Note on Trigonometrical and Rademacher's Series

by

A. Zygmund.

§ 1.

A lacunary trigonometrical series is any series of the form

$$\sum_{\nu=1}^{\infty} (a_\nu \cos n_\nu x + b_\nu \sin n_\nu x), \tag{1}$$

where the integers n_ν satisfy an inequality $n_{\nu+1}/n_\nu > q > 1$, with q independent of ν. Lacunary series have a number of curious properties. For example, if the series

$$\sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2) \tag{2}$$

diverges, the series (1) is almost everywhere non-summable by any linear method of summation, and, in particular, is not a Fourier series. On the contrary, if (2) is finite, (1) is the Fourier series of a function $f(x)$ which is integrable in every power; more generally, the function $\exp \lambda f^2(x)$ is integrable for every $\lambda > 0$[1]). One of the objects of this

[1]) For the proofs of these results, see *e. g.* Zygmund, *Trigonometrical Series*, Warszawa, 1935, pp. II+331, esp. p. 119 sq. We shall refer to this book by the letters TS.

We avail ourselves of the opportunity to correct an obvious slip on page 61, line 19 of that book:

for $\quad \Phi_x(h) = o(h) \qquad$ read $\quad \int_0^h |f(x \pm t) - f(x)|\, dt = o(h).$

note is to obtain some theorems about the series (1) with

$$(3) \qquad \sum_{\nu=1}^{\infty} (|a_\nu|^r + |b_\nu|^r)$$

convergent for an $r < 2$.

Let $\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t), \ldots$ $(0 \leqslant t \leqslant 1)$ denote the sequence of Rademacher's functions, *i. e.*

$$\varphi_\nu(t) = \operatorname{sign} \sin(2^{\nu+1} \pi t), \quad \nu = 0, 1, \ldots; \; 0 \leqslant t \leqslant 1.$$

The Rademacher series

$$(4) \qquad \sum_{\nu=0}^{\infty} c_\nu \varphi_\nu(t),$$

where the complex constants c_ν are independent of t, have many properties similar to those of lacunary trigonometrical series, and the study of the former is a little simpler than that of the latter. It will therefore be more convenient to begin with the series (4).

§ 2.

Suppose that the series $\Sigma |c_\nu|^2$ converges. Then, by the Riesz-Fischer theorem, (4) is the Fourier series of a function $f(t) \varepsilon L^2$, and it is known that the series (4) converges to $f(x)$ almost everywhere [2]. It is also known that $f(t)$ belongs to L^k for every $k > 0$, which is a corollary of the following more precise proposition.

Lemma A. *If* $\Sigma |c_\nu|^2 < \infty, k \geqslant 2$, *the sum* $f(t)$ *of* (4) *satisfies the inequality*

$$(5) \qquad \left\{ \int_0^1 |f(t)|^k \, dt \right\}^{1/k} \leqslant k^{1/2} \left(\sum_{\nu=0}^{\infty} |c_\nu|^2 \right)^{1/2}.$$

In the case $k = 2m$, $m = 1, 2, \ldots$, the proof of this lemma will be found in *TS*, p. 123 — 124; we have, then, even a little stronger

[2]) *TS*. p. 123.

result, for we may replace the factor $k^{1/2}$ on the right of (5) by $(k/2)^{1/2}$. To obtain (5) in the general case, let $2m - 2 \leqslant k < 2m$, $m = 2, 3, \ldots$ Then

$$\left\{ \int_0^1 |f|^k \, dt \right\}^{1/k} \leqslant \left\{ \int_0^1 |f|^{2m} \, dt \right\}^{1/2m} \leqslant m^{1/2} (\Sigma |c_\nu|^2)^{1/2}$$

$$\leqslant \left(\frac{1}{2} k + 1 \right)^{1/2} (\Sigma |c_\nu|^2)^{1/2} \leqslant k^{1/2} (\Sigma |c_\nu|^2)^{1/2}.$$

We shall now prove the following proposition, where r' denotes the number connected with r by the relation $1/r + 1/r' = 1$. (This notation will be used throughout the paper).

