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Dedicated to Professor J. Klamka
on the occasion of his 70th birthday

Controllability, reachability and minimum energy control of fractional discrete-time linear systems with multiple delays in state

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Abstract. In the paper the problems of controllability, reachability and minimum energy control of a fractional discrete-time linear system with delays in state are addressed. A general form of solution of the state equation of the system is given and necessary and sufficient conditions for controllability, reachability and minimum energy control are established. The problems are considered for systems with unbounded and bounded inputs. The considerations are illustrated by numerical examples. Influence of a value of the fractional order on an optimal value of the performance index of the minimum energy control is examined on an example.

Key words: linear systems, fractional, discrete-time, time-delay, controllability, reachability, minimum energy control.

1. Introduction

Controllability and observability are two major concepts of the modern control theory introduced by R. Kalman [1]. These concepts play a crucial role in many control problems, such as stabilization by feedback, realization or optimal control.

The problems of controllability and reachability of dynamic systems have been considered in book [2] and in many recent papers, see for example [3-4] for standard systems and [5–13] for fractional order systems. An overview of the recent developments in theory of controllability problems for a wide class of dynamical systems has been given in [14].

Dynamical systems described by fractional order differential or difference equations have been investigated in several areas such as viscoelasticity, electrochemistry, diffusion processes, automatic control, etc. (see [15–19], for example, and references therein).

The problem of stability of linear fractional order discrete-time systems has attracted considerable attention recently, see [20–24].

The problem of minimum energy control for standard systems has been introduced by J. Klamka [25–27]. This problem for standard systems has been investigated in [2, 28–30] and in [31–34] for fractional order systems.

The aim of the paper is to give the general form of solution of the state equation of the fractional discrete-time linear system with delays in state, necessary and sufficient conditions for controllability and reachability and a solution of the minimum energy control problem. We also consider the above problems for systems with bounded inputs. Recently, the problem of minimum energy control with bounded inputs has been formulated and solved in [32, 34]. Influence of a value of the fractional order on an optimal value of the performance index of the minimum energy control we examine on an example.

The paper is organized as follows. Preliminaries and formulation of the problem are given in Sec. 2. Necessary and sufficient conditions for controllability and reachability are established in Sec. 3. Minimum energy control problem of the system is solved in Sec. 4 and concluding remarks are given in Sec. 5.

2. Preliminaries and problem formulation

Consider the discrete-time linear system with delays described by the state equation

$$\Delta^\alpha x_{i+1} = A_0 x_i + \sum_{k=1}^{h} A_k x_{i-k} + B u_i, \tag{1}$$

with the initial conditions

$$x_{-k} \in \Re^n, \qquad k = 0, 1, ..., h, \tag{2}$$

where h is a positive number, $x_i \in \Re^n$, $u_i \in \Re^m$ are the state and input vectors, respectively, $A_k \in \Re^{n\times n}$ $(k = 0, 1, ..., h)$, $B \in \Re^{n\times m}$,

$$\Delta^\alpha x_i = \sum_{k=0}^{i} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} x_{i-k} \tag{3}$$

is the fractional difference of order $\alpha \in \Re$ of the discrete-time function x_i and

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\alpha!}{k!(\alpha-k)!}. \tag{4}$$

Substitution of (3) for $i+1$ in (1) gives the equation

$$x_{i+1} = F_0 x_i + \sum_{k=1}^{h} A_k x_{i-k} + \sum_{k=1}^{i} c_k(\alpha) x_{i-k} + B u_i, \tag{5}$$

where

$$F_0 = A_0 + I_n \alpha, \tag{6}$$

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I_n is the $n \times n$ identity matrix and

$$c_k(\alpha) = (-1)^k \begin{pmatrix} \alpha \\ k+1 \end{pmatrix}, \quad k = 1, 2, ... \tag{7}$$

The coefficients (7) can be computed by the following algorithm [21]

$$c_{k+1}(\alpha) = c_k(\alpha)\frac{k+1-\alpha}{k+2}, \qquad k = 1, 2, ... \tag{8}$$

with $c_1(\alpha) = 0.5\alpha(1-\alpha)$.

