

## Teilerprobleme. Fünfte Abhandlung.

Von

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Es sei  $\sigma(n)$  die Summe der reziproken Teiler einer natürlichen Zahl  $n$

$$\sigma(n) = \sum_{d|n} \frac{1}{d}.$$

also  $n\sigma(n)$  die Teilersumme

$$n\sigma(n) = \sum_{d|n} d.$$

Ich betrachte ferner die beiden summatorischen Funktionen

$$S_0(x) = \sum_{n \leq x} \sigma(n), \quad S(x) = \sum_{n \leq x} n\sigma(n)$$

(falls die untere Summationsgrenze nicht ausdrücklich angegeben wird, ist sie stets Eins) und setze

$$S_0(x) = \frac{\pi^2}{6}x - \frac{1}{2}\log x + T_0(x),$$

$$S(x) = \frac{\pi^2}{12}x^2 + T(x).$$

Es sei  $\nu$  eine der Zahlen 0, 1, 2, 3, 4;

$$(1) \quad V_\nu(x) = \int_0^x T(2^\nu u) T(u) du.$$

Mit  $Q_0, Q_1, Q_2$  bezeichne ich die quaternären quadratischen Formen

$$Q_0 = n_1^2 + n_2^2 + n_3^2 + n_4^2,$$

$$Q_1 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2,$$

$$Q_2 = n_1^2 + 2n_2^2 + 2n_3^2 + 4n_4^2.$$

Dann sind  $0 \leq Q_0 \leq x, 0 \leq Q_1 \leq x, 0 \leq Q_2 \leq x$  vierdimensionale Ellipsoide vom Volumen  $W_0(x) = \frac{\pi^2}{2} x^2, W_1(x) = \frac{\pi^2}{4} x^2, W_2(x) = \frac{\pi^2}{8} x^2$ . Es seien  $P_0(x), P_1(x), P_2(x)$  die entsprechenden Gitterreste

$$P_m(x) = \sum_{Q_m \leq x} 1 - W_m(x) \quad (m=0, 1, 2).$$

Im folgenden sei stets  $x \geq 3$ . Mit  $B$  bezeichne ich unterschiedslos Zahlen, deren absolute Beträge unterhalb numerischer Schranken liegen.  $C$  sei die Eulersche Konstante.

Das Hauptziel dieser Arbeit besteht in dem Nachweis der folgenden Abschätzungen:

$$(I) \quad \int_0^x T_0^2(u) du = \left( \left( \frac{C + \log 2\pi}{2} \right)^2 + \frac{5\pi^2}{144} \right) x + Bx^{\frac{1}{2}},$$

$$(II) \quad \int_0^x T^2(u) du = \frac{36 + 5\pi^2}{432} x^3 + Bx^{\frac{5}{2}},$$

$$(III) \quad \int_0^x P_0^2(u) du = \frac{2}{3} \pi^2 x^3 + Bx^{\frac{5}{2}},$$

$$(IV) \quad \int_0^x P_1^2(u) du = \frac{\pi^2}{6} x^3 + Bx^{\frac{5}{2}},$$

$$(V) \quad \int_0^x P_2^2(u) du = \frac{\pi^2}{24} x^3 + Bx^{\frac{5}{2}}.$$

Gemeinsame Quelle für (I) — (V) ist

$$(VI) \quad V(x) = \left( \frac{2\pi^2}{12} + \frac{(5+3\gamma)\pi^2}{432} \right) x^3 + Bx^{\frac{5}{2}}.$$

Zum Beweise von (VI) benutze ich im wesentlichen die Methode der dritten gleichnamigen Abhandlung (Journal für die reine und angewandte Mathematik 169 (1933), S. 111 — 130). Dabei mache ich auch von der zweiten und vierten Abhandlung (Mathematische Zeitschrift 34 (1931) S. 448 — 472 und Annali della R. Scuola Normale Superiore di Pisa, serie 2,5 (1936) S. 289 — 298) Gebrauch, sowie von der fünften Abhandlung „Über Gitterpunkte in mehrdimensionalen Ellipsoiden“ (Acta Arithmetica 1 (1936), S. 222 — 283). Ich nenne diese Arbeiten kurz  $T_{III}, T_{II}, T_{IV}$  und  $E_V$ .

Ich möchte noch erwähnen, dass (I) — (VI), mit den etwas ungünstigeren Restgliedern  $Bx^{\frac{1}{2}} \log x$  und  $Bx^{\frac{5}{2}} \log x$  in  $T_{II}$  ((II) — (III)) und  $E_V$  ((II) — (VI)) stehen.

Bevor ich zur Sache übergehe, schalte ich die folgende Bemerkung zu  $E_V$  ein:

Es sei

$$Q = \sum_{g,h=1}^4 a_{gh} n_g n_h \quad (a_{gh} = a_{hg} \text{ rational})$$

eine positiv definite quadratische Form der Determinante  $D$ ,

$$P_Q(x) = \sum_{Q \leq x} 1 - \frac{\pi^2}{2\sqrt{D}} x^2$$

der Gitterrest des vierdimensionalen Ellipsoids  $0 \leq Q \leq x$ .