Theorem 1. *If* $1 < r \leqslant 2$, $k \geqslant 1$, *and the series*

$$(6) \qquad \sum_{\nu=1}^{\infty} |c_\nu|^r$$

converges, the sum $f(t)$ *of* (4) *satisfies the inequality*

$$(7) \qquad \left\{ \int_0^1 |f(t)|^k \, dt \right\}^{1/k} \leqslant k^{1/r'} \left(\sum_{\nu=0}^{\infty} |c_\nu|^r \right)^{1/r}.$$

It is sufficient to prove the inequality (7) for the n-th partial sum $f_n = c_0 \varphi_0 + \ldots + c_n \varphi_n$, $n = 0, 1, 2, \ldots$, of the series (4). We fix n, and for simplicity, write f for f_n.

Given any non-negative numbers α and β, let $M_{\alpha\beta}$ denote the upper bound of the ratio

$$\left\{ \int_0^1 |f(t)|^{1/\beta} \, dt \right\}^{\beta} \Bigg/ \left(\sum_{\nu=0}^{n} |c_\nu|^{1/\alpha} \right)^{\alpha}$$

for all possible values of $c_0, c_1, \ldots, c_n$. The fundamental theorem of M. Riesz asserts that $\log M_{\alpha\beta}$ is a convex function on any segment lying in the triangle

$$(\triangle) \qquad 0 \leqslant \alpha \leqslant 1, \qquad 0 \leqslant \beta \leqslant \alpha \; {}^{3)}.$$

[3]) M. Riesz, *Acta Math.*, 49 (1926), p. 465—497; *T S*, p. 192 — 202.

In view of Lemma A, we have $M_{1/2,1/l} \leqslant l^{1/2}$ for $l \geqslant 2$; moreover $M_{1,0} \leqslant 1$. Therefore, for any point $P(\alpha, \beta)$ lying on the segment joining $P_1(1/2, 1/l)$ and $P_2(1, 0)$, we have

$$M_{\alpha\beta} \leqslant M_{1/2,1/l}^{\frac{1-\alpha}{1/2}} M_{1,0}^{\frac{\alpha-1/2}{1/2}} \leqslant l^{1-\alpha}.$$

From the equation of the line $P_1 P_2$ we obtain $1-\alpha=\beta l/2$ and so

$$M_{\alpha\beta} = \left\{\frac{2(1-\alpha)}{\beta}\right\}^{1-\alpha} \leqslant \left\{\frac{1}{\beta}\right\}^{1-\alpha}. \tag{8}$$

Putting $\alpha=1/r$, $\beta=1/k$, we obtain the inequality (7).

It must however be observed that, since the point P_1 is on the segment $\left(\frac{1}{2}, 0\right)\left(\frac{1}{2}, \frac{1}{2}\right)$, the values of α, β for which the inequality has been established satisfy the conditions $\frac{1}{2} \leqslant \alpha \leqslant 1$, $\beta \leqslant 1-\alpha$, or, what is the same thing,

$$1 \leqslant r \leqslant 2, \qquad k \geqslant r'.$$

In order to prove (7) for any $k \geqslant 1$, we observe that for $k=r$, the inequality (7) is true without the factor $k^{1/r'}$ (the Hausdorff-Young inequality), and that the left-hand side of (7) decreases with k. This completes the proof.

Theorem 2. *If the series* (6) *converges and* $1 < r \leqslant 2$, *the function*

$$\exp \lambda |f(t)|^{r'}$$

is integrable for every $\lambda > 0$. *If* $2\lambda e r' C^{r'} \leqslant 1$, *where* C^r *is the sum of the series* (6), *then*

$$\int_0^1 \exp \lambda |f|^{r'} dt < 2. \tag{9}$$

From the inequality (7), with k replaced by $r'\nu$, $\nu=1, 2, \ldots$, we obtain

$$\frac{\lambda^\nu}{\nu!}\int_0^1 |f|^{\nu r'} dt \leqslant \frac{\lambda^\nu}{\nu!}(\nu r' C^{r'})^\nu$$

$$\int_0^1 \exp \lambda |f|^{r'} dt \leqslant 1 + \sum_{\nu=1}^{\infty} \frac{\lambda^\nu}{\nu!}(\nu r' C^{r'})^\nu. \tag{10}$$

