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THEOREMS RELATING TO QUOTIENT-GROUPS.

(TWIERDZENIA O GRUPACH ILORAZOWYCH).

As there is a (1, 1) correspondence between the invariant subgroups and the quotient groups [1]), it follows that the theorems which relate to quotient groups are directly useful in the theory of invariant subgroups and vice versa. In what follows we shall emphasize the quotient group or the invariant subgroup as the theorems under consideration seem to bring out the properties of the one or the other of these groups in a prominent manner. The group under consideration will be denoted by G, the invariant subgroup by H, and the quotient group by I. The corresponding small letters will be employed to represent the orders of these groups.

When I is abelian H includes the commutator subgroup of G and vice versa. Hence each of the invariant subgroups of G which leads to an abelian quotient group corresponds to a subgroup of the same index in the commutator quotient group; i. e. in the quotient group which corresponds to the invariant subgroup generated by the commutators of G. In particular, the quotient groups of prime order (p) correspond to quotient groups of order p in the commutator quotient-group. The number of the latter is $\frac{p^n-1}{p-1}$, p^n being the order of the group generated by the operators of order p in the commutator quotient group of G [2]).

1) Two invariant subgroups of the same type may correspond to two quotient-groups of different types, and two invariant subgroups of different types may correspond to two quotient-groups of the same type, but there is always a (1, 1) correspondence between the number of these groups since each invariant subgroup leads to one and only one quotient-group.

2) Cf. Weber. Algebra, vol. 2. 1896, p. 49.

In other words, the number of the invariant subgroups of order g/p which are contained in G is either 0 or $1+p+p^2+\ldots+p^{n-1}$. Since every subgroup of order $g/2$ is invariant under G, it follows that there is either no subgroup of order $g/2$ in G, or the number of these subgroups is $1+2+\ldots+2^{n-1}$. In particular, if the order of a Sylow subgroup in G is 2^m, there cannot be more than $1+2+\ldots+2^{m-1}$ subgroups of order $g/2$ in G, and there are groups in which this limit is attained for every value of m.

Since every invariant subgroup of index p under G contains the p^{th} power of every operator of G as well as the commutator subgroup, it follows that the quotient group with respect to the common operator of all the invariant subgroups of index p is of type $(1, 1, 1, \ldots)$. This furnishes another proof of the fact that the number of the invariant subgroups of index p in G is always of the form $1+p+p^2+\ldots$, whenever there is at least one such invariant subgroup [1].

Several theorems in regard to non-invariant subgroups of index p may be given here as they constitute elegant generalizations of the theorem that every subgroup of order p^{m-1} is invariant under a group of order p^m. If H' is such a non-invariant subgroup of G, it has just p conjugates under G. Each of these conjugates contains all the operators of H' whose ordes are powers of p, since such an operator cannot transform the remaining conjugates among themselves. Hence the operators of H whose order are any power of p generate an invariant subgroup of G [2]. In particular, if g is divisible by p^2 and if G contains a subgroup of index p, it must be a composite group. If $g=p^n$, H' is generated by operators whose orders are powers of p and hence it is invariant under G.

The other generalization in question may be stated as follows. If G contains a non-invariant subgroup (H_1) of index k, then G is isomorphic with a non-regular transitive substitution group of degree k [3]. In this isomorphism H_1 corresponds to a group whose degree cannot exceed $k-1$. Hence H_1 contains a subgroup whose index under H_1 is less than k. If the orders of G and H_1 were p^m and p^{m-1} respectively, H_1 could not include a subgroup of a smaller index than p, and hence H_1 would have to be invariant. In particular, if a simple group has a subgroup of index k, then this subgroup must in turn contain a subgroup which is of lower index than k.

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[1]) Bauer. Nouvelles Annales, vol. **19** (1900), p. 509.

[2]) Bulletin of the American Mathematical Society, vol. 3 (1896), p. 115.

[3]) The necessary and sufficient condition that this is a simple isomorphism is that H_1 does not include an invariant subgroup of G besides the identity. Cf. Dyck. Mathematische Annalen, vol. **22** (1883), p. 89. If H_1 were invariant the corresponding substitution group would be regular.

From the second paragraph it follows that the theory of abelian quotient-groups of G is dependent upon the theory of subgroups in the commutator quotient-group of G. As the number of cyclic quotient-groups of order p^α, $\alpha>1$, in the latter is always a multiple of p [1]), t h e n u m b e r o f c y c l i c q u o t i e n t - g r o u p s o f o r d e r p^α i n a n y g r o u p i s a l w a y s s o m e m u l t i p l e o f p. In fact, this number is simply the number of the cyclic subgroups of order p^α in the commutator quotient-group of G. Combining this result with the recent theorems relating to cyclic subgroups [2]), it follows that when $p>2$, $\alpha>1$ both the number of cyclic subgroups of order p^α in G and the number of its cyclic quotient-groups of this order are multiples of p. Instead of saying „the number of cyclic quotient-groups", we may say, the number of invariant subgroups which give rise to cyclic quotient-groups. The number of non-cyclic abelian quotient-groups of order p^α is $\equiv 1 \bmod p$.

