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p-Isomorphisms of an abelian group of order p^m.

(Izomorfizmy p-stopniowe grupy abelowej rzędu p^m).

If G is an abelian group of order p^m, p being a prime number, and if I represents a subgroup of order p^α which is contained in the group of isomorphisms of G then G involves a series of subgroups $H_1, H_2, \ldots, H_{m-1}$ of orders $p, p^2, \ldots, p^{m-1}$ respectively, such that each of these subgroups is included in the one which follows it and is invariant under I. The isomorphisms of G which correspond to the operators of I are called p-isomorphisms of G.

In particular, there is at least one such series of subgroups which corresponds to a Sylow subgroup of the group of isomorphisms of G. In what follows we shall confine our attention to the possible series of such subgroups which correspond to a particular Sylow subgroup I_1 in the group of isomorphisms of G. One such series may be selected as follows.

Let $s_1, s_2, \ldots, s_\lambda$ be a set of independent generators of G arranged so that the order of s_α is at least equal to that of $s_{\alpha+1}$, $\alpha=1, 2, \ldots, \lambda-1$. From the ordinary method of finding a set of independent generators of any abelian group, it results directly that a p-automorphism of G may be obtained by multiplying s_α by any operator of the group generated by all the operators of G whose orders are less than that of s_α together with those operators whose orders are equal to that of s_α but are generated by the independent generators of G which precede s_α in the given set. By this method we associate with each one of a set of independent generators of G a certain subgroup composed of all the operators which may be multiplied into this independent generator to obtain one of the automorphisms in question. This subgroup may be called the associated p-subgroup of the given generator.

By multiplying each of the operators $s_1, s_2, \ldots, s_\lambda$ independently of each other, by all the operators in their associated p subgroups we obtain a group of p-automorphisms whose order is equal to the product of the orders of the associated subgroups. This group of p-automorphisms is a Sylow subgroup of the group of isomorphisms of G, since the given arrangement of independent generators is clearly possible under such a subgroup. A series of subgroups $H_1, H_2, \ldots, H_{m-1}$ corresponding to I_1 may therefore always be selected so that the order of every operator of H_β which is not contained in $H_{\beta-1}$ is at least as large as the largest order of an operator of $H_{\beta-1}$, $\beta = 2, 3, \ldots, m-1$. A series of subgroup satisfying this condition will be called a *principal series* of subgroups corresponding to the Sylow subgroup I_1 of the group of isomorphisms of G.

We proceed to prove that I_1 has only one principal series of subgroups $H_1, H_2, \ldots, H_{m-1}$. Suppose that $H_1', H_2', \ldots, H'_{m-1}$ were a second principal series corresponding to I_1. As H_1 and H_1' are both generated by the operators of highest order in G which are transformed under I_1 into themselves multiplied only by operators of lower order, it results that $H_1 \equiv H_1'$. In fact, H_1 is clearly contained in the fundamental characteristic subgroup of G, and it constitutes this subgroup whenever G has only one independent generator of highest order. In fact, all the operators of H_α which are not contained in $H_{\alpha-1}$ have the same order and they are completely determined by their orders and the number of their conjugates under I_1. Hence $H_\alpha \equiv H_\alpha$, $\alpha = 1, 2, \ldots, m-1$. That is, *the number of the Sylow subgroups of order p^m in the group of isomorphisms of an abelian group of order p^m is equal to the number of distinct principal series of subgroups $H_1, H_2, \ldots, H_{m-1}$.*

As a very elementary illustration of this theorem it may be observed that if G represents the abelian group of order 8 and of type (1, 1, 1) we may select H_1 in seven ways and as H_2 involves H_1 it can be selected in three ways after H_1 has been fixed. It therefore follows that this group has 21 principal series of subgroups. Its group of isomorphisms is known to be the simple group of order 168, and this has clearly 21 Sylow subgroups of order 8. On the other hand, the abelian group of order 8 and of type (2, 1) has only one principal series of subgroups and its group of isomorphisms is the octic group, which is in accord with the given theorem.

When all the invariants of G are equal to each other each series of subgroups corresponding to a Sylow subgroup of the group of isomorphisms of G is clearly a principal series. This is also true when G has only two sets of equal invariants whose quotient is p. In all other cases G involves a characteristic subgroup which does not include all its operators of a given order, and hence a Sylow subgroup of order p^m in its group of isomorphisms has more than one series of corresponding subgroups $H_1, H_2, \ldots, H_{m-1}$. Hence the theorem: *A necessary and sufficient condition that an abelian group of order p^m involves a non-principal series of subgroups $H_1, H_2, \ldots, H_{m-1}$ corresponding to a Sylow subgroup of order p^m in its group of isomorphisms is that it has at least two invariants whose quotient exceeds p.*

If G has λ_1 equal largest invariants the first λ_1 and the last λ_1 of its series of subgroups corresponding to I_1 must be identical in all the possible series. As a simple illustration of non-principal series of subgroups $H_1, H_2, \ldots, H_{m-1}$ we may consider the case when G is of type $(1, m-1)$, $m>3$. In this case, H_1 is the subgroup of order p generated by any one of the operators of highest order, while H_{m-1} is composed of all the operators of G whose orders divide p^{m-2}. The subgroup H_α, $1 < \alpha < m-1$, can clearly be chosen in two ways, as it is either cyclic or non-cyclic for every value of α within the given limits. If H_α is non-cyclic then $H_{\alpha+1}$ is also non-cyclic. Hence the series $H_1, H_2, \ldots, H_{m-1}$ can be chosen in $m-2$ ways so us to correspond to I_1 in this special case. That is, *in an abelian group of order p^m, $m>2$, and of type $(1, m-1)$ the series of subgroups of orders $p, p^2, \ldots, p^{m-1}$ which are separately invariant under a Sylow subgroup of order p^m in the group of isomorphisms, can be chosen in $m-2$ different ways.*

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[1]) American Journal of Mathematics, vol. **27**, p. 16 (1905).