Observing that $\nu^\nu \leqslant e^\nu \nu!$ [3a]), and assuming that $\lambda e r' C^{r'} \leqslant 1/2$, we obtain (9). To show that the left-hand side of (9) is finite for every $\lambda > 0$, we put $f = s_n + r_n$, where s_n is the n-th partial sum of the series (4). In view of the second part of Theorem 2, if λ is fixed and n is large enough, the function $\exp \lambda 2^{r'} |r_n(t)|^{r'}$ is integrable; and it remains to observe that

$$\exp \lambda |f|^{r'} \leqslant \exp \lambda 2^{r'} \{|s_n|^{r'} + |r_n|^{r'}\} \leqslant \text{Const.} \exp \lambda 2^{r'} |r_n|^{r'},$$

since the function $s_n(t)$ is bounded.

We complete Theorem 2 by the following remark: *If* (6) *is finite, and* $s_n(t)$ *denotes the* n-*th partial sum of the series* (4), *then, for any fixed* $\lambda > 0$,

$$\int_0^1 \exp \lambda |f - s_n|^{r'} dt \to 1 \text{ as } n \to \infty \text{ [4]}).$$

This follows from the fact that $f - s_n$ is the sum of a Rademacher series, and that the right-hand side of (10) tends to 1 as $C \to 0$.

§ 3.

Results analogous to Theorem 1 and 2 hold for lacunary trigonometrical series.

Lemma B. *Let the series* (2) *converge, and* $f(x)$ *denote the sum of the series* (1). *Then, for any* $k \geqslant 1$, *we have*

[3a]) This inequality follows from the fact that the product of the ν numbers $\left(1+\frac{1}{n}\right)^n$, $n=1, 2, \ldots, \nu$, is less than e^ν

[4]) For $r=2$ this remark was proved by Kaczmarz and Steinhaus, *Studia Math.* 2 (1930), 331 — 247.

$$\left\{\int_0^{2\pi} |f|^k\, dx\right\}^{1/k} \leqslant A_q k^{1/2} \left\{\sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2)\right\}^{1/2}, \tag{11}$$

where A_q depends on q only[5]).

Assuming the truth of this lemma, and supposing, as we may, that the constant A_q of (11) is not less than 1, we obtain, by an argument similar to that which led to Theorems 1 and 2, the following

Theorem 3. *If the series* (3) *converges,* $1 < r \leqslant 2$, $k > 1$, *then*

$$\left\{\int_0^{2\pi} |f(x)|^k\, dx\right\}^{1/k} \leqslant A_q k^{1/r'} \left\{\sum_{\nu=1}^{\infty} (|a_\nu|^r + |b_\nu|^r)\right\}^{1/r}$$

where the constant A_q depends on q only.

Theorem 4. *If the series* (3) *converges,* $1 < r \leqslant 2$, *the function* $\exp \lambda\, |f(x)|^{r'}$ *is integrable for every* $\lambda > 0$. *If* $0 < \lambda \leqslant \lambda_0 = \lambda_0(r, q)$, *then*

$$\frac{1}{2\pi}\int_0^{\pi} \exp \lambda\, |f(x)|^{r'}\, dx \leqslant 2.$$

For any fixed $\lambda > 0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} \exp \lambda\, |f - s_\nu|^{r'}\, dx \longrightarrow 1 \quad as \quad \nu \longrightarrow \infty,$$

where s_ν is the ν-th partial sum of (1).

§ 4.

Theorem 5. *Let $g(x)$ be any function such that $g \cdot (\log^+ |g|)^{1/r}$, $r \geqslant 2$, be integrable over $(0, 2\pi)$, and let*

$$g(x) \sim \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx). \tag{12}$$

Then, if $n_{\nu+1}/n_\nu > q > 1$, $\nu = 1, 2, \ldots$, we have

$$\left\{\sum_{\nu=1}^{\infty} (|\alpha_{n_\nu}|^r + |\beta_{n_\nu}|^r)\right\}^{1/r} \leqslant A \int_0^{2\pi} g \cdot (\log^+ |g|)^{1/r}\, dx + B, \tag{13}$$

where the numbers $A = A_{r,q}$ and $B = B_{r,q}$ depend on r and q only.

The theorem does not hold for $r < 2$[6]).

