_2)\to(\bar{y}_1,\bar{y}_2)\\ \bar{y}_1^2+\bar{y}_2^2>0}} [y_1, y_2]\cdot\operatorname{colon}[f_1(x, y_1, y_2), f_2(x, y_1, y_2)] = 0.$$

The general solution of (7) has the form

$$y_1 = \frac{c_1}{x}, \qquad y_2 = c_2 e^{c_1^2/2x^2}. \tag{8}$$

It follows from (8) that the solution $y_1 = 0$, $y_2 = 0$ of system (7) is stable, although not asymptotically.

Remark 2. If

$$\limsup_{\substack{x\to\infty\\ Y\to\bar{Y}\\ \|\bar{Y}\|>0}} Y\cdot F(x, Y) = 0$$

for some $\bar{Y} \neq 0$, then the zero solution of system (1) may be asymptotically stable as the following example shows:

The system of equations

$$\frac{dy_1}{dx} = -\frac{y_1}{x}, \qquad \frac{dy_2}{dx} = -y_2 \tag{9}$$

considered in the same set D has the solution $y_1 = 0$, $y_2 = 0$ which is asymptotically stable, although

$$\limsup_{\substack{x\to\infty\\ Y\to\bar{Y}\\ \|\bar{Y}\|>0}} Y\cdot F(x, Y) = -\bar{y}_2^2$$

(the limit is equal to zero for $\bar{y}_2 = 0$).

The asymptotic stability follows from the form of the general solution of system (9)

$$y_1 = \frac{c_1}{x}, \qquad y_2 = c_2 e^{-x}.$$

We shall prove a theorem on the asymptotic stability uniform with respect to any initial conditions. We shall apply in the theorem Lemma 2 from [3], p. 314, which we quote here in an appropriate form as the following

LEMMA 3. *Assume that*

1° *the functions* $f_i(x, Y)$ $(i = 1, 2, \ldots, n)$ *are continuous in* D,

2° *exactly one solution of system* (1) *passes through each point of* D,

3° $Y \cdot F(x, Y) \leqslant 0$ *for* $(x, Y) \in D$.

If we denote by C *the part of integral curves of system* (1) *issuing from* $\bar{K}_{x_0\beta}$ *for* $x_0 \leqslant x \leqslant x_0 + A$, *then under these conditions the curves* C *issuing from* $\bar{K}_{x_0\beta}$ *and defined for* $x \in \langle x_0, x_0+A \rangle$ *form in* D *a closed domain. The bound of this domain consists of the set* $\bar{K}_{x_0\beta}$, *the surface* Γ *formed of the curves* C *issuing from the set*

$$\{(x, Y)\colon\ x = x_0,\ \|Y\| = \beta\}$$

$x_0 \in \Delta_a$, $0 < \beta < a$ *and of the set* Z *consisting of points of curves* C *corresponding to the coordinate* $x = x_0 + A$.

THEOREM 2. *Assume that*

1° *the functions* $f_i(x, Y)$ $(i = 1, 2, \ldots, n)$ *are defined and continuous in the set* D,

2° *exactly one solution of system* (1) *passes through each point of* D,

3° $F(x, 0) = 0$ *for* $x \in \Delta_a$ *and* $Y \cdot F(x, Y) \leqslant 0$ *for each* $(x, Y) \in D$,

4° *there exists the limit*

$$\limsup_{\substack{x\to\infty \\ Y\to\bar{Y} \\ \|\bar{Y}\|>0}} Y \cdot F(x, Y) = \delta, \qquad \delta < 0,$$

5° $Y \cdot F(x+h, Y) < Y \cdot F(x, Y)$ *for* $(x, Y) \in D$ *such that* $(x+h, Y) \in D$ *and for any* $h > 0$.

