A l'aide des symboles introduits nous représentons le nombre des systèmes des nombres entiers m et n, vérifiant les inégalités:

$$0 < m \leqslant \sqrt{x}\,, \quad 0 < n \leqslant \sqrt{x}\,, \quad m^2 + n^2 > x\,,$$

c'est à dire le nombre des points du triangle MAN, sous la forme

$$\tau\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} - \sum_{i=0}^{i=k} \left[s\begin{pmatrix} a_i & , b_i \\ a_{i+1} & , b_{i+1} \end{pmatrix} - \tau\begin{pmatrix} a_i & , b_i \\ a_{i+1} & , b_{i+1} \end{pmatrix} \right]$$

ce qui donne:

$$F(x) = \pi x - 4\vartheta \sum_{i=0}^{i=k} \left[R\,(a_i^2 + b_i^2) + R\, a_{i+1}^2 + b_{i+1}^2) \right], \quad \text{où } |\vartheta| < 1\,.$$

En posant le paramètre t, qui entre dans l'éxpression $R(u)$, et qui restait arbitraire jusqu' ici, égal à $\sqrt[3]{x}$, nous trouvons:

$$F(x) = \pi x + O(\sqrt[3]{x})\,.$$

Ainsi, en appliquant la méthode de M. Voronoï nous avons démontré le théorème suivant:

„La fonction πx représente la fonction numérique $F(x)$ avec une erreur dont l'ordre ne surpasse pas celui de la fonction $\sqrt[3]{x}$".

G. A. MILLER,

GROUPS GENERATED BY TWO OPERATORS WHICH TRANSFORM EACH OTHER INTO THE SAME POWER.

(O GRUPACH UTWORZONYCH PRZEZ DWA OPERATORY PRZEKSZTAŁCAJĄCE SIĘ WZAJEMNIE NA TĘ SAMĄ POTĘGĘ).

If a group is generated by a single operator it is cyclic and its properties are comparatively elementary, but the groups generated by two operators include an infinite number of different types and their structures are so different that very little has been done towards a useful classification. When the orders of the two generators and that of their product are represented by l, m, n respectively, the group is completely determined by these numbers provided two of them are equal to 2; or one is 2, the other 3 while the third is one of the three numbers 3, 4, 5 [1]). Several other special cases have been discussed quite recently [2]). The present article is devoted to the special case noted in the heading, which presents several results of unusual interest.

Let t_1, t_2 be any two operators such that

$$t_1^{-1}\, t_2\, t_1 = t_2^{\alpha} \quad \text{and} \quad t_2^{-1}\, t_1\, t_2 = t_1^{\alpha}\,.$$

From these conditions it follows that the commutator of these operators $t_1^{-1}\, t_2^{-1}\, t_1\, t_2 = t_1^{\alpha-1} = t_2^{1-\alpha}$, is commutative with t_1, t_2 and hence the commutator subgroup of the group (G), generated by t_1, t_2 is composed of invariant operators under G. The number of the operators generated by t_1 which are invariant under t_2 divides α—1. Since the order of the quotient group of the group generated by t_1 with respect to these invariant operators

[1]) American Journal of Mathematica, vol. 24 (1902), p. 96; American Mathematical Monthly, vol 11 (1904), p 184. It should be obsewed that the conditions $a^3 = 1$, $b^3 = 1$, $ba = a^2 b^2$; Netto. Journal für reine und angewandte Mathematik, vol. 128 (1904), p. 255, are equivalent to $l = 3$, $m = 3$, $n = 2$. Several other groups discussed by Netto in this article are well know but the author fails to give any references.

[2]) Archiv der Mathematik und Physik, vol. 9 (1905), p. 6; Netto, loc. cit.

must also divide $\alpha-1$ [1]) it follows that $(\alpha-1)^2$ is a multiple of the orders of t_1, t_2. In particular, two distinct operators cannot transform each other into their squares. In fact, if two operators transform each other into their squares each of these operators is the identity. In what follows it will be assumed that t_1, t_2 are distinct and that neither of them is the identity.

When $\alpha=3$, the orders of t_1, t_2 must divide 4. When one of these operators is of order two, it must be commutative with the other and hence each one is of order two. In this case G is clearly the well known four group. In the other possible case each of the operators t_1, t_2 is of order 4. If $t_1^2=t_2^{-2}=t_2^2$ it follows that G is the quaternion group. In other words, the quaternion group is the only non-abelian group which is generated by two operators which transform each other into their third powers. From the same considerations it follows that the quaternion group is the only non-abelian group which is generated by two operators which transform each other into their inverses. These two definitions of the quaternion group are of especial interest in view of the fact of the prominent place which this group occupies in mathematical literature. The results may also be stated as follows. It two distinct operators transform each other either into their inverses or into their third powers, they generate the four group if they are commutative, but if they are not commutative they must always generate the quaternion group.