Let $a_1, b_1, a_2, b_2, \ldots$ be any sequence of numbers such that

$$\sum_{\nu=1}^{\infty} (|a_\nu|^{r'} + |b_\nu|^{r'}) = 1. \tag{14}$$

The left-hand side of (13) is the upper bound of the expression

$$\sum_{\nu=1}^{\infty} (\alpha_{n_\nu} a_\nu + \beta_{n_\nu} b_\nu) \tag{15}$$

for all sequences $\{a_\nu, b_\nu\}$ satisfying (14). By $f(x)$ we shall denote thet sum of the corresponding series (1). Since $1 < r' \leqslant 2$, using Theorem 4 with r replaced by r', we see that

$$\frac{1}{\pi}\int_0^{2\pi} \exp \lambda\, |f(x)|^r\, dx \leqslant 4, \tag{16}$$

where $\lambda = \lambda_0(r', q)$.

Let $\Phi(u) = \exp \lambda u^r - 1$, and let $\psi(u)$ denote the function inverse o $\varphi(u) = \Phi'(u) = \lambda r u^{r-1} \exp \lambda u^r$. If $\Psi(u)$ denotes the integral of ψ over he interval $(0, u)$, then

$$\Psi(u) \leqslant u\, \psi(u) \leqslant A' u (\log^+ |u|)^{1/r} + B',$$

where A' and B' depend on r and q only.

In view of Young's inequality[7])

[5]) This result is established in the author's paper in the *Journal of the London Math. Soc.* 5 (1930), pp. 138 — 145, esp. pp. 141 — 142. The proof uses some rather deep results of Littlewood. It would be interesting to obtain (11) by the same argument which gave (5), but a straightforward application of that argument seems to give a bigger constant than $A_q k^{1/2}$ on the right of (11), which would not be sufficient for our purposes.

[6]) It was shown by Paley, *Annals of Mathematics*, 34 (1933), p. 615—616, that, if $g(x)$ is integrable, and the series conjugate to (12) is a Fourier series, then the expression on the left of (13) is finite for $r = 2$.

[7]) See *e. g.* *T S*, p. 64.

$$\left|\frac{1}{\pi}\int_0^{2\pi} f g\, dx\right| \leqslant \frac{1}{\pi}\int_0^{2\pi} |f| \cdot |g|\, dx \leqslant \frac{1}{\pi}\int_0^{2\pi} \Phi(|f|)\, dx + \frac{1}{\pi}\int_0^{2\pi} \Psi(|g|)\, dx$$

$$(17) \qquad \leqslant 2 + \frac{1}{\pi}\int_0^{2\pi} \Psi(|g|)\, dx$$

$$\leqslant A \int_0^{2\pi} |g| (\log^+ |g|)^{1/r} dx + B,$$

where $\qquad A = A'/\pi, \qquad B = 2B' + 2.$

Now we observe that, since $\Phi(|f|)$ and $\Psi(|g|)$ are integrable, we have

$$\sum_{\nu=1}^{\infty} (\alpha_{n_\nu} a_\nu + \beta_{n_\nu} b_\nu) = \frac{1}{\pi}\int_0^{2\pi} f g\, dx,$$

the series on the left being summable $(C, 1)$[8]. Taking into account that the signs of the numbers a_ν, b_ν may be arbitrary, we see that (15) converges for every $\{a_\nu, b_\nu\}$ satisfying (14), and its sum does not exceed the last expression in (17). This completes the proof of the first part of Theorem 5.

To show that this result does not hold for any $0 < r < 2$, we shall show that there is a function g, such that $g(\log^+ |g|)^{1/r} \varepsilon L$, $0 < r < 2$, and yet the left-hand side of (13) is infinite. For simplicity we restrict ourselves to the case $1 < r < 2$; the case $0 < r \leqslant 1$ requires small modifications only. Let $\Phi(u)$ and $\Psi(u)$ have the same meaning as before, except that we now suppose that $\lambda = 1$. It is not difficult to see that

$$\Psi(u) \sim u(\log u)^{1/r} \quad \text{as } u \to \infty.$$

Let $\sigma_n(x)$ denote the first Cesàro means of the series

$$(18) \qquad \sum_{\nu=1}^{\infty} \frac{\cos n_\nu x}{\sqrt{\nu}}$$

[8]) $T\,S$, p. 88.