Under these assumptions the solution $Y = 0$ *of system* (1) *is asymptotically stable uniformly with respect to any arbitrary initial conditions, which means that*

1) $Y = 0$ *is a stable solution of* (1) *in* Δ_a *and that*

2) *for each* $\varepsilon > 0$ *there exists a number* $A > 0$ *depending exclusively on* ε *that for* $x > x_0 + A$ *the inequality*

$$\|\Phi(x)\| < \varepsilon$$

holds for all solutions $Y = \Phi(x)$ *of system* (1) *issuing from any arbitrary point of the set*

$$\bar{K}_{x_0\beta} = \{(x, Y)\colon\ x = x_0,\ \|Y\| \leqslant \beta\}$$

for any $x_0 \geqslant a$ *and for fixed value* $\beta \in (0, a)$.

Proof. It follows from assumptions 1°–4° and from Theorem 1 that the solution $Y = 0$ of system (1) is asymptotically stable uniformly with respect to the initial conditions given on the set $\bar{K}_{x_0\beta}$ for a fixed x_0. It follows hencefrom that for any $\varepsilon > 0$ and $x_0 = a$ there exists a number $A > 0$ such that the inequality

$$\|Y(x)\| < \varepsilon$$

holds for $x > a+A$ and for any solution $Y = Y(x)$ satisfying the initial inequality

$$\|Y(a)\| \leqslant \beta.$$

We shall demonstrate that for the same values of ε and A and for any $x_1 \epsilon (a, +\infty)$ the inequality

$$\|\Psi(x)\| < \varepsilon$$

holds for $x > x_1 + A$ and for any solution $Y = \Psi(x)$ satisfying the initial inequality

$$\|\Psi(x_1)\| \leqslant \beta.$$

It follows from Lemma 3 that integral curves of system (1) for $a \leqslant x \leqslant a+A$ issuing from the initial set

$$\{(x, Y) \colon x = a, \|Y\| \leqslant \beta\}$$

form a closed and bounded domain Ω.

Let $Y = \Psi(x)$ be an arbitrary solution of system (1) satisfying the initial condition

$$\Psi(x_1) = \overline{Y}, \quad \text{where } x_1 \epsilon \Delta_a, \ \overline{Y} = [\overline{y}_1, \ldots, \overline{y}_n], \ \|\overline{Y}\| = \beta. \tag{10}$$

We shall show now that the curve C defined by the equation

$$Y = \overline{\Psi}(x), \quad a \leqslant x \leqslant a+A,$$

where

$$\overline{\Psi}(x) = \Psi(x+h), \quad h = x_1 - a \tag{11}$$

is contained in Ω for $a \leqslant x \leqslant a+A$.

Denote by $v(x)$ the function

$$v(x) = \sum_{i=1}^{n} \overline{\psi}_i^2(x), \quad a \leqslant x \leqslant a+A. \tag{12}$$

Then

$$v'(x) = 2\sum_{i=1}^{n} \psi_i(x) \frac{d\overline{\psi}_i(x)}{dx} = 2\sum_{i=1}^{n} \overline{\psi}_i(x) \cdot f_i\big(x+h, \overline{\Psi}(x)\big).$$

For $x = a$ we have by (10) and (11)

$$v'(a) = 2\sum_{i=1}^{n} \overline{y}_i f_i(a+h, \overline{Y}).$$

It follows from Assumption 5° that the function

$$\sum_{i=1}^{n} \overline{y}_i f_i(a+h, \overline{Y}) - \sum_{i=1}^{n} y_i f_i(x, Y) \tag{13}$$

takes a negative value for $(x, Y) = (a, \bar{Y})$. As the function is continuous (with respect to (x, Y)), there exists a set U defined in the following way

$$U = \{(x, Y)\colon \|Y - \bar{Y}\| \leqslant \delta,\ \|Y\| = \beta,\ a \leqslant x \leqslant \delta_1\}$$

such that the function defined by formula (13) is negative for $(x, Y) \in U$.