The equation (5) describes the discrete-time linear system with increasing number of delays.

Definition 1. The fractional system with delays (1) is called controllable in N steps if for every given final state $x_f \in \Re^n$ and every given initial conditions (2) there exists a sequence of inputs $u_i \in \Re^m$, $i = 0, 1, ..., N-1$, which transfers the system (1) from initial conditions (2) to the state x_f.

Definition 2. The fractional system with delays (1) is called reachable in N steps if for every given final state $x_f \in \Re^n$ there exists a sequence of inputs $u_i \in \Re^m$, $i = 0, 1, ..., N-1$, which transfers the system (1) from zero initial conditions (i.e. $x_{-k} = 0$ for $k = 0, 1, ..., h$) to the state x_f.

The problems of controllability and reachability of dynamic systems, standard and fractional, have been recently considered in many papers (see [3–13], for example). A survey of recent results in theory of controllability of dynamical systems is given in [14].

In this paper we give the necessary and sufficient conditions for controllability and reachability of the fractional system (1) and methods for computing the control sequences which transfer the system (1) from prescribed initial conditions (2) (zero and non-zero) to the given final state. We consider the problem of computing the control sequences with unbounded and bounded amplitudes. The control sequences with bounded amplitudes satisfy the assumption

$$u_i \in U \subset \Re^m \qquad \text{for} \quad i = 0, 1, ..., N-1 \tag{9}$$

with the set U defined by

$$U = \{u_i \in \Re^m : |u_{ij}| \le M, \ \forall j = 1, 2, ..., m\}, \tag{10}$$

where M is a positive number.

Moreover, we give the solution of the minimum energy control problem (with unbounded and bounded inputs) formulated as follows. Find a control sequence $u_i \in \Re^m$, $i = 0, 1, ..., N-1$, which transfers the system (1) from zero initial conditions to the desired final state $x_f \in \Re^n$ and minimises the performance index

$$I(u) = \sum_{i=0}^{N-1} u_i^{\mathrm{T}} Q u_i, \tag{11}$$

where Q is a symmetric positive definite weighting matrix.

The minimum energy control problem for standard systems has been introduced by Professor J. Klamka [25–27].

Recently, the problem of minimum energy control with bounded inputs has been formulated and solved in [32] for fractional positive continuous-time linear systems and in [34] for positive fractional discrete-time linear systems without delays.

The paper is organized as follows. In Sec. 3 the general form of solution of the fractional system (1) with initial conditions (2) is derived and necessary and sufficient conditions for controllability and reachability are established. The solution of the minimum energy control problem is given in Sec. 4. Moreover, influence of the fractional order α on optimal value of the performance index (11) is analysed on the base of example. Concluding remarks are given in Sec. 5.

3. Controllability and reachability

Taking the Z-transform to both sides of (5) with initial conditions (2) leads to

$$\begin{aligned} zX(z) - zx_0 = F_0 X(z) &+ \sum_{k=1}^{h} A_k z^{-k} \left[X(z) + \sum_{r=-k}^{-1} x_r z^{-r} \right] \\ &+ \sum_{k=1}^{i} c_k z^{-k} X(z) + BU(z), \end{aligned} \tag{12}$$

where $X(z) = Z\{x_i\}$, $U(z) = Z\{u_i\}$.

The above equation can be written in the form

$$\Delta(z) X(z) = zx_0 + \sum_{k=1}^{h} A_k z^{-k} \sum_{r=-k}^{-1} x_r z^{-r} + BU(z), \tag{13}$$

where

$$\Delta(z) = zI_n - F_0 - \sum_{k=1}^{h} A_k z^{-k} - \sum_{k=1}^{i} I_n c_k(\alpha) z^{-k} \tag{14}$$

is the characteristic matrix.

