Tylko pierwszy wyraz naszego wzoru ma wartość teoretyczną, następne stanowią identyczność ze wzorem doświadczalnym i odnaleść w nich można tylko to, co się w spółczynnikach doświadczalnych już mieściło.

To też zarówno nasz wzór, jak i wzór Helmholtza służyć może jedynie do stwierdzenia przyjętego przez nas założenia, o ile się zna zmianę siły elektrycznej stosu z temperaturą, nie zaś do wyznaczenia tej siły przy pewnej temperaturze z góry dla każdego stosu.

G. A. MILLER.

Groups generated by two operators satisfying the condition $s_1 s_2 = s_2^{-1} s_1^n$.

(GRUPY, UTWORZONE PRZEZ DWA OPERATORY, SPEŁNIAJĄCE WARUNEK $s_1 s_2 = s_2^{-1} s_1^n$).

§ 1. General considerations.

Since $(s_1 s_2)^2 = s_1 s_2 s_1 s_2 = s_1 s_2 . s_2^{-1} s_1^n = s_1^{n+1} = (s_2 s_1)^2$, and $s_2^{-1} s_1^{n+1} s_2 = s_2^{-1} s_1^n . s_1 s_2 = (s_1 s_2)^2 = s_1^{n+1}$, it results that s_1^{n+1} is invariant under the group G generated by s_1, s_2 Hence G contains at least one invariant operator besides the identity whenever $n \neq -1$, and G is generated by the three operators $s_1 s_2$, $s_2 s_1$, s_1,—the first two of which have a common square and this square is the $n+1$ power of the third. If the condition $s_1 s_2 = s_2^{-1} s_1^n$ is transformed by s_2^{-1}, and the letters of the resulting equation are interchanged, it becomes $s_1 s_2 = s_2^n s_1^{-1}$. Hence these two conditions are equivalent. As $s_1 s_2 = s_2^{-1} s_1^n = s_2^{-1} s_1^{n-1} . s_1 s_2 . s_2^{-1}$ it is evident that $s_2^{-1} s_1^{n-1}$ is of the same order as s_2.

It seems desirable to state explicitly two elementary theorems which are of frequent use in the developments which follow, viz.

If two commutative operators satisfy the condition $s_1^\alpha = s_2^\beta$ they generate the direct product of two cyclic groups whose orders are respectively the lowest common multiple of the orders of s_1, s_2, and a divisor of the highest common factor of the four numbers α, β, and the orders of s_1, s_2. The latter cyclic group may be the identity, and it must be the identity whenever unity is the highest common factor of the orders of s_1, s_2, and α, β.

If s^m is of order n, and if k is the smallest number such that m/k is prime to n, then the order of s is both a multiple of kn and a divisor of mn; moreover, s may be so selected that its order is an arbitrary number satisfying these conditions. In particular, when n is divisible by all the prime factors of m the order of s is exactly mn.

§ 2. Several values of n close to zero.

When $n=-1$ the order of s_1s_2 is 2 but no further restrictions are imposed upon s_1, s_2. Since every possible symmetric group is included among the groups which may be generated by two operators whose product is of order 2[1], it is clear that the condition $s_1s_2=s_2^{-1}s_1^{-1}$ is too general to lead to a simple category of groups. On the contrary, a very simple category results when $n=-2$. In this case we have the condition

$$s_1s_2=s_2^{-1}s_1^{-2}.$$

Since s_1^{n+1} is invariant under G it results that G is abelian. The condition imposed upon s_1, s_2 may therefore be replaced by $s_1s_2=s_1^{-2}s_2^{-1}$, and hence $s_1^3=s_2^{-2}$; that is, G is generated by two commutative operators s_1, s_2^{-1} such that the square of one is equal to the cube of the other, and therefore G is cyclic in accord with a theorem of § 1. These results may be stated as follows: *If two operators s_1, s_2 satisfy the condition $s_1s_2=s_2^{-1}s_1^{-2}$ they generate the cyclic group whose order is the lowest common multiple of their orders.*

When $n=-3$ the given condition becomes

$$s_1s_2=s_2^{-1}\,s_1^{-3}.$$

From the fact that $s_1s_2=s_2^{-1}s_1^{-4}.s_1s_2.s_2^{-1}$ and that s_1^{-2} is invariant under G, it results that the order of s_1^4 divides that of s_2; i. e. the order of s_1 divides the order of s_2 multiplied by 4.