(the vacant terms of (18) must be replaced by zeros). Suppose, contrary to what we want to prove, that the series $\Sigma(|\alpha_{n_\nu}|^r + |\beta_{n_\nu}|^r)$, $1 < r < 2$, converges for every g with $g(\log^+ |g|)^{1/r} \varepsilon L$, or, what is the same thing, $\Psi(|g|) \varepsilon L$. Then, by Hölder's inequality, the series

$$(19) \qquad \sum_{\nu=1}^{\infty} \frac{\alpha_{n_\nu}}{\sqrt{\nu}}$$

converges, and, à fortiori, is summable $(C, 1)$. The first arithmetic means of (19) are equal to

$$(20) \qquad \frac{1}{\pi}\int_0^{2\pi} g(x)\, \sigma_n(x)\, dx.$$

The sequence (20) being bounded for every g with $\Psi(|g|) \varepsilon L$, there is a constant $\mu > 0$ such that

$$\int_0^{2\pi} \Phi(\mu\, |\sigma_n(x)|)\, dx = O(1)\,^{9)},$$

and so (18) would be a Fourier series[10]), which as we know (cf. § 1 of this paper) is false.

§ 5.

Theorems 3 and 4 have applications to series of the form

$$(21) \qquad \sum_{\nu=0}^{\infty} \pm c_\nu f_\nu(x),$$

where the $\pm$ are chosen in a quite arbitrary manner, the coefficients c are constants, and $\{f_\nu(x)\}$, $0 \leqslant x \leqslant 1$, is a sequence of measurable and uniformly bounded functions; to fix ideas we suppose that $|f_\nu(x)| \leqslant 1$, $\nu = 0, 1, \ldots$ The series

$$(22) \qquad \pm \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} \pm (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

are special cases of (21).

[9]) $T\,S$, p. 100.
[10]) $T\,S$, p. 83.

Neglecting sequences $\{\pm\}$ which are constant from some place onwards, we may write the series (21) and (22) in the forms

$$\sum_{\nu=0}^{\infty} c_\nu f_\nu(x)\,\varphi_\nu(t), \tag{23}$$

$$\frac{1}{2}a_0\varphi_0(t)+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x)\,\varphi_\nu(t) \tag{24}$$

respectively, where the φ_ν are Rademacher's functions. If for almost every value of t the series (23) has a property P, we shall say that almost all the series (23) have the property P. The sum, wherever it exists, of the series (23) (and also of the series (24)), will be denoted by $F_t(x)$.

Theorem 6. *If the expression*

$$\left(\sum_{\nu=0}^{\infty}|c_\nu|^r\right)^{1/r} \qquad (1<r\leq 2) \tag{25}$$

is finite, then, for almost every value of t, the series (23) converges almost everywhere in x. Moreover, for almost every value of t, the function

$$\exp\lambda\,|F_t(x)|^{r'}$$

is integrable over $0\leqslant x\leqslant 1$, however large λ may be.

Theorem 7. *The preceding theorem holds for the series (24), provided that*

$$\left(\sum_{\nu=1}^{\infty}|a_\nu|^r+|b_\nu|^r\right) \qquad (1<r\leqslant 2)$$

is finite.

In the case $r=2$, Theorem 6 is known[11]). The proof of the general result does not differ essentially from that of the special case. Let E denote the set of the points of convergence of (23), situated in the square $0\leqslant t\leqslant 1$, $0\leqslant x\leqslant 1$. Using the theorem of Rademacher[12])

[11]) Paley and Zygmund, *Proc. Cambridge Phil. Society*, 26 (1930), p. 337—357; *T S*, p. 125.

[12]) *T S*, p. 123.

that the series (4) converges almost everywhere, provided that $\Sigma|c_\nu|^2<\infty$, we see that, under the hypothesis of Theorem 6, the intersection of E with every line $x=x_0$, $0\leqslant x_0\leqslant 1$, is of linear measure 1. The set E being measurable, its plane measure is equal to 1, and so the intersection of E with almost every line $t=t_0$ is of linear measure 1. This proves the first part of Theorem 6.