The results of the preceding paragraph can be extended directly to the general case in which $\alpha-1$ is an prime number. Let p represent this prime number. Since the orders of t_1, t_2 divide p^2 and since it is impossible that one of these orders is p when the other is p^2, it follows that G is either the abelian group of order p^2 and of type $(1, 1)$ or it is of order p^3 and conformal with the abelian group of type $(2, 1)$ whenever $p>2$ [2]). The case when $p=2$ was considered above. Although this group is well known it seems desirable to exhibit some of its properties which are especially useful in what follows.

It is clear that this G contains operators of order p^2 which transform t_2 into any arbitrany power $\beta\equiv 1 \bmod p$. Let s, be such an operator; that is, let $s_1^{-1}t_2 s_1=t_2^\beta=t_2^{\beta-1}t_2$. Hence $s^{-1}t_2^\gamma s_1=t_2^{\gamma(\beta-1)}t_2^\gamma$, or $t_2^{-\gamma}s_1 t_2^\gamma=t_2^{\gamma(1-\beta)}s_1$. That is, γ can be so chosen that $t_2^{\gamma(1-\beta)}s_1$ is any

[1]) Bulletin of the American Mathematical Society, vol. 6 (1900) p. 337.

[2]) As such a group is known to exist the existence proof is omitted, Cf. Burnside, Theory of groups of finite order. 1897, p. 75.

power $\delta\equiv 1 \bmod p$ of s_1. In other words, G contains two operators of order p^2 which transform each other into any arbitrary powers β, δ which are congruent to unity modulo p. In particular, these powers may have the values $\beta=\delta=p+1$. Moreover, G is completely defined by the fact that it is generated by two non-commutative operators which transform each other into the $p+1$ powers, as was observed above. The numbers β, δ define a non-abelian group only when each of them is equal to $p+1$. In all other cases there is more than one non-abelian group which is generated by two operators which transform each other into their β, δ powers respectively. The same remark applies clearly to abelian groups generated by two such operators.

When $\alpha-1=p^m$, where p is a prime, the orders of t_1, t_2 must divide p^{2m} and hence G is of order p^n. If the order of t_1 exceeds p^m the order of t_2 must also exceed p^m. In fact, t_1, t_2 are necessarily of the same order whenever they are non-commutative and $\alpha-1=p^m$. This interesting fact results directly from the fact that

$$t_1^{-1}t_2^{-1}t_1t_2=t_1^{\alpha-1}=t_2^{1-\alpha}\neq 1\ ^{1)}.$$

When t_1, t_2 are commutative they may have different orders. In this case these orders are arbitrany divisors of p^m.

When the order of t_1, t_2 is p^{m+1} and G is of order p^{m+2}, G is conformal with the abelian group of type $(m+1, 1)$ except when $m=1$ and $p=2$, as noted above. This G contains pairs of operators of order p^m which transform each other into any arbitrary powers β, δ which are congruent to unity modulo p^m, and it is generated by any such pair provided the operators are non-commutative. It is completely defined by the facts that it is generated by two non-commutative operators which transform each other into their p^m+1 power and that its order does not exceed p^{m+2}. The number of the distinct groups which are generated by two operators of order p^{m+1} which transform each other into their p^m+1 power is m and the orders of these groups are respectively $p^{m+2}, p^{m+3}, \ldots, p^{2m+1}$. The group of cogredient isomorphisms of each one of these groups is of order p^2, and all of these group with the single exceptioned mentioned above are conformal with abelian groups generated by two operators.

When the order of t_1, t_2 is p^{m+2}, $m>1$, the order of G may have any one of the following $m-1$ values: $p^{m+4}, p^{m+5}, \ldots, p^{2m+2}$. There is

[1]) From these relations it follows that if two non-commutative operators, whose orders are powers of the same prime, transform each other into then β, δ powers respectively, these operators must be of the same order whenever $\beta-1, \delta-1$ are divisible by the same highest power ob this prime.

only one G of each of these orders and the group of cogredient isomorphisms of all of these G_s is of order p^4. Moreover, all of these groups are conformal with abelian groups generated by two operators [1]). Continuing this process it is easy to see that t_1, t_2 may generate any one of $1+2+\ldots+m-1+m$ non-abelian groups and that all of these groups are conformal with abelian groups having just two generators, except when $p=2$ and $m=1$. All of them contain two operators of highest order which transform each other into their β, δ powers, where β, δ are arbitrany numbers satisfying the condition that they are congruent to unity modulo p^m.