Solving Eq. (13) for $X(z)$ we obtain

$$\begin{aligned} X(z) &= \Delta^{-1}(z) z x_0 + \Delta^{-1}(z) \sum_{k=1}^{h} A_k z^{-k} \sum_{r=-k}^{-1} x_r z^{-r} \\ &+ \Delta^{-1}(z) BU(z) = [\Delta^{-1}(z) z] x_0 \\ &+ [\Delta^{-1}(z) z] \sum_{k=1}^{h} A_k \sum_{r=0}^{k-1} x_{r-k} z^{-r-1} \\ &+ [\Delta^{-1}(z) z] B z^{-1} U(z). \end{aligned} \tag{15}$$

Taking the inverse Z-transform to (15) gives the solution of Eqs. (5) and (1) in the form

$$\begin{aligned} x_i = \Phi_i x_0 &+ \sum_{k=1}^{h} \sum_{r=0}^{k-1} \Phi_{i-r-1} A_k x_{r-k} \\ &+ \sum_{k=0}^{i-1} \Phi_{i-1-k} B u_k, \end{aligned} \tag{16}$$

where

$$\Phi_i = Z^{-1}\{z \Delta^{-1}(z)\} \tag{17}$$

is the state-transition matrix for Eqs. (5) and (1).

From (17) and (14) it follows that the state-transition matrix Φ_i satisfies the equation

$$\Phi_{i+1} = F_0\Phi_i + \sum_{k=1}^{h} A_k\Phi_{i-k} + \sum_{k=1}^{i} c_k(\alpha)\Phi_{i-k} \tag{18}$$

with the initial conditions

$$\Phi_0 = I_n, \qquad \Phi_i = 0 \quad \text{for} \quad i < 0. \tag{19}$$

Lemma 1. The state-transition matrix Φ_i also satisfies the equation

$$\Phi_{i+1} = \Phi_i F_0 + \sum_{k=1}^{h} \Phi_{i-k}A_k + \sum_{k=1}^{i} c_k(\alpha)\Phi_{i-k} \tag{20}$$

with initial conditions (19).

Proof. Consider the equation

$$y_{i+1} = F_0^{\mathrm{T}} y_i + \sum_{k=1}^{h} A_k^T y_{i-k} + \sum_{k=1}^{i} c_k(\alpha) y_{i-k}, \tag{21}$$

where $y_i \in \Re^n$.

The state-transition matrix Ψ_i for the Eq. (21) can be computed from the formula

$$\Psi_i = Z^{-1}\{z\Delta_1^{-1}(z)\}, \tag{22}$$

where

$$\Delta_1(z) = zI_n - F_0^{\mathrm{T}} - \sum_{k=1}^{h} A_k^{\mathrm{T}} z^{-k} - \sum_{k=1}^{i} I_n c_k(\alpha) z^{-k}. \tag{23}$$

Hence, the state-transition matrix (22) satisfies the equation

$$\Psi_{i+1} = F_0^{\mathrm{T}}\Psi_i + \sum_{k=1}^{h} A_k^{\mathrm{T}}\Psi_{i-k} + \sum_{k=1}^{i} c_k(\alpha)\Psi_{i-k} \tag{24}$$

with the initial conditions $\Psi_0 = I_n$, $\Psi_i = 0$ for $i < 0$.

From (14) and (23) it follows that $\Delta(z) = \Delta_1^{\mathrm{T}}(z)$. Hence

$$\begin{aligned}\Phi_i &= Z^{-1}\{z\Delta^{-1}(z)\} = Z^{-1}\{(\Delta_1^{-1}(z))^{\mathrm{T}} z\} \\ &= [Z^{-1}\{\Delta_1^{-1}(z)z\}]^{\mathrm{T}} = \Psi_i^{\mathrm{T}}\end{aligned} \tag{25}$$

and from (24) one obtains

$$(\Phi_{i+1})^{\mathrm{T}} = F_0^{\mathrm{T}}\Phi_i^{\mathrm{T}} + \sum_{k=1}^{h} A_k^{\mathrm{T}}(\Phi_{i-k})^{\mathrm{T}} + \sum_{k=1}^{i} c_k(\alpha)(\Phi_{i-k})^{\mathrm{T}} \tag{26}$$

which shows that Φ_i satisfies the Eq. (20). This completes the proof.