[1] Bulletin of the American Mathematical Society, vol. 7 (1901), p. 426

Moreover, the group $\{s_1s_2, s_2s_1\}$ generated by s_1s_2, s_2s_1 is invariant under G since $s_1^{-1}.s_2s_1.s_1=s_1^{-2}.s_1s_2.s_1^{2}=s_1s_2$, and it involves s_1^2 since $(s_2s_1)^2=s_1^{-2}$. From this it results that the order of G divides the order of $\{s_1s_2, s_2s_1\}$ multipliedby 2, and that G is always solvable. It is known[1] that

$$s_1s_2s_1^{-1}s_2^{-1}=s_2^{-1}s_1^{-4}s_2^{-1}=s_1^{-4}s_2^{-2}$$

is transformed into its inverse both by s_1s_2 and by s_2s_1. From this equation it may also be observed that the order of $s_1s_2s_1^{-1}s_2^{-1}$ divides the lowest common multiple of the orders s_1^4, s_2^2, and hence this order divides that of s_2.

In case s_1, s_2 are commutative the given condition reduces to $s_1^4s_2^2=1$ and hence G is either cyclic or the direct product of a cyclic group and the group of order 2, according to a theorem of § 1. Moreover, every cyclic group and every such direct product may be generated by two commutative operators satisfying the condition $s_1s_2=s_2^{-1}s_1^{-3}$. In fact, if G is generated by the operator t, we may let $s_1=t$, $s_2=t^{-2}$. On the other hand, if G is generated by the two independent operators t_1, t_2 (where t_2 is of order 2) we may assume $s_1=t_1t_2$, $s_2=t_1^{-2}t_2$. When s_1, s_2 are non-commutative, the order of s_1 is even, since it transforms s_1s_2, s_2s_1 into each other. The order of s_1s_2 must also be even since its square is s_1^{-2} and it is not cummutative with s_1. Hence s_1 and s_1s_2 are of the same order whenever s_1, s_2 are non commutative

If s_1, s_2 are of orders 2 and 8 respectively, and satisfy the condition $s_2s_1s_2=s_1^5$, it is clear that they also satisfy the condition $s_1s_2=s_2^{-1}s_1^{-3}$ and that they generate a group of order 16. This is a special case of the general theorem that the order of G divides four times the square of the order of s_2 whenever this order is even. To prove this theorem it is only necessary to observe that the order of G divides twice the product of the orders of $s_1s_2s_1^{-1}s_2^{-1}$ and s_1s_2, and that these orders divide the order of s_2 and twice this order respectively when the order of s_1 divides twice the order of s_2; when this condition is not satisfied the order of s_1 must divide the order of s_2 multiplied by 4, and the two cyclic groups generated by $s_1s_2s_1^{-1}s_2^{-1}$ and s_1s_2 respectively have at least two common operators. Hence the order of G divides four times the square of that of s_2 in each case and the example given above proves that the order of G may actually be equal to the square of the order of s_2 multiplied by 4.

When s_2 is of odd order $\{s_1s_2, s_2s_1\}$ is identical with G since it involves all the operators of odd order contained in G and hence includes s_2. From this it results that *the order of G divides four times the square of the order of s_2* regardless of the form of the order of this operator. It has been observed that $\{s_1s_2, s_2s_1\}$ is generated

[1] Archiv der Mathematik und Physik, vol 9 (1905), p. 6.

by two operators $s_1s_2s_1^{-1}s_2^{-1}$ and s_1s_2 such such that the second transforms the first into its inverse. It is of interest to observe that every group that can be generated in this way may also be generated by two operators satisfying the condition under consideration. In fact, if $t_2^{-1}t_1t_2=t_1^{-1}$ and

$$s_1=t_2\,, \qquad s_2=t_1t_2^{-2}$$

it is easy to verify that $s_2^{-1}s_1^{-3}=t_2^2t_1^{-1}t_2^{-3}=t_2t_1t_2^{-2}=s_1s_2$. That is, the group generated by t_1, t_2 is identical with one generated by two operators satisfying the condition under consideration.