To prove the second part, we observe that, in view of Theorem 2, we have, for every x, $0\leqslant x\leqslant 1$,

$$\int_0^1 \exp\lambda\,|F_t(x)|^{r'}\,dt<2, \tag{26}$$

provided that $2\lambda e r' C^{r'}\leq 1$, where C is the value of (25). Integrating (26) with respect to x, and inverting the order of integration, we obtain

$$\int_0^1 dt\int_0^1 \exp\lambda\,|F_t(x)|^{r'}\,dx<2.$$

It follows that the inner integral is finite for almost every t, provided that $\lambda\leqslant 1/2er'C^{r'}$. To remove the last restriction, we argue as in he proof of Theorem 2.

Theorem 7 is a corollary of Theorem 6.

For further applications we shall require the following extensions of Theorem 2.

Theorem 8. *Assuming that (6) is finite, and $(1<r\leqslant 2)$, let*

$$F^*(t)=\operatorname*{Sup}_n\left|\sum_{\nu=0}^{n}c_\nu\varphi_\nu(t)\right|.$$

Then, for every $\lambda>0$, the function

$$\exp\lambda\,F^{*r'}(t) \tag{27}$$

is integrable. If $2\lambda e r' C^{r'}\leqslant 1$, then

$$\int_0^1 \exp\lambda\,F^{*r'}(t)\,dt<2A, \tag{28}$$

where A is an absolute constant.

For $r=2$ this result is known[13]). The proof of the general case does not require new devices. Let $s_n(t)$ denote the n-th partial sum of (4), and let $t_0 \neq p/2^q$. If $t_0 \in I_k = (l\,2^{-k}, (l+1)\,2^{-k})$, then, since $s_{k-1}(t)$ is constant over I_k, we have [14])

$$s_{k-1}(t_0) = \frac{1}{|I_k|}\int\limits_{I_k} s_{k-1}(t)\,dt = \frac{1}{|I_k|}\int\limits_{I_k} f(t)\,dt \text{ }^{15})$$

Therefore

$$(29) \qquad F^*(t_0) \leqslant \sup_{\xi_1<t_0<\xi_2} \frac{1}{\xi_2-\xi_1}\int\limits_{\xi_1}^{\xi_2} |f(t)|\,dt = g(t_0).$$

By the well-known theorem of Hardy and Littlewood [16])

$$\int\limits_0^1 g^s(t)\,dt \leqslant 2\left(\frac{s}{s-1}\right)^s \int\limits_0^1 |f(t)|^s\,dt \qquad (s>1),$$

and so

$$(30) \qquad \int\limits_0^1 g^s(t)\,dt \leqslant A \int\limits_0^1 |f(t)|^s\,dt \qquad (s \geqslant 2),$$

where A is an absolute constant. From (29), (30), and (7) we obtain

$$(31) \qquad \int\limits_0^1 F^{*s}(t)\,dt \leqslant A \int\limits_0^1 |f(t)|^s\,dt \leqslant A\, s^{s/r'}\left(\sum_{\nu=0}^{\infty} |c_\nu|^r\right)^{s/r}$$

for $1<r\leqslant 2\leqslant s$. Since $r' \geqslant 2$, starting from (30), and arguing as in the proof of Theorem 2, we obtain (28).

Theorem 9. *Let*

$$F_t^*(x) = \sup_n \left|\sum_{\nu=0}^{n} c_\nu f_\nu(x)\,\varphi_\nu(t)\right|,$$

[13]) Paley and Zygmund, *Proc. Cambridge Phil. Soc.*, 28 (1932) p. 190—205, esp. p. 190 — 191.

[14]) By $|I_k|$ we denote the measure of I_k.

[15]) TS, p. 123.

[16]) TS, p. 244.

where $|f_\nu(x)| \leqslant 1$, *and the expression* (6) *is finite. Then, for almost every* t, *the function*

$$\exp \lambda\, \{F_t^*(x)\}^{r'} \qquad (1<r\leqslant 2)$$

is integrable over $0\leqslant x\leqslant 1$, *however large* λ *may be.*

This follows from Theorem 8 in the same way as Theorem 6 follows from Theorem 2.

As a special case, we obtain a similar result for the series (24). More interesting, however, is the following

Theorem 10. *Let*

$$s_{t,n}(x) = \frac{1}{2}a_0\varphi_0(t) + \sum_{\nu=1}^{n}(a_\nu \cos\nu x + b_\nu \sin\nu x)\,\varphi_\nu(t).$$

If (3) *is finite, and* $1<r\leqslant 2$, *then, for almost every* t *we have*

$$s_{t,n}(x) = o\,(\log n)^{1/r'}$$

uniformly in x.