From the above considerations we have the following theorem.

Theorem 1. The solution of the fractional system (1) with initial conditions (2) has the form (16), where the state-transition matrix can be computed from the recursive formula (18) or (20).

The form of solution of the state equation of the fractional discrete-time linear systems with delays in state and control has been given in [17]. This form has been obtained in a different way than the solution (16). Moreover, the solution (16) has different form than the solution proposed in [17] in the case of delays in state variables only.

From (16) it follows that the solution of the Eq. (5) (and (1)) for $i = N$ has the form

$$x_N = S_N + R_N u^N, \tag{27}$$

where

$$R_N = [\; B \quad \Phi_1 B \quad \cdots \quad \Phi_{N-1}B \;], \tag{28}$$

$$S_N = \Phi_N x_0 + \sum_{k=1}^{h}\sum_{r=0}^{k-1} \Phi_{N-r-1}A_k x_{r-k}, \tag{29}$$

$$u^N = \begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_0 \end{bmatrix} \in \Re^{Nm}. \tag{30}$$

Theorem 2. The system (1) is controllable in N steps if and only if there exists integer number N such that rank of the controllability matrix (28) has full row rank, i.e. rank $R_N = n$.

Proof. Equation (27) can be written in the form

$$P_N = R_N u^N, \tag{31}$$

where

$$P_N = x_N - S_N. \tag{32}$$

The left hand side of (31) is known. Therefore, for every $x_f = x_N \in \Re^n$ the control u^N can be computed form (31) if and only if rank $R_N = n$. This completes the proof.

Theorem 3. If there exists N such that rank $R_N = n$ then the control sequence $u_i \in \Re^m$, $i = 0, 1, ..., N-1$, which transfers the system (1) from initial conditions (2) to the desired final state $x_f = x_N \in \Re^n$, can be computed from the formula

$$u^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} P_N, \tag{33}$$

where P_N is defined by (32).

Proof. If rank $R_N = n$ then $\det(R_N R_N^{\mathrm{T}}) \neq 0$ and matrix $R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1}$ is well definite. If (33) holds then

$$P_N = R_N u^N = R_N R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} P_N = P_N. \tag{34}$$

This completes the proof.

The control u^N can be computed also from the formula [35]

$$u^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} P_N + (I_n - R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N)k, \tag{35}$$

where $k \in \Re^{mN}$ is an arbitrary vector.

To show that control (35) transfers the system (1) from initial conditions (2) to the final state x_f, we substitute (35) in (31) and then we obtain

$$\begin{aligned}P_N &= R_N u^N = R_N R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} P_N \\ &+ (R_N - R_N R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N)k = P_N.\end{aligned} \tag{36}$$

It is easy to see that if $k = 0$ then from (35) one obtains (33).

If the initial conditions (2) are equal to zero then from Theorems 2 and 3 we obtain the following conditions of reachability of the fractional system (1).

Theorem 4. The system (1) is reachable in N steps if and only if there exists integer number N such that rank of the controllability (reachability) matrix (28) is equal to n. If this holds, then control u^N which transfers the system (1) from zero initial conditions to the desired final state $x_f \in \Re^n$, can be computed from

$$u^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f \tag{37}$$

or

$$u^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f + (I_n - R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N)K, \tag{38}$$

where $k \in \Re^{mN}$ is an arbitrary vector.

Now we consider the controllability (reachability) problem with bounded inputs, i.e. with the assumption (9).

The energy of control is closely related with the amplitudes of control signals. Energy of practical systems is limited, therefore the amplitudes of control signals can be smaller for large number N of steps in comparison with small number of steps. Hence, if controls do not satisfy the condition (9), then increase N by one and repeat computation of control sequence (from (33) or (37)) for $N+1$. After some limited number of such computations we obtain the input sequence satisfying the condition (9).

The above procedure for computation of minimum energy control with bounded inputs has been proposed in [34] for positive fractional discrete-time linear systems without delays.