Ich brauchte in  $E_V$  die Abschätzung

$$\int_0^x P_Q^2(u) du = O(x^3)$$

und gab dafür einen langen und sehr umständlichen Beweis. Einer freundlichen Mitteilung Prof. Jarníks entnehme ich, dass diese Abschätzung bereits bekannt ist und auf ganz einfache Weise in wenigen Zeilen erledigt werden kann. Vgl. V. Jarník „Über die Mittelwertsätze der Gitterpunktlehre. I.“ (Mathematische Zeitschrift 33 (1931), S. 62 — 84), S. 83.

Die folgenden Festsetzungen mögen für die ganze vorliegende Arbeit gelten:

Für reelle  $u$  sei

$$(2) \quad R(u) = u - [u], \phi(u) = \phi_1(u) = R(u) - \frac{1}{2},$$

$$\phi_2(u) = R^2(u) - R(u) + \frac{1}{6}.$$

Die nicht kleinere der beiden Zahlen  $v$  und  $w$  heisse  $\text{Max}(v; w)$ , die nicht grössere  $\text{Min}(v; w)$ ; dagegen bedeute  $(v, w)$  den grössten gemeinsamen Teiler der beiden ganzen Zahlen  $v$  und  $w$ .

Ferner sei

$$(3) \quad E_1 = 0 \text{ oder } 1, E_2 = 0 \text{ oder } 1, \delta = \frac{1}{100}.$$

Die fetten Zahlen geben Nummern von Formeln an, die an den betreffenden Stellen benutzt werden, sind also als Hinweise zu verstehen.

Hilfssatz 1:

$$(4) \quad T(u) - u T_0(u) = \frac{1}{2}(C + \log 2\pi - 1)u + Bu^{\frac{1}{3}} \quad (u \geq 1).$$

Beweis: E. Landau „Über einige zahlentheoretische Funktionen“ (Göttinger Nachrichten 1924, S. 116—134) und etwas einfacher in  $T_{IV}$ .

Hilfssatz 2:

$$(5) \quad -T_0(u) = \frac{1}{2}(C + \log 2\pi) + \sum_{n \leq \sqrt{u}} \frac{1}{n} \phi\left(\frac{u}{n}\right) + \frac{1}{u} \sum_{n \leq \sqrt{u}} n \phi\left(\frac{u}{n}\right)$$

$$+ \frac{1}{2u^2} \sum_{n \leq \sqrt{u}} n^2 \phi_2\left(\frac{u}{n}\right) + \frac{2}{\sqrt{u}} \phi_2(\sqrt{u}) + \frac{1}{12\sqrt{u}} + Bu^{-1} \quad (u \geq 1).$$

Beweis: Das ist  $T_{IV}$  (4).

Hilfssatz 3:

$$(6) \quad -T(u) = \frac{u}{2} + u \sum_{n \leq \sqrt{u}} \frac{1}{n} \phi\left(\frac{u}{n}\right) + \sum_{n \leq \sqrt{u}} n \phi\left(\frac{u}{n}\right) + 2\sqrt{u} \phi_2(\sqrt{u})$$

$$+ \frac{\sqrt{u}}{12} + Bu^{\frac{1}{3}} \quad (u \geq 1).$$

Beweis: Es sei  $u \geq 1$ ,

$$(7) \quad -T(u) = \frac{u}{2} + u \sum_{n \leq \sqrt{u}} \frac{1}{n} \phi\left(\frac{u}{n}\right) + \sum_{n \leq \sqrt{u}} n \phi\left(\frac{u}{n}\right) + 2\sqrt{u} \phi_2(\sqrt{u})$$

$$+ \frac{\sqrt{u}}{12} + \frac{1}{2u} \sum_{n \leq \sqrt{u}} n^2 \phi_2\left(\frac{u}{n}\right) + Bu^{\frac{1}{3}} \quad (4, 5).$$

$$(8) \quad (1 \leq v \leq \sqrt{u}) \sum_{n \leq v} \phi_2\left(\frac{u}{n}\right) = Bu^{\frac{1}{3}} \quad (T_{IV}, \text{ Schluss des Beweises von (2)}),$$

$$(9) \quad \sum_{n \leq \sqrt{u}} n^2 \phi_2\left(\frac{u}{n}\right) = Bu^{\frac{4}{3}} \quad (8).$$

(6) folgt aus (7) und (9).

