$$+ (qx - \mathrm{E}qx)(qy - \mathrm{E}qy)\left[f\left(\frac{\mathrm{E}qx+1}{q}, \frac{\mathrm{E}qy+1}{q}\right) - f\left(\frac{\mathrm{E}qx}{q}, \frac{\mathrm{E}qy}{q}\right)\right]$$

skąd, wobec zakładanej ciągłości funkcyi $f(x, y)$, z łatwością wnosimy, że dla $|x| \leqslant 1$ oraz $|y| \leqslant 1$ funkcya $f_q(x, y)$ jednostajnie zmierza do $f(x, y)$. Przechodząc wreszcie od funkcyj $\varphi(qx - p')$ oraz $\varphi(qy - p'')$ do odpowiednich przybliżeń przez wielomiany, dojdziemy do wniosku, że do każdej danej liczby dodatniej ε można dobrać wielomian całkowity $P(x, y)$ taki, iż dla $|x| \leqslant 1$ oraz $|y| \leqslant 1$ stale:

$$|f(x, y) - P(x, y)| < \varepsilon.$$

Stąd wynika natychmiast twierdzenie Weierstrassa dla funkcyj dwuch zmiennych.

G. A. MILLER.

Extension of a group by operators of orders two and four.

(Rozszerzenie grupy za pomocą operatorów rzędów 2 i 4).

It frequently happens that a group H of order h is extended by means of operators of orders two or four so as to obtain a group G of order $g = 2h$. For instance, the alternating group of order 12 is extended by 6 operators of order 2 and 6 operators of order 4 to obtain the symmetric group of order 24, and the cyclic group of order h may be extended by h operators of order 2 to obtain the dihedral group of order $2h$. The present article is devoted to a study of some groups which may be extended by operators of orders 2 or 4 so as to obtain a group whose order is the double of the order of the original group. Many questions relating to such groups remain unsolved, and the present brief article deals only with a few of the simpler cases.

A necessary and sufficient condition that a group H of order h may be extended by means of h operators of order 2 so as to obtain a group of order $2h$ is that H be abelian. This theorem results direclty from the known fact that the abelian group is the only group in which all the operators correspond to their inverses in an automorphism of the group. Hence the theory of extending groups by means of operators of order 2 is very simple, but if we extend a group by means of operators of order 4, or by means of operator of orders 2 and 4 the matter becomes very much more difficult.

Among the simplest cases is the one when H is extended by means of h operators of order 4 which have a common square. If the group G is obtained in this way it involves the group of order 2 generated by this common square as an invariant subgroup, and the corresponding quotient group comes under the theorem of the preceding paragraph. Each of the h given operators of order 4 in G must transform every operator of H into its inverse, as it could not transform such an operator into its inverse multiplied by the given common square. From this it results that H must be an abelian group of even order. As such a G can always be constructed when H is an abelian group of even order we have arrived at the theorem: A necessary and sufficient condition that a group H of order h may be extended by h operators of order 4 which have a common square so as to obtain a group of order $2h$ is that H be an abelian group of even order. When H is cyclic the groups of this theorem are the dicycle groups.

In the case which has just been considered the h operators of order 4 by means of which H was extented had a common square. Another extreme case which is equally simple is the one in which H is extended by means of h operators of order 4 which have $h/2$ distinct squares, so as to obtain a group of order $2h$. These squares must generate H since no two of them can be commutative; for if one of these $h/2$ operators of order 2 were commutative with more than two of the given operators of order 4, it would follow that more than two of these operators of order 4 would have a common squares and one of these operators of order 2 could not transform one of the given operators of order 4 into its inverse. Hence each of these $h/2$ operators of order 2 transforms into a different subgroup each of $h/2-1$ of the given cyclie subgroups of order 4, and it is therefore not commutative with any other one of these $h/2$ operators of order 2. This proves that the order of H is twice an odd number, and hence its substitutions of odd order constitute an abelian subgroup. Hence the theorem: A necessary and sufficient condition that a group H of order h may be extended by means of h operators of order 4 having $h/2$ distinct squares so as to obtain a group of order $2h$, is that H involve an abelian subgroup of order $h/2$ composed of all of its operators of odd order and that the group of isomorphisms of this abelian subgroup involves an operators of order 4 whose squares is in H and transforms each operator of odd order in H into its inverse.

It has been observed that a necessary and sufficient condition that a group be abelian is that more than threefourths of the operators of the group correspond to their inverses in some automorphism of the group [1]; and if more than half of the operator of an abelian group correspond to their inverses in an automorphism of the group, all of its operators must correspond to thier

[1]) Annals of Mathematics, vol. 7 (1906), p. 55.

inverses in this automorphism. Hence it results that when H is extended so as to obtain a group of order $2h$ the number of operators of order 2 which have been added is h whenever this number exceeds $3/4\, h$. If this number is exactly $3/4\, h$ the remaining $1/4\, h$ operators are of order 4 and the group of cogredient isomorphisms of H is the four-group. Hence the theorem: A necessary and sufficient condition that a group H of order h can be extended by $3/4\, h$ operators of order 2 and $1/4\, h$ operators of order 4 so as to obtain a group of order $2h$ is that the group of cogredient isomorphisms of H be the fourgroup.