Example 1. Consider the system (1) with $\alpha = 0.5$, $h = 2$ and the matrices

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & -0.7 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & -0.8 \\ 0 & 0 & 0 \end{bmatrix}, \tag{39}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Check controllability of the system and find control that transfers the system from initial conditions

$$x_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

$$x_{-1} = \begin{bmatrix} -2 \\ 0.5 \\ 0.7 \end{bmatrix}, \quad x_{-2} = \begin{bmatrix} -2.5 \\ 1 \\ 0 \end{bmatrix}, \tag{40}$$

to the final state $x_f = [\ 1 \quad 1 \quad 1\ ]^{\mathrm{T}}$. Moreover, find control that transfers the system from initial conditions (40) to desired final state and satisfies the assumption

$$u_i \in U = \{u_i \in \Re^2 : \ |u_{ij}| \leq 1.1, \ \forall j = 1,2\} \tag{41}$$

for all $i = 0, 1, ..., N-1$.

It is easy to check that the controllability (reachability) matrix (28) has full row rank for $N \geq 4$. By Theorem 2, the system is controllable in four steps.

The matrix (28) for $N = 4$ has the form

$$R_N = \begin{bmatrix} B & \Phi_1 B & \Phi_2 B & \Phi_3 B \end{bmatrix}, \tag{42}$$

where matrices Φ_i $(i = 1,2,3)$ are computed from the recursive formula (18) or (20).

Computing from (33) the control sequence which transfers the system (1), (39) from initial conditions (40) to the desired final state one obtains

$$u_0 = \begin{bmatrix} -2.0662 \\ 1.1106 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 0.1954 \\ 0.8383 \end{bmatrix},$$
$$u_2 = \begin{bmatrix} -0.2056 \\ 0.6907 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0.4113 \\ 0.6279 \end{bmatrix}. \tag{43}$$

The control sequence (43) does not satisfy the condition (41). Therefore, we repeat computations for $N = 5, 6, ...$ and we obtain that control sequence satisfies the condition (41) for $N = 5$. This sequence has the form

$$u_0 = \begin{bmatrix} 0.5924 \\ 1.0646 \end{bmatrix}, \quad u_1 = \begin{bmatrix} -0.8183 \\ 0.8080 \end{bmatrix},$$
$$u_2 = \begin{bmatrix} 0.1632 \\ 0.6099 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -0.1718 \\ 0.5026 \end{bmatrix}, \tag{44}$$
$$u_4 = \begin{bmatrix} 0.3435 \\ 0.4569 \end{bmatrix}.$$

Computing from (5) the state vector x_i for $i = 1, 2, \cdots, 5$ with initial conditions (40) and the above control sequence we obtain $x_5 = x_f = [\ 1 \quad 1 \quad 1\ ]^{\mathrm{T}}$. This means that the control sequence (44) is determined correctly.

If we consider the performance index (11) and assume $Q = I_2$ then we obtain that value of this index is equal to 7.3260 for control sequence (43) and 3.8142 for control sequence (44). This means that smaller value of the performance index is obtained for larger number of steps.

4. Minimum energy control

If the initial conditions (2) are equal to zero, then the solution (16) takes the form

$$x_i = \sum_{k=0}^{i-1} \Phi_{i-1-k} B u_k, \tag{45}$$

where the state-transition matrix Φ_i is computed form the recursive formula (18) or (20).

Knowing form (45) of solution of the fractional system (1), we solve the minimum energy control problem applying the classical approach, exactly the same as in the case of standard (i.e. non-fractional) systems, see [2, 17, 26, 30], for example. This approach has been applied in [34] for fractional positive discrete-time linear systems without delays.

Define the matrix

$$W = R_N \widetilde{Q}_N R_N^{\mathrm{T}} \in \Re^{n\times n}, \tag{46}$$

where R_N is the reachability (controllability) matrix of the form (28) and

$$\widetilde{Q}_N = \mathrm{diag}[Q^{-1}, ..., Q^{-1}] \in \Re^{Nm\times Nm}, \tag{47}$$

where Q is a symmetric positive definite weighting matrix of the performance index (11).

The matrix W defined by (46) is non-singular if and only if the matrix R_N has full row rank.