Hilfssatz 4: Mit

$$(10) \quad c_{E_1, E_2} = c_{E_1, E_2, v} = \begin{cases} 2^v & \text{für } E_1 = 0, E_2 = 0 \text{ oder für } E_1 = 0, E_2 = 1 \\ 1 & \text{für } E_1 = 1, E_2 = 0 \text{ oder für } E_1 = 1, E_2 = 1, \end{cases}$$

$$(11) \quad r_v = \text{Max}\left(\frac{m^2}{2^v}; n^2\right)$$

ist

$$(12) \quad V_v(x) = \frac{2^v}{12} x^3 + \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \int_{r_v}^x \phi\left(\frac{2^v u}{m}\right) \phi\left(\frac{u}{n}\right) u^{2-E_1-E_2} du$$

$$+ 2^{v+1} \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \int_{\text{Max}\left(\frac{m^2}{2^v}, 1\right)}^x \phi\left(\frac{2^v u}{m}\right) \phi_2(\sqrt{u}) u^{\frac{3}{2}} du$$

$$+ 2^{\frac{v}{2}+1} \sum_{n \leq \sqrt{x}} \frac{1}{n} \int_{\frac{n^2}{2^v}}^x \phi\left(\frac{u}{n}\right) \phi_2(\sqrt{2^v u}) u^{\frac{3}{2}} du + Bx^{\frac{5}{2}}.$$

Beweis: Es sei  $u \geq 1$ .

$$(13) \quad -T(2^v u) = 2^{v-1} u + 2^v u \sum_{m \leq \sqrt{2^v u}} \frac{1}{m} \phi\left(\frac{2^v u}{m}\right) + \sum_{m \leq \sqrt{2^v u}} m \phi\left(\frac{2^v u}{m}\right)$$

$$+ 2^{\frac{v}{2}+1} \sqrt{u} \phi_2(\sqrt{2^v u}) + 2^{\frac{v}{2}} \frac{\sqrt{u}}{12} + B u^{\frac{1}{3}} \quad (6).$$

$$(14) \quad T(2^v u) T(u) = 2^{v-2} u^2 + \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v u} \\ n \leq \sqrt{u}}} m^{2E_1-1} n^{2E_2-1} \phi\left(\frac{2^v u}{m}\right) \phi\left(\frac{u}{n}\right) u^{2-E_1-E_2} \\ + 2^{v+1} \sum_{m \leq \sqrt{2^v u}} \frac{1}{m} \phi\left(\frac{2^v u}{m}\right) \phi_2(\sqrt{u}) u^{\frac{3}{2}} + 2^{\frac{v}{2}+1} \sum_{n \leq \sqrt{u}} \frac{1}{n} \phi\left(\frac{u}{n}\right) \phi_2(\sqrt{2^v u}) u^{\frac{3}{2}} \\ + 2^{v-1} \sum_{n \leq \sqrt{u}} \frac{1}{n} \phi\left(\frac{u}{n}\right) u^2 + 2^{v-1} \sum_{n \leq \sqrt{u}} n \phi\left(\frac{u}{n}\right) u + 2^{v-1} \sum_{m \leq \sqrt{2^v u}} \frac{1}{m} \phi\left(\frac{2^v u}{m}\right) u^2 \\ + \frac{2^v}{12} \sum_{m \leq \sqrt{2^v u}} \frac{1}{m} \phi\left(\frac{2^v u}{m}\right) u^{\frac{3}{2}} + \frac{1}{2} \sum_{m \leq \sqrt{2^v u}} m \phi\left(\frac{2^v u}{m}\right) u + \frac{2^{\frac{v}{2}}}{12} \sum_{n \leq \sqrt{u}} \frac{1}{n} \phi\left(\frac{u}{n}\right) u^{\frac{3}{2}} \\ + B u^{\frac{3}{2}} \quad (13, 6, 10).$$

$$V_v(x) = \int_1^x T(2^v u) T(u) du + B \quad (1)$$

$$= \frac{2^v}{12} x^3 + \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \int_{\tau_v}^x \phi\left(\frac{2^v u}{m}\right) \phi\left(\frac{u}{n}\right) u^{2-E_1-E_2} du \\ + 2^{v+1} \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \int_{\text{Max}\left(\frac{m^2}{2^v}, 1\right)}^x \phi\left(\frac{2^v u}{m}\right) \phi_2(\sqrt{u}) u^{\frac{3}{2}} du \\ + 2^{\frac{v}{2}+1} \sum_{n \leq \sqrt{x}} \frac{1}{n} \int_{\frac{n^2}{2^v}}^x \phi\left(\frac{u}{n}\right) \phi_2(\sqrt{2^v u}) u^{\frac{3}{2}} du \\ + 2^{v-1} \sum_{n \leq \sqrt{x}} \frac{1}{n} \int_{\frac{n^2}{2^v}}^x \phi\left(\frac{u}{n}\right) u^2 du + 2^{v-1} \sum_{n \leq \sqrt{x}} n \int_{\frac{n^2}{2^v}}^x \phi\left(\frac{u}{n}\right) u du$$

$$+ 2^{v-1} \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \int_{\text{Max}\left(\frac{m^2}{2^v}, 1\right)}^x \phi\left(\frac{2^v u}{m}\right) u^2 du + \frac{2^v}{12} \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \int_{\text{Max}\left(\frac{m^2}{2^v}, 1\right)}^x \phi\left(\frac{2^v u}{m}\right) u^{\frac{3}{2}} du \\ + \frac{1}{2} \sum_{m \leq \sqrt{2^v x}} m \int_{\text{Max}\left(\frac{m^2}{2^v}, 1\right)}^x \phi\left(\frac{2^v u}{m}\right) u du + \frac{2^{\frac{v}{2}}}{12} \sum_{n \leq \sqrt{x}} \frac{1}{n} \int_{\frac{n^2}{2^v}}^x \phi\left(\frac{u}{n}\right) u^{\frac{3}{2}} du \\ + B x^{\frac{5}{2}} \quad (14, 11),$$