The preceding considerations can readily be applied to the case when $a-1$ is not some power of a prime. From the fact that the commutators of G are invariant it follows that G is the direct product of its Sylow subgroups. Hence the number of the distinct G_s is equal to the product of the numbers of distinct Sylow subgroups. Since these Sylow subgroups are conformal with abelian groups except when one is the quaternion group, it follows that G *is always conformal with an abelian group except when its Sylow subgroup whose order is a power of 2 is the quaternion group*. In this case G is the direct product of the quaternion group and a group which is conformal with an abelian group.

When $a-1=2^{m_0} p_1^{m_1} p_2^{m_2} \ldots p_\lambda^{m_\lambda}$ these are just

$$1/2^{\lambda+1}\, m_0 m_1 m_2 \ldots m_\lambda (m_0+1)(m_1+1)(m_2+1)\ldots(m_\lambda+1)$$

groups which are generated by two operators transforming each other into their a power and which involve only non-abelian Sylow subgroups. In all of these groups the operators which transform each other into their a power are of the same order. There is one and only one such group of order $2^{m_0+2} p_1^{m_1+2} \ldots p_\lambda^{m_\lambda+2}$ and the order of each of the others is divisible by this number.

The necessary and sufficient condition that a Sylow subgroup of order p_β^{γ}, contained in a group which is generated by two operators which transform each other into their a power, be abelian is that $\gamma < m_\beta + 1$ the order of each of its two generators be less than $p_\beta^{m_\beta+1}$. The number of such groups which have one or more than one abelian Sylow subgroups can readily be obtained from the number of those in which the Sylow subgroups are non-abelian, since an abelian Sylow subgroup has not to satisfy any condition except that it has only two generators and that the order of each of these divides p^m.

[1]) Bulletin of the American Mathematical Society, vol. 7 (1901), p. 351.

A. PRZEBORSKI

O CAŁKACH NIEANALITYCZNYCH
równań różniczkowych liniowych o pochodnych cząstkowych rzędu pierwszego.

Jednem z najważniejszych i najtrudniejszych pytań Analizy jest pytanie o charakterze całek równań różniczkowych. Znane jest twierdzenie Cauchy'ego, że wszelkie równanie różniczkowe

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \ldots \frac{d^{n-1} y}{dx^{n-1}}\right),$$

gdzie F jest funkcyą analityczną argumentów w pewnym obszarze (ε), ma w tym obszarze *tylko* całki analityczne. Także wszelki układ równań różniczkowych jednoczesnych

$$\frac{dy_1}{dx} = \omega_1(x, y_1, y_2, \ldots, y_n), \ \frac{dy_2}{dx} = \omega_2(x, y_1, y_2 \ldots y_n) \ldots \frac{dy_n}{dx} = \omega_n(x, y_1, y_2, \ldots, y_n),$$

gdzie $\omega_1, \omega_2, \ldots \omega_n$ są funkcye analityczne w pewnym obszarze (ε), ma zawsze w tym obszarze *tylko* całki analityczne. Inaczej rzecz się ma z równaniami o pochodnych cząstkowych rzędu pierwszego.

W artykule niniejszym zastanawiam się wyłącznie nad jednem równaniem różniczkowem liniowem o pochodnych cząstkowych rzędu pierwszego:

$$(1) \quad \begin{aligned} & A_0(x_0, x_1, \ldots, x_n) \frac{\partial x_n}{\partial x_0} + A_1(x_0, x_1, \ldots, x_n) \frac{\partial x_n}{\partial x_1} + \ldots \\ & + A_{n-1}(x_0, x_1, \ldots, x_n) \frac{\partial x_n}{\partial x_{n-1}} = A_n(x_0, x_1, \ldots, x_n), \end{aligned}$$

gdzie $A_0, A_1, A_2, \ldots, A_n$ są funkcye analityczne w pewnym obszarze (ε), przyczem zakładamy, że dla punktów tego obszaru funkcya $A_0(x_0, x_1, \ldots, x_n)$ nie jest równa zeru.