Also this result is known for $r=2$ [17]). The proofs of the general theorem and of the case $r=2$ do not differ essentially, if we take account of Theorem 9, with $c_\nu f_\nu(x) = a_\nu \cos\nu x + b_\nu \sin\nu x$. For this reason we omit the proof of Theorem 10 here, and refer the reader to the paper quoted in the last footnote.

We add that the theorem similar to Theorem 10 would be false for *general* series of the form (23), with $|f_\nu(x)|\leqslant 1$, $\nu=0, 1, \ldots$ For let $c_0=0$, $c_\nu=1/\nu$ for $\nu>0$, $f_\nu(x)=\varphi_\nu(x)$, and consider the series

$$(32) \qquad \sum_{\nu=1}^{\infty} \frac{\varphi_\nu(x)\,\varphi_\nu(t)}{\nu}.$$

If $t\neq p/2^q$, and $x=t$, the n-th partial sum of (32) is exactly of the order $\log n$, although $\Sigma|c_\nu|^r<\infty$ for every $r>1$.

Theorem 11. *If the expression* (3) *is finite, then almost all the series*

$$\sum_{\nu=2}^{\infty} \frac{a_\nu\cos\nu x + b_\nu \sin\nu x}{(\log\nu)^{\frac{1}{r'}+\varepsilon}}\,\varphi_\nu(t)$$

converge uniformly in x, *and so are Fourier series of continuous functions.*

This follows from Theorem 10 by partial summation.

[17]) Paley and Zygmund, *Proc. Cambridge Phil. Soc.* 28 (1932) p. 190. The result is proved there for *power* series, but this point is without importance.

Theorem 12. *If* $\Sigma(|a_\nu|^r+|b_\nu|^r)(\log\nu)^{r-1+\varepsilon}$, $1<r\leqslant 2$, $\varepsilon>0$, *is finite, then almost all the series* (24) *converge uniformly in* x.

The theorem follows from Theorem 11.

§ 6.

In the previous paragraphs we studied properties of the series of the form (23). Introducing, the *complex* unit factors $e^{2\pi i\theta}$ $0\leqslant\theta\leqslant 1$, instead of the real unit-factors ± 1, we obtain the series

$$\sum_{\nu=0}^{\infty} c_\nu f_\nu(x)\, e^{2\pi i\theta_\nu}, \tag{33}$$

where the independent variables θ_ν vary within the interval (0, 1). Every sequence of numbers $\theta_0, \theta_1, \theta_2, \ldots$ belonging to (0, 1) may be considered as a point in a space Ω of infinitely many dimensions, which is defined by the inequalities

$$0\leqslant\theta_j\leqslant 1, \qquad (j=0,\ 1,\ 2,\ \ldots) \tag{Ω}$$

A theory of measure and integration may be constructed for this space[18], and if the series (33) has a property P for any point $\theta=(\theta_0,\theta_1,\ldots)$ except for those which belong to a set of measure 0, we shall say that almost all the series (33) have the property P. The results which we obtained for the series (23), hold also for the series of the form (33); and the proofs remain, substantially, the same as before. This is due to the fact that the theorems which we established for Rademacher's series (4) remain valid for the series of the form

$$\sum_{\nu=0}^{\infty} c_\nu e^{2\pi i\theta_\nu}\ ^{19)} \qquad (0\leqslant\theta_\nu\leqslant 1) \tag{34}$$

As an example, we state the following theorem, which is an analogue of Theorem 8.

Theorem 8'. *Suppose that* $\Sigma|c_\nu|^r<\infty$, $1<r\leqslant 2$, *and let*

$$F^*(\theta)=\operatorname*{Sup}_n\left|\sum_{\nu=0}^{n} c_\nu e^{2\pi i\theta_\nu}\right|.$$

[18]) See H. Steinhaus, *Math. Zeitschrift*, 31 (1930), 408—416, B. Jessen, *Acta Mathematica*, 63 (1934), 249—323; in the last paper a further bibliography of the subject will be found.

[19]) The reader interested in the subject will find more details in the papers quoted in the last two footnotes.