und hierin sind nach (2) die sechs letzten Summen, wenn über geeignete Strecken integriert wird,

$$= B x^3 \sum_{n \leq \sqrt{x}} \frac{1}{n} \left| \int \phi\left(\frac{u}{n}\right) du \right| + B x \sum_{n \leq \sqrt{x}} n \left| \int \phi\left(\frac{u}{n}\right) du \right| \\ + B x^3 \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \left| \int \phi\left(\frac{2^v u}{m}\right) du \right| + B x^{\frac{3}{2}} \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \left| \int \phi\left(\frac{2^v u}{m}\right) du \right| \\ + B x \sum_{m \leq \sqrt{2^v x}} m \left| \int \phi\left(\frac{2^v u}{m}\right) du \right| + B x^{\frac{3}{2}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \left| \int \phi\left(\frac{u}{n}\right) du \right| \\ = B x^3 \sum_{n \leq \sqrt{x}} 1 + B x \sum_{n \leq \sqrt{x}} n^2 + B x^2 \sum_{m \leq \sqrt{2^v x}} 1 + B x^{\frac{3}{2}} \sum_{m \leq \sqrt{2^v x}} 1 \\ + B x \sum_{m \leq \sqrt{2^v x}} m^2 + B x^{\frac{3}{2}} \sum_{n \leq \sqrt{x}} 1 = B x^{\frac{5}{2}}.$$

Hilfssatz 5: Sind in dem Intervall  $u_1 \leq u \leq u_2$  die quadratisch integrierbaren Funktionen  $g(u)$ ,  $h(u)$ ,  $s_k(u)$ ,  $t_k(u)$  ( $k \geq 1$ ) gegeben, für die bei wachsendem  $k$

$$\int_{u_1}^{u_2} s_k(u) du, \int_{u_1}^{u_2} t_k(u) du \quad \text{beschränkt,}$$

$$\int_{u_1}^{u_2} (g(u) - s_k(u))^2 du \rightarrow 0, \int_{u_1}^{u_2} (h(u) - t_k(u))^2 du \rightarrow 0,$$

so ist

$$(15) \quad \int_{u_1}^{u_2} g(u) h(u) du = \lim_{k \rightarrow \infty} \int_{u_1}^{u_2} s_k(u) t_k(u) du.$$

Beweis: Das ist Hilfssatz 10 von  $T_{II}$ .

Hilfssatz 6:

$$(16) \quad \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \int_{\text{Max} \left( \frac{m^2}{2^v}; 1 \right)}^x \psi \left( \frac{2^v u}{m} \right) \psi_2(\sqrt{u}) u^{\frac{3}{2}} du$$

$$= -\frac{1}{2\pi^3} \lim_{k \rightarrow \infty} \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \sum_{a, b=1}^k \frac{1}{a b^2} \left( \int_{\text{Max} \left( \frac{m^2}{2^v}; 1 \right)}^x u^{\frac{3}{2}} \sin \left\{ 2\pi \left( \frac{2^v a u}{m} + b \sqrt{u} \right) \right\} du \right.$$

$$\left. + \int_{\text{Max} \left( \frac{m^2}{2^v}; 1 \right)}^x u^{\frac{3}{2}} \sin \left\{ 2\pi \left( \frac{2^v a u}{m} - b \sqrt{u} \right) \right\} du \right),$$

$$(17) \quad \sum_{n \leq \sqrt{x}} \frac{1}{n} \int_{\frac{n^2}{2^v}}^x \psi \left( \frac{u}{n} \right) \psi_2(\sqrt{2^v u}) u^{\frac{3}{2}} du$$

$$= -\frac{1}{2\pi^3} \lim_{k \rightarrow \infty} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{a, b=1}^k \frac{1}{a b^2} \left( \int_{\frac{n^2}{2^v}}^x u^{\frac{3}{2}} \sin \left\{ 2\pi \left( \frac{a u}{n} + b \sqrt{2^v u} \right) \right\} du \right.$$

$$\left. + \int_{\frac{n^2}{2^v}}^x u^{\frac{3}{2}} \sin \left\{ 2\pi \left( \frac{a u}{n} - b \sqrt{2^v u} \right) \right\} du \right).$$