Since there is a $(1, 1)$ correspondence between the subgroups and the quotient-groups of an abelian group and since the number of subgroups of any type which are contained in auch a group can be readily found [3]), the determination of all the abelian quotient-groups of G is practically reduced to the determination of its commutator quotient-groups. Hence we shall not pursue this subject any further at this place. In regard to non-abelian quotient-groups the matter becomes much more difficult both because there is no longer such an intimate relation between subgroups and quotient-groups, and also because the theory of the non-abelian groups is so much more difficult than that of abelian groups.

To illustrate the application of the preceding theory we proceed to give a method by means of which all the quotient-groups of order pq, p and q being different primes, may be determined. When the quotient-group is cyclic the matter is included in the preceding developments. If there is any such quotient-group their total number is $(1+p+p^2+\ldots+p^{n_1-1})(1+q+q^2+\ldots+q^{n_2-1})$, p^{n_1} and q^{n_2} being the orders of the groups generated by the operators of orders p and q respectively in the commutator quotient-group [4]).

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[1]) Throughout this paragraph it is assumed that the Sylow subgroup of the commutator quotient-group is non-cyclic

[2]) Proceedings of the London Mathematical Society, vol. 2 (1904, p. 142.

[3]) Annals of Mathematics, vol. **6** (1904), p. 1. Cf. Zeigmondy. Monatshefte für Mathematik und Physik, vol. **7** (896), p. 207.

[4]) If G involves at least one invariant subgroup of each of the prime indices p, q, r, ..., the number of its cyclic quotient-groups of order $pqr\ldots$ is $(1+p+p^2+\ldots+p^{n_1}-1)(1+q+q^2+\ldots+q^{n_2}-1)(1+r+r^2+\ldots+r^{n_4}-1)(\ldots)\ldots$; p^{n_1}, q^{n_2}, r^{n_4}, ... being the orders of the groups generated by the operators of orders $p, q, r, \ldots$ respectively in the commutator quotient-group of G.

Non-cyclic quotient-groups can exist only when $n_2 > 0$. Hence this is a necessary condition for the existence of any quotient-group of order pq. Such quotient-groups may however exist when $n_1 = 0$. To find the total number of these groups we have to consider separately the H's which correspond to the $1 + q + q^2 + \ldots + q^{n_2-1}$ quotient-groups of order q. In such ou H the number of invariant subgroups of index p under this H is of the form $1 + p + p^2 + \ldots$ We need only consider those subgroups which have both of the following properties: 1) They are also invariant under G, 2) They do not include the commutator subgroup of G. Each of the invariant subgroups of index p in H, which satisfies these conditions gives rise to just one non-cyclic quotient-group of order pq. If the group generated by the operators of order p in the commutator quotient-group of H is of order p^{n_3}, the number of these subgroups is $1+p+p^2+\ldots+p^{n_3-1} - (1+p+p^2+\ldots+p^{n_1-1}) - kq$.

WŁ. GORCZYŃSKI,

O SPOSOBACH WYPROWADZENIA PRAWA KIRCHHOFFA.

I.

§ **1.** Pamiętna rozprawa Kirchhoffa „Ueber den Zusammenhang zwischen Emission und Absorption von Licht und Wärme" była ogłoszona jeszcze w roku 1859. Badane doświadczalnie fakty nad prążkami pochłonięcia i odkrycie widm odwróconych przedstawił Kirchhoff, jako konieczną konsekwencyę prawa ogólnego, formułującego stosunek wzajemny emisyi i zdolności absorbcyjnej danego rodzaju energii promienistej przy pewnych ograniczeniach szczegółowych.

W tem prawie nie był Kirchhoff zupełnie bez poprzedników, już bowiem de la Provostaye i Desains, a także Stokes, Stewart i A. Ångström poświęcili poprzednio wiele prac badaniom w tym kierunku, a zwłaszcza interesującemi są poglądy A. Ångströma, który już w roku 1852 próbował powiązać zaobserwowane zjawiska z zasadą rezonansu.

Dopiero jednak Kirchhoff wskazał dobitnie, że istnieje związek ilościowy między emisyą i zdolnością absorbcyjną dla każdej radiacyi monochromatycznej; ograniczając się do czysto kalorycznego przebiegu zjawisk t. j. do przypadku, gdy źródłem promieniowania jest jedynie energia cieplna i gdy odwrotnie energia promienista przechodzi przy absorbcyi wyłącznie i całkowicie w ciepło, sformułował on prawo swoje w sposób ściśle określony, który można krótko wyrazić tak:

Stosunek emisyi $(e_{\lambda,T})$ jakiegokolwiek ciała (K) do jego odpowiedniej zdolności absorbcyjnej $(a_{\lambda,T})$ jest wartością stałą dla wszystkich ciał w tej samej temperaturze