Assume that $Y = Y(x)$ is the solution of system (1) satisfying the initial condition

$$Y(a) = \tilde{Y}, \quad (a, \tilde{Y}) \in U.$$

Then the function

$$u(x) = \sum_{i=1}^{n} y_i^2(x)$$

has a derivative given by the formula

$$u'(x) = 2\sum_{i=1}^{n} y_i(x) f_i\big(x, Y(x)\big).$$

As $(a, \tilde{Y}) \in U$, then

$$v'(a) < u'(a).$$

We obtain from the above inequality and from the fact that $u(a) = v(a) = \beta$ the following condition

$$v(x) < u(x)$$

for values of x belonging to some interval $(a, \bar{a})$, where $\bar{a} > a$. This inequality means that the curve $Y = \bar{\Psi}(x)$ lies nearer the x-axis for $x \in (a, \bar{a})$ then the curve $Y = Y(x)$.

As any arbitrary solution $Y = Y(x)$ issuing from a sufficiently small neighbourhood of the point $(a, \bar{Y})$ and lying on the bound of Ω has the above mentioned property therefore the curve $Y = \bar{\Psi}(x)$ lies inside Ω for $x \in (a, \bar{a})$.

We shall show in the further part of the proof that this curve lies entirely in Ω for $x \in \langle a, a+A\rangle$.

Assume on the contrary that there exist on C points not belonging to Ω. Then there exists a point (x^*, Y^*) such that $a < x^* < a + A$ and

1) $\bar{\Psi}(x^*) = Y^*$,
2) the curve $Y = \bar{\Psi}(x)$ lies in Ω for $a \leqslant x \leqslant x^*$,
3) the curve $Y = \bar{\Psi}(x)$ lies outside Ω for $x \in (x^*, \alpha^*)$ $\alpha^* > x^*$.

Then the point (x^*, Y^*) is the point of intersection of C and the surface Γ being a bound part of Ω and formed of integral curves of system (1) issuing from the set

$$\{(x, Y)\colon x = a,\ \|Y\| = \beta\}.$$

The derivative of $v(x)$ defined by formula (12) has in the point $x = x^*$ the value

$$v'(x^*) = 2\sum_{i=1}^{n} \psi_i(x^*)\cdot f_i\big(x^*+h,\, \overline{\Psi}(x^*)\big).$$

The function

$$\sum_{i=1}^{n} y_i^* f_i(x^*+h,\, Y^*) - \sum_{i=1}^{n} y_i f_i(x,\, Y)$$

takes a negative value for $x = x^*$, $Y = Y^*$. As it is continuous function of the variables (x, Y) hence there exists a set

$$U^* = \{(x, Y):\ \|Y - Y^*\| \leqslant \delta^*,\ Y \epsilon \Gamma,\ x^* \leqslant x \leqslant \delta_1^*\}$$

such that the above mentioned function is negative for $(x, Y) \epsilon U^*$. We can state by means of a method similar to that applied in the first part of the proof that there exists an interval $(x^*, \tilde{a})$, $\tilde{a} > x^*$ such that for $x \epsilon (x^*, \tilde{a})$ the points $\big(x, \overline{\Psi}(x)\big)$ lie inside the set Ω.

Hence we obtained a contradiction with the assumption that there exists an interval (x^*, a^*) such that $\big(x, \overline{\Psi}(x)\big) \notin \Omega$ for $x \epsilon (x^*, a^*)$. Therefore the curve $Y = \overline{\Psi}(x)$ lies in Ω. It follows therefrom and from the uniform stability of the solution $Y = 0$ with respect to the initial conditions given on the set $\overline{K}_{x_0\beta}$ for a fixed x_0, that

$$\|\overline{\Psi}(x)\| < \varepsilon \qquad \text{for } x > a + A$$

and hence and from (11) it follows that

$$\|\Psi(x)\| < \varepsilon \qquad \text{for } x > x_1 + A$$

which means that the stability is uniform with respect to arbitrary initial conditions. The proof of the theorem is complete.