Beweis: 1) Mit  $u_1 = \text{Max} \left( \frac{m^2}{2^v}; 1 \right)$ ,  $u_2 = x$ ,  $g(u) = \psi \left( \frac{2^v u}{m} \right)$ ,

$$h(u) = \psi_2(\sqrt{u}) u^{\frac{3}{2}},$$

$$(18) \quad s_k(u) = -\frac{1}{\pi} \sum_{a=1}^k \frac{1}{a} \sin \frac{2\pi a 2^v u}{m}, t_k(u) = \frac{u}{\pi^2} \sum_{b=1}^k \frac{1}{b^2} \cos \{ 2\pi b \sqrt{u} \}$$

sind wegen (2) die Bedingungen von Hilfssatz 5 erfüllt. Daher liefert (15) für die linke Seite von (16)

$$-\frac{1}{\pi^3} \lim_{k \rightarrow \infty} \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \sum_{a, b=1}^k \frac{1}{a b^2} \int_{\text{Max} \left( \frac{m^2}{2^v}; 1 \right)}^x u^{\frac{3}{2}} \sin \frac{2\pi a 2^v u}{m} \cos \{ 2\pi b \sqrt{u} \} du,$$

was der rechten Seite von (16) gleich ist.

$$2) \text{ Mit } u_1 = n^2, u_2 = x, g(u) = \psi \left( \frac{u}{n} \right), h(u) = \psi_2(\sqrt{2^v u}) u^{\frac{3}{2}},$$

$$(19) \quad s_k(u) = -\frac{1}{\pi} \sum_{a=1}^k \frac{1}{a} \sin \frac{2\pi a u}{n}, t_k(u) = \frac{u}{\pi^2} \sum_{b=1}^k \frac{1}{b^2} \cos \{ 2\pi b \sqrt{2^v u} \}$$

sind wegen (2) die Bedingungen von Hilfssatz 4 erfüllt. Daher liefert (15) für die linke Seite von (17)

$$-\frac{1}{\pi^3} \lim_{k \rightarrow \infty} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{a, b=1}^k \frac{1}{a b^2} \int_{\frac{n^2}{2^v}}^x u^{\frac{3}{2}} \sin \frac{2\pi a u}{n} \cos \{ 2\pi b \sqrt{2^v u} \} du,$$

was der rechten Seite von (17) gleich ist.

Hilfssatz 7: Für  $v=0, 1, 2, 3, 4$  und  $v'=1, 2, 3, 4$  ist

$$(20) \quad S_{v, v'}^{\pm} = \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \sum_{a, b=1}^{\infty} \frac{1}{a b^3} \left| \int_{\text{Max} \left( \frac{m^2}{2^v}; 1 \right)}^x u^{\frac{3}{2}} \sin \left\{ 2\pi \left( \frac{2^v a u}{m} \pm b \sqrt{2^{v'} u} \right) \right\} du \right|$$

$$(21) \quad = B x^{\frac{9}{4}}.$$

Beweis: Es sei zur Abkürzung (nur für den vorliegenden Beweis)

$$(22) \quad M^2 = \text{Max} \left( \frac{m^2}{2^v}; 1 \right), M > 0; \rho = \frac{2^{v+1} \pi a}{m}, \omega = \frac{2^{\frac{v'}{2}-v-1} b m}{a}.$$

$$S_{v, v'}^{\pm} = 2 \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \sum_{a, b=1}^{\infty} \frac{1}{a b^2} \left| \int_M^{\sqrt{x}} u^4 \sin \left\{ 2\pi \left( \frac{2^v a u^2}{m} \pm b 2^{\frac{v'}{2}} u \right) \right\} du \right| \quad (20, 22)$$

$$\leq 2 \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \sum_{a, b=1}^{\infty} \frac{1}{a b^2} \left| \int_M^{\sqrt{x}} u^4 \exp \left\{ 2\pi i \left( \frac{2^v a u^2}{m} \pm b 2^{\frac{v'}{2}} u \right) \right\} du \right|$$

$$= 2 \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \sum_{a, b=1}^{\infty} \frac{1}{a b^2} \left| \int_M^{\sqrt{x}} u^4 e^{i \rho \frac{(u \pm \omega)^2}{2}} du \right| \quad (22)$$

$$(23) = 2 \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \left( \sum_{a, b=1}^{\infty} + \sum_{a, b=1}^{\infty} + \sum_{a, b=1}^{\infty} \right) \frac{1}{a b^2} \left| \int_M^{\sqrt{x}} u^4 e^{i \rho \frac{(u \pm \omega)^2}{2}} du \right|$$

$$(24) = S' + S'' + S'''.$$

$$\left( \omega \leq \frac{M}{2} \right) \int_M^{\sqrt{x}} u^4 e^{i \rho \frac{(u \pm \omega)^2}{2}} du = \frac{1}{2 i \rho} \int_M^{\sqrt{x}} \frac{u^4}{u \pm \omega} d e^{i \rho \frac{(u \pm \omega)^2}{2}}$$

$$(25) = \frac{B x^2}{\rho M} = \frac{B x^2}{\rho m} = \frac{B x^2}{a} \quad (22),$$

$$(26) S' = B x^2 \sum_{m \leq \sqrt{2^v x}} \frac{1}{m} \sum_{a, b=1}^{\infty} \frac{1}{a^2 b^2} = B x^2 \log x \quad (23-25).$$