Then, for every $\lambda>0$, *the function* $\exp\lambda\{F^*(\theta)\}^{r'}$ *is integrable over the space* Ω. *If* λ *is small enough*, $\lambda\leqslant\lambda_0(r,C)$, *the integral*

$$\int_\Omega \exp\lambda\{F^*(\theta)\}^{r'}\,d\theta$$

does not exceed an absolute constant.

We add that, as regards applications of the complex unit-factors it is more natural to consider not the trigonometrical series, but the power series

$$\sum_{\nu=0}^{\infty} c_\nu z^\nu e^{2\pi i\theta_\nu}.$$

§ 7.

We conclude this paper by a few remarks about the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{\lambda_\nu s} \qquad (s=\sigma+it),$$

where $\{\lambda_\nu\}$ is sequence of different real numbers, and the coefficients c_ν are independent of the variable s. It is well-known that the theory of such series is closely connected with the theory of almost periodic functions. The sum, if it exists, of the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{\lambda_\nu s} e^{2\pi i\theta_\nu} \qquad (s=\sigma+it) \tag{35}$$

will be denoted by $f_\theta(s)$. A number of interesting properties of almost all the series (35) was found by Jessen[20]). Combining Jessen's argument with Theorem 8', we obtain a number of further results.

Let $f(t)$, $-\infty<t<\infty$, be a function integrable over any finite interval. We shall say that f has a mean value $M_t\{f(t)\}$ over $(-\infty,\infty)$, if the limits

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\,dt, \qquad \lim_{T\to\infty}\frac{1}{T}\int_{-T}^0 f(t)\,dt$$

[20]) Jessen, *loc. cit.*

exist and are equal; this common value will, by definition, be $M_t\{f(t)\}$. In the case when $f(t)$ is non-negative, the greater of the two numbers.

$$\overline{\lim_{T\to\infty}}\ \frac{1}{T}\int_0^T f(t)\,dt,\quad \overline{\lim_{T\to\infty}}\ \frac{1}{T}\int_{-T}^0 f(t)\,dt$$

will be denoted by $\overline{M}_t\{f(t)\}$.

Theorem 13. *Let the series*

$$\sum_{\nu=0}^{\infty}|c_\nu|^r \qquad 1<r\leqslant 2$$

be convergent; then, for almost all points θ in Ω, the series

$$\sum_{\nu=0}^{\infty} c_\nu e^{i\lambda_\nu t}\, e^{2\pi i\theta_\nu} \tag{36}$$

s convergent everywhere in t, to a sum $f_\theta(t)$ such that

$$\overline{M}_t\{e^{\lambda|f_\theta(t)|^{r'}}\}$$

is finite for every $\lambda>0$. Moreover, for almost every θ, and every $\lambda>0$

$$\lim_{n\to\infty}\overline{M}_t\{e^{\lambda|f_\theta(t)-f_{n,\theta}(t)|^{r'}}\}\longrightarrow 1,$$

where $f_{n,\theta}(t)$ is the n-th partial sum of the series (36).

Theorem 14. *Suppose that the series*

$$\sum_{\nu=0}^{\infty}|c_\nu|^r e^{\lambda_\nu r\sigma} \qquad (1<r\leqslant 2)$$

converges in every interval interior to $\alpha<\sigma<\beta$. Then, for almost every point θ in Ω, the series (35) *is uniformly convergent in every closed and bounded domain D interior to the strip $\alpha<\sigma<\beta$, and so represents a function $f_\theta(s)$ regular in that strip.*

Denoting by $f_{n,\theta}(s)$ the n-th partial sum of (35), *we have*

$$\lim_{n\to\infty}\overline{M}\left\{\int_{\alpha_1}^{\beta_1}\exp\lambda|f_\theta(\sigma+it)-f_{n,\theta}(\sigma+it)|^{r'}\,d\sigma\right\}=1,$$

for every $\lambda>0$, every finite (α_1,β_1) interior to (α,β), and almost every θ. Moreover, for almost all θ,

$$f_\theta(\sigma+it)=o(\log|t|)^{1/r'}$$

uniformly in (α_1,β_1).

For $r=2$ these theorems have been established by Jessen (*loc. cit.*). Using Theorem 8′, the proofs of the general results are mere repetitions of the Jessen argument, and so may be omitted here.