When $n=1$ there results the equation $s_1s_2=s_2^{-1}s_1$ or $s_1^{-1}s_2s_1=s_2^{-1}$. From the preceding paragraph it results that this category of elementary groups is included among those which satisfy the given condition when $n=-3$. Some of the preceding results are expressed in the following theorem: *If $s_1s_2=s_2^{-1}s_1^{-3}$, the group generated by s_1, s_2 belongs to one of the following two categories of solvable groups: 1) Those which are generated by two operators such that one of them transforms the other into its inverse, 2) Those which result when the groups of the preceding category are extended by means of operators which transform them into themselves and also have their squares in the groups to be extended.* The former of these categories is composed of the groups which result when $s_1s_2=s_2^{-1}s_1$, and if we we allow the identity to be one of the operators it also includes the cyclic groups, which result when $n=0$.

§ 3. Conclusion.

The values of n which were considered in the preceding section are 1, 0, -1, -2, -3. In the case then $n=-1$, the system of possible groups is, however, very complex. In fact, it includes every possible group of finite order either directly or as *a* subgroup since it includes every symmetric group. In view of this fact it does not appear likely that this case will give rise to theorems of special interest. On the contrary, the other four cases relate to very elementary groups. In fact, for the two values of n which are adjacent to -1 (viz., 0 and -2) the groups are cyclic, being generated by s_2 and s_1s_2 respectively. Each of these groups is therefore completely determined by the order of one of these generating operators.

When $m=1$, s_1 transforms s_2 into its inverse and each group is completely determined by the orders of s_1, s_2 and the number of the common

operators in the cyclic groups generated by s_1, s_2 respectively. Finally, when $n=-3$, s_1s_2 transforms $s_1^4s_2^2$ into its inverse, and s_1 transforms into itself the group generated by these operators, and it has its square in this group. For values of n which differ from -1 by more than 2 the considerations again become so complex that it does not seem probably that it is possible to establish theorems of special interest. In fact. if the condition $s_1s_2=s_2^{-1}s_1^{n}$ be put in the form $s_1s_2=(s_1s_2)^{-1}\cdot s_1^{n+1}$, and if it be assumed that s_1, s_1s_2 are of orders $n+1$ and 2 respectivly, it is evident that the condition is satisfied. As G is generated by s_1, s_1s_2 it is clear that G includes all groups that may be generated by an operator of order 2 and an operator whose order exceeds 2 whenever n differs from -1 by more than 2. As this includes an infinite number of symmetric groups even when the latter operator is of order 3[1]. it is clear that these systems include very complex groups, and that all the cases leading to categories composed of elementary groups are included in the above considerations.

One of the theorems expressed in § 1 can readily be extended by observing that *if a power of the product of two operators is equal to some power of one of these operators, this power is commutative with each of these two operators and it is independent of the order of the factors in this product.* In particular, the condition under consideration is $s_1s_2=(s_1s_2)^{-1}\cdot s_1^{n+1}$ and hence s_1^{n+1} is the square of s_1s_2. According to this elementary theorem s_1^{n+1} must be commutative with s_2 and $(s_1s_2)^2=(s_2s_1)^2$. It may be added that the present paper has close contact with the one on groups which are generated by two operators satisfying the condition $s_1s_2=s_2^{-2}s_1^{-2}$, published in a recent number of the „Bulletin of the American Mathematical Society“.

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[1]) Bulletin of the American Mathematical Society, vol. 7 (1905), p. 6.