$$\left( \frac{M}{2} < \omega < \sqrt{2x} \right) \int_M^{\sqrt{x}} u^4 e^{i \rho \frac{(u \pm \omega)^2}{2}} du = \int_M^{\sqrt{x}} + \int_M^{\sqrt{x}} \\ \left| \int \cos(\rho u^2) du \right| + \left| \int \sin(\rho u^2) du \right| + \frac{1}{2 i \rho} \int_M^{\sqrt{x}} \frac{u^4}{u \pm \omega} d e^{i \rho \frac{(u \pm \omega)^2}{2}}$$

$$(27) = B(\omega^4 + x^2) \rho^{-\frac{1}{2}} + B \rho^{-1} x^{2-\frac{1}{4}} = B x^2 m^{\frac{1}{2}} a^{-\frac{1}{2}} + B x^{\frac{7}{4}} m a^{-1} \quad (22).$$

$$S'' = B x^2 \sum_{m \leq \sqrt{2^v x}} m^{-\frac{1}{2}} \sum_{a=1}^{\infty} a^{-\frac{3}{2}} \sum_{b=1}^{\infty} b^{-2} + B x^{\frac{7}{4}} \sum_{m \leq \sqrt{2^v x}} 1 \cdot \sum_{a, b=1}^{\infty} (a b)^{-2}$$

$$(28) = B x^{\frac{9}{4}} \quad (23, 24, 27)'$$

$$\left( \omega \geq \sqrt{2x} \right) \int_M^{\sqrt{x}} u^4 e^{i \rho \frac{(u \pm \omega)^2}{2}} du = \frac{1}{2 i \rho} \int_M^{\sqrt{x}} \frac{u^4}{u \pm \omega} d e^{i \rho \frac{(u \pm \omega)^2}{2}}$$

$$(29) = B \rho^{-1} x^{2-\frac{1}{2}} = B x^{\frac{3}{2}} m a^{-1} \quad (22).$$

$$(30) S''' = B x^{\frac{3}{2}} \sum_{m \leq \sqrt{2^v x}} 1 \cdot \sum_{a, b=1}^{\infty} (a b)^{-2} = B x^2 \quad (23, 24, 29).$$

(21) folgt aus (24), (26), (28) und (30).

Hilfssatz 8:

$$(31) V_v(x) = \frac{2^v}{12} x^3 + \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \int_{r_v}^x \psi \left( \frac{2^v u}{m} \right) \psi \left( \frac{u}{n} \right) u^{2-E_1-E_2} du \\ + B x^{\frac{5}{2}}.$$

Beweis: (12), (16), (17), (21) mit  $v'=0$ , (21) mit  $v=0$ ,  $v'=v$ ,  $m=n$ .

Hilfssatz 9:

$$(32) \sum_{\substack{m, n=1 \\ (m, n)=1}}^{\infty} \frac{1}{m^2 n^2} = \frac{5}{2},$$

$$(33) \sum_{\substack{m, n=1 \\ (m, n)=1, m \equiv 1 \pmod{2}}}^{\infty} \frac{1}{m^2 n^2} = 2,$$

$$(34) \sum_{\substack{m=1 \\ (m, n)=1}}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \sum_{s|n} \frac{\mu(s)}{s^2},$$

$$(35) \quad \sum_{\substack{m=1 \\ (m, n)=1, m \equiv 1 \pmod{2}}}^{\infty} \frac{1}{m^2} = \begin{cases} \frac{\pi^2}{8} \sum_{s|n} \frac{\mu(s)}{s^2} & \text{für } n \equiv 1 \pmod{2} \\ \frac{\pi^2}{6} \sum_{s|n} \frac{\mu(s)}{s^2} & \text{für } n \equiv 0 \pmod{2}, \end{cases}$$

$$(36) \quad \sum_{s=1 \pmod{2}}^{\infty} \frac{\mu(s)}{s^3} = \frac{8}{7 \zeta(3)},$$

$$(37) \quad \sum_{s \equiv 1 \pmod{2}}^{\infty} \frac{\mu(s)}{s^3} = -\frac{1}{7 \zeta(3)},$$

$$(38) \quad \sum_{m=1}^{\infty} \frac{\varphi(m)}{m^3} = \frac{\pi^2}{6 \zeta(3)},$$

$$(39) \quad \sum_{m=1 \pmod{2}}^{\infty} \frac{\varphi(m)}{m^3} = \frac{\pi^2}{7 \zeta(3)},$$

$$(40) \quad \sum_{n=1}^{\infty} \frac{\varphi(2n)}{n^3} = \frac{4\pi^2}{21 \zeta(3)}.$$

Beweis: (32) ist  $T_{II}$  (39), (33) ist  $T_{II}$  (115).