Assumption 5° in Theorem 2 is essential, as the following example shows:

EXAMPLE. Consider the following system of differential equations

$$(14) \qquad \frac{dy_1}{dx} = -y_1 - \frac{y_2}{\sqrt{x}}, \qquad \frac{dy_2}{dx} = -y_2 - \frac{y_2}{\sqrt{x}}.$$

Let us investigate the position of its solutions in the set

$$D = \{(x, y_1, y_2):\ x \epsilon \langle 1, +\infty),\ y_1^2 + y_2^2 < +\infty\}.$$

Functions appearing on the right-hand side of system (14) satisfy assumptions 1°–4° of Theorem 2. Indeed,

1° the functions $f_i(x, y_1, y_2)$ $(i = 1, 2)$ are continuous in D,

2° exactly one solution of (14) passes through each point of D, 3° $Y \cdot F(x, Y) = -y_1^2 - y_2^2 - \frac{y_2^2}{\sqrt{x}} - \frac{y_1 y_2}{\sqrt{x}} \leqslant 0$ for each $(x, y_1, y_2) \in D$ and

$$F(x, 0) = 0 \quad \text{for } x \in \langle 1, +\infty),$$

4° $\limsup\limits_{\substack{x\to\infty \\ Y\to\bar{Y} \\ \|\bar{Y}\|>0}} Y \cdot F(x, Y) = -\bar{y}_1^2 - \bar{y}_2^2 < 0.$

The function $Y \cdot F(x, Y)$ of the variable x is not decreasing for any fixed Y.

The general solution of system (14) has the form

$$y_1 = (c_1 + c_2 e^{-2\sqrt{x}}) e^{-x}, \qquad y_2 = c_2 e^{-2\sqrt{x}-x}.$$

The integral curve $Y = \Phi(x)$ passing through the point $P_0(1, y_1^0, y_2^0)$ satisfies the equations

$$y_1 = (y_1^0 - y_2^0) e^{1-x} + y_2^0 e^{3-2\sqrt{x}-x}, \qquad y_2 = y_2^0 e^{3-2\sqrt{x}-x}.$$

Consider an arbitrary solution $Y = \Psi(x)$ satisfying the initial condition

$$\Psi(x_1) = \bar{Y}, \qquad \text{where } x_1 > 1,\ \bar{Y} = [y_1^0, y_2^0].$$

This solution has the form

$$y_1 = (y_1^0 - y_2^0) e^{x_1 - x} + y_2^0 e^{2\sqrt{x_1}+x_1-2\sqrt{x}-x}, \qquad y_2 = y_2^0 e^{2\sqrt{x_1}+x_1-2\sqrt{x}-x}.$$

The integral curve $Y = \Psi(x)$ when translated to the point P_0 has the form

$$Y = \Psi(x + x_1 - 1) = \bar{\Psi}(x),$$

where

$$y_1 = (y_1^0 - y_2^0) e^{1-x} + y_2^0 e^{2\sqrt{x_1}-2\sqrt{x+x_1-1}+1-x}, \qquad y_2 = y_2^0 e^{2\sqrt{x_1}-2\sqrt{x+x_1-1}+1-x}.$$

As can be easily seen that the inequality

$$\|\bar{\Psi}(x)\|^2 > \|\Phi(x)\|^2$$

holds, and therefore the solution $y_1 = 0$, $y_2 = 0$ is not uniformly stable with respect to the arbitrary initial conditions.

References

[1] W. Hahn, *Über typen des Stabilitätsverhaltens*, Monath. Math. 71 (1967), p. 7–13.

[2] W. Pawelski, *On a simple case of asymptotic stability*, Comm. Math. 13 (1970), p. 233–239.

[3] – *Remarques sur des inégalités mixtes entre les intégrales des équation aux dérivées partielles du premier ordre*, Ann. Polon. Math. 13 (1963), p. 309–326.