Theorem 5. If the system (1) is reachable in N steps then the sequence of inputs $\widehat{u}_0, \widehat{u}_1, ..., \widehat{u}_{N-1}$ defined by

$$\widehat{u}^N = \begin{bmatrix} \widehat{u}_{N-1} \\ \widehat{u}_{N-2} \\ \vdots \\ \widehat{u}_0 \end{bmatrix} = \widetilde{Q}_N R_N^{\mathrm{T}} W^{-1} x_f \tag{48}$$

transfers the system (1) from zero initial conditions to the final state $x_f \in \Re^n$ and minimises the performance index (11). Moreover, the minimal (optimal) value of this index is given by

$$I(\widehat{u}) = x_f^{\mathrm{T}} W^{-1} x_f. \tag{49}$$

Proof. The proof is similar to the proof of classical minimum energy problem (see, for example [2, 17, 26, 30, 34]) and it is omitted here.

Optimal control which minimises the performance index (11) depends on the weighting matrix Q. From comparison (37) and (48), (46) it follows that control sequence (37) minimises the performance index (11) with $Q = I_m$. This means that u^N computed from (37) is a minimum energy control with the performance index (11) with $Q = I_m$.

Similarly as in [30], it can be showed that if the weighting matrix has the form $Q = qI_m$ with $q > 0$ then the optimal value (49) of the performance index (11) can be computed from the formula

$$I(\widehat{u}) = q x_f^{\mathrm{T}} [R_N R_N^{\mathrm{T}}]^{-1} x_f. \tag{50}$$

If the optimal control computed form (48) does not satisfy the condition (9) then (similarly as in the case of control sequences (33) and (37)) increase N by one and repeat computation of control sequence (48) for $N+1$. After limited number of such computations we obtain the input sequence (48) satisfying the condition (9).

Example 2. Consider the following problems for the system (1) with $\alpha = 0.5$, $h = 2$ and the matrices (39):

Problem 1. Find control that transfers the system from zero initial conditions to the final state $x_f = [\ 1 \quad 1 \quad 1\ ]^{\mathrm{T}}$.

Problem 2. Solve Problem 1 for optimal control which minimises the performance index (11) with

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}. \tag{51}$$

Problem 3. Solve Problem 2 with bounded inputs, i.e. with the assumption

$$u_i \in U = \{u_i \in \Re^2 : \ |u_{ij}| \le 1, \ \forall j = 1, 2\} \tag{52}$$

for $i = 0, 1, ..., N-1$.

Problem 4. Solve Problem 2 and find optimal value (49) of the performance index (11) for several fixed values $\alpha \in (0,\ 2]$.

Solution of Problem 1. From Example 1 it follows that controllability (reachability) matrix (28) has full row rank for $N \ge 4$. This matrix for $N = 4$ has the form (42).

Computing from (37) the control sequence which transfers the system (1), (39) from zero initial conditions to the desired final state one obtains

$$u_0 = \begin{bmatrix} -2 \\ 0.2484 \end{bmatrix}, \qquad u_1 = \begin{bmatrix} 0.1368 \\ 0.1875 \end{bmatrix},$$
$$u_2 = \begin{bmatrix} -0.1440 \\ 0.1545 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 0.2880 \\ 0.1405 \end{bmatrix}. \tag{53}$$

Solution of Problem 2. Computing the optimal control sequence from (48) one obtains

$$\widehat{u}_0 = \begin{bmatrix} -2 \\ 0.5452 \end{bmatrix}, \quad \widehat{u}_1 = \begin{bmatrix} 0.1224 \\ 0.0036 \end{bmatrix},$$
$$\widehat{u}_2 = \begin{bmatrix} -0.1655 \\ 0.0695 \end{bmatrix}, \quad \widehat{u}_3 = \begin{bmatrix} 0.2841 \\ -0.0405 \end{bmatrix}. \tag{54}$$

According to (49), minimal value of the performance index (11) is $I(\widehat{u}) = 7.234$.