Alle Kongruenzen im folgenden seien mod 2 gemeint,  $p$  möge über Primzahlen laufen.

$$\sum_{\substack{m=1 \\ (m, n)=1}}^{\infty} \frac{1}{m^2} = \prod_{p \nmid n} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p|n} \left(1 - \frac{1}{p^2}\right) = \frac{\pi^2}{6} \sum_{s|n} \frac{\mu(s)}{s^2}.$$

$$(n \equiv 1) \quad \sum_{\substack{m=1 \\ (m, n)=1, m \equiv 1 \pmod{2}}}^{\infty} \frac{1}{m^2} = \prod_{\substack{p \nmid n \\ p \neq 2}} \left(1 - \frac{1}{p^2}\right)^{-1} = \left(1 - \frac{1}{2^2}\right)^{-1} \prod_{p|n} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p|n} \left(1 - \frac{1}{p^2}\right) \\ = \frac{\pi^2}{8} \sum_{s|n} \frac{\mu(s)}{s^2},$$

$$(n \equiv 0) \quad \sum_{\substack{m=1 \\ (m, n)=1, m \equiv 1 \pmod{2}}}^{\infty} \frac{1}{m^2} = \prod_{p \nmid n} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$$

$$= \frac{\pi^2}{6} \sum_{s|n} \frac{\mu(s)}{s^2},$$

$$\sum_{s=1}^{\infty} \frac{\mu(s)}{s^3} = \prod_{p \geq 2} \left(1 - \frac{1}{p^3}\right)^{-1} = \left(1 - \frac{1}{2^3}\right)^{-1} \prod_p \left(1 - \frac{1}{p^3}\right)^{-1} = \frac{8}{7 \zeta(3)},$$

$$\sum_{\substack{s=1 \\ s \neq 0}}^{\infty} \frac{\mu(s)}{s^3} = \frac{1}{\zeta(3)} - \frac{8}{7 \zeta(3)} = -\frac{1}{7 \zeta(3)} \quad (36),$$

$$\sum_{m=1}^{\infty} \frac{\varphi(m)}{m^3} = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{1}{p^3}\right) = \frac{\pi^2}{8 \zeta(3)},$$

$$\sum_{\substack{m=1 \\ m \neq 0}}^{\infty} \frac{\varphi(m)}{m^3} = \prod_{p \geq 2} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{1}{p^3}\right) \\ = \frac{6}{7} \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{1}{p^3}\right) = \frac{\pi^2}{7 \zeta(3)},$$

$$\sum_{n=1}^{\infty} \frac{\varphi(2n)}{n^3} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} + 2 \sum_{n=0}^{\infty} \frac{\varphi(n)}{n^3} = 2 \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} - \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} = \frac{4\pi^2}{21 \zeta(3)}$$

(38, 39).

Hilfssatz 10:

$$(41) \quad \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{(m, 2^v n)^3}{m^2 n^2} =$$

$$= \frac{5\pi^2}{12} + \frac{v\pi^2}{4} - \frac{\pi^2}{\zeta(3)} \left( \frac{1+2^{\frac{v}{2}}}{12} + \frac{v}{28} + \frac{2^{\frac{v}{2}}-2^{-\frac{v}{2}}}{42} \right) \log x + \frac{B}{\sqrt{x}}.$$

Beweis: Ich bezeichne mit  $m_0, m_1, \dots, m_v$  die Restklassen

$$(42) \quad m_0 \equiv 0 \pmod{2^v}; m_\rho \equiv 2^{\rho-1} \pmod{2^\rho} \text{ für } 1 \leq \rho \leq v.$$

Kongruenzen, deren Modul nicht angegeben ist, sind mod 2 gemeint.

$\rho$  sei eine ganze Zahl mit  $1 \leq \rho \leq v$ . Dieselben Festsetzungen mögen auch beim Beweise des nächsten Hilfssatzes gelten.

$$\begin{aligned}
& \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{(m, 2^v n)^2}{m^2 n^2} = \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r}} \frac{1}{m^2 n^2} \\
(43) \quad & = \sum_{\nu=0}^v \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{\substack{m, n \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r}} \frac{1}{m^2 n^2} = \sum_{\nu=0}^v S_{\nu} = S_0 + \sum_{\nu=1}^v S_{\nu}. \\
& S_0 = \sum_{\substack{r \leq \sqrt{2^v x} \\ r \equiv 0 \pmod{2^v}}} r^2 \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r, m \equiv 0 \pmod{2^v}}} \frac{1}{m^2 n^2} \quad (42, 43) \\
& = \sum_{r \leq \sqrt{x/2^v}} 2^{2\nu} r^2 \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n) = r}} \frac{1}{2^{2\nu} m^2 n^2} \\
& \text{(ich habe } m, n, r \text{ durch } 2^v m, n, 2^v r \text{ ersetzt)} \\
& = \sum_{r \leq \sqrt{x/2^v}} r^2 \sum_{\substack{m \leq \frac{1}{r} \sqrt{x/2^v}, n \leq \frac{1}{r} \sqrt{x} \\ (m, n) = 1}} \frac{1}{r^4 m^2 n^2} \\
& \text{(ich habe } m, n, r \text{ durch } r m, r n, r \text{ ersetzt)} \\
& = \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n) = 1}} \frac{1}{m^2 n^2} \sum_{r \leq \text{Min}\left(\frac{1}{m} \sqrt{x/2^v}, \frac{1}{n} \sqrt{x}\right)} \frac{1}{r^2} \\
(44) \quad & = \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n) = 1}} \frac{1}{m^2 n^2} \sum_{r=1}^{\infty} \frac{1}{r^2} - \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n) = 1}} \frac{1}{m^2 n^2} \sum_{r > \sqrt{x} \text{ Min}\left(\frac{1}{m} \sqrt{x/2^v}, \frac{1}{n} \sqrt{x}\right)} \frac{1}{r^2} \\
(45) \quad & = S_{01} - S_{02}. \\
(46) \quad & S_{01} = \frac{\pi^2}{6} \sum_{\substack{m, n=1 \\ (m, n)=1}}^{\infty} \frac{1}{m^2 n^2} + B \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m > \sqrt{x/2^v}} \frac{1}{m^2} = \frac{5\pi^2}{12} + \frac{B}{\sqrt{x}} \quad (44, 45, 32). \\
& S_{02} = \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n)=1}} \frac{1}{m^2 n^2} \frac{\text{Max}(m \sqrt{2^v}, n)}{\sqrt{x}} + B \sum_{m, n \leq \sqrt{x}} \frac{1}{m^2 n^2} \frac{\text{Max}(m^2, n^2)}{x} \\
& \quad (44, 45)
\end{aligned}$$