Control sequence which also transfers the system from zero initial conditions to the final state $x_f = [\ 1 \quad 1 \quad 1\ ]^{\mathrm{T}}$ has the form (53). Value of the performance index (11) for this control is equal to $I(u) = 7.9009 > I(\widehat{u}) = 7.234$.

Solution of Problem 3. The control sequence (54) does not satisfy the condition (52). Therefore, we repeat computations for $N = 5, 6, ...$ and we obtain that optimal control sequence satisfies the condition (52) for $N = 7$. This sequence has the form

$$\widehat{u}_0 = \begin{bmatrix} 0.3592 \\ 0.0234 \end{bmatrix}, \quad \widehat{u}_1 = \begin{bmatrix} -0.6660 \\ 0.2521 \end{bmatrix},$$

$$\widehat{u}_2 = \begin{bmatrix} 0.6037 \\ -0.086 \end{bmatrix}, \quad \widehat{u}_3 = \begin{bmatrix} -0.9192 \\ 0.2791 \end{bmatrix},$$

$$\widehat{u}_4 = \begin{bmatrix} 0.1207 \\ 0.0070 \end{bmatrix}, \quad \widehat{u}_5 = \begin{bmatrix} -0.1670 \\ 0.0724 \end{bmatrix}, \tag{55}$$

$$\widehat{u}_6 = \begin{bmatrix} 0.2830 \\ -0.0429 \end{bmatrix}.$$

Minimal (optimal) value (49) of the performance index (11) for control sequence (55) is equal to $I(\widehat{u}) = 3.4525$.

Solution of Problem 4. Taking account fixed values $\alpha \in [0.01,\ 2]$ of the fractional order α and solving the minimum energy control problem in $N = 4$ steps, we obtain the plot of optimal values (49) of the performance index shown in Fig. 1. From this figure it follows that optimal values are largest for $\alpha \approx 0$ and $\alpha \approx 1$. Moreover, the plot of optimal values has minima for $\alpha \approx 0.4$ and $\alpha \approx 1.7$.

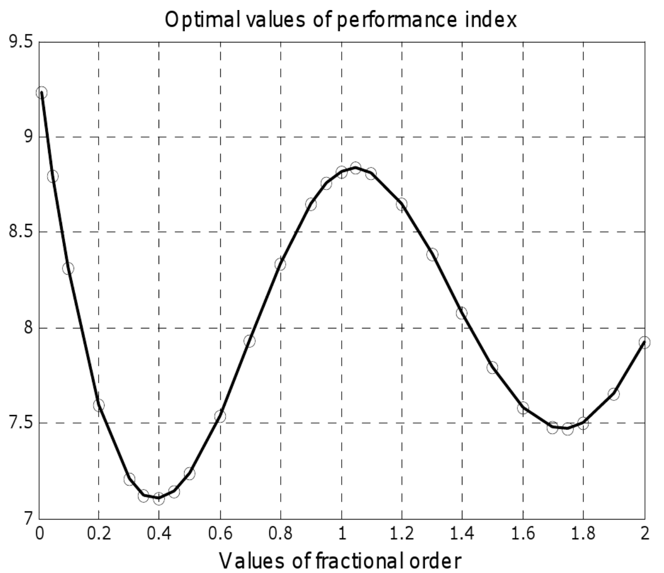

Fig. 1. Optimal value of the performance index versus fractional order $\alpha \in [0.01,\ 2]$

5. Concluding remarks

The problems of controllability, reachability and minimum energy control of the fractional system (1) with delays have been addressed. The general form of solution of the state equation (1) has been given in Theorem 1. Necessary and sufficient conditions for controllability (Theorems 2 and 3), reachability (Theorem 4) and minimum energy control (Theorem 5) have been established and illustrated by examples. The above problems in the case of bounded inputs also have been considered and a method for computation of bounded control sequences has been proposed.

Moreover, in Example 2 it has been shown that if fractional order takes values from interval (0, 2] (including orders 1 and 2) then optimal values (49) of the performance index (11) are largest for α near to the values of zero and one. An open problem is answer to the question: does this hold in the general case?

The considerations can be extended to fractional discrete-time linear systems with delays in state and control and for positive systems with delays.

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