$$\begin{aligned}
& = \frac{1}{\sqrt{x}} \sum_{\substack{m \sqrt{2^v} \leq n \leq \sqrt{x} \\ (m, n)=1}} \frac{1}{m^2 n} + \sqrt{\frac{2^v}{x}} \sum_{\substack{n \leq m \sqrt{2^v} \leq \sqrt{x} \\ (m, n)=1}} \frac{1}{m n^2} \\
& + \frac{B}{\sqrt{x}} \sum_{\substack{m \sqrt{2^v} \leq n \leq \sqrt{x} \\ (m, n)=1}} \frac{1}{m^3} + \frac{B}{x} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n \leq \sqrt{x}} 1 \\
& = \frac{1}{\sqrt{x}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{\substack{m \leq \frac{n}{\sqrt{2^v}} \\ (m, n)=1}} \frac{1}{m^2} + \sqrt{\frac{2^v}{x}} \sum_{n \leq \sqrt{x/2^v}} \frac{1}{n} \sum_{\substack{m \leq n \sqrt{2^v} \\ (m, n)=1}} \frac{1}{m^2} + \frac{B}{\sqrt{x}}
\end{aligned}$$

(im zweiten Glied habe ich  $m, n$  durch  $n, m$  ersetzt)

$$\begin{aligned}
& = \frac{1}{\sqrt{x}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \left( \sum_{\substack{m=1 \\ (m, n)=1}}^{\infty} \frac{1}{m^2} + \frac{B}{n} \right) + \sqrt{\frac{2^v}{x}} \sum_{n \leq \sqrt{x/2^v}} \frac{1}{n} \left( \sum_{\substack{m=1 \\ (m, n)=1}}^{\infty} \frac{1}{m^2} + \frac{B}{n} \right) + \frac{B}{\sqrt{x}} \\
& = \frac{\pi^2}{6 \sqrt{x}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{s|n} \frac{\mu(s)}{s^2} + \frac{\pi^2}{6} \sqrt{\frac{2^v}{x}} \sum_{n \leq \sqrt{x/2^v}} \frac{1}{n} \sum_{s|n} \frac{\mu(s)}{s^2} + \frac{B}{\sqrt{x}} \quad (34) \\
& = \frac{\pi^2}{6 \sqrt{x}} \sum_{sn \leq \sqrt{x}} \frac{1}{n} \frac{\mu(s)}{s^3} + \frac{\pi^2}{6} \sqrt{\frac{2^v}{x}} \sum_{sn \leq \sqrt{x/2^v}} \frac{1}{n} \frac{\mu(s)}{s^3} + \frac{B}{\sqrt{x}}
\end{aligned}$$

(ich habe  $n, s$  durch  $sn, s$  ersetzt)

$$\begin{aligned}
& = \frac{\pi^2}{6 \sqrt{x}} \sum_{s \leq \sqrt{x}} \frac{\mu(s)}{s^3} \sum_{n \leq \sqrt{x/s}} \frac{1}{n} + \frac{\pi^2}{6} \sqrt{\frac{2^v}{x}} \sum_{s \leq \sqrt{x/2^v}} \frac{\mu(s)}{s^3} \sum_{n \leq \frac{1}{s} \sqrt{x/2^v}} \frac{1}{n} + \frac{B}{\sqrt{x}} \\
& = \frac{\pi^2}{6 \sqrt{x}} \sum_{s \leq \sqrt{x}} \frac{\mu(s)}{s^3} \left( \frac{1}{2} \log x + B \log 2 s \right) + \\
& + \frac{\pi^2}{6} \sqrt{\frac{2^v}{x}} \sum_{s \leq \sqrt{x/2^v}} \frac{\mu(s)}{s^3} \left( \frac{1}{2} \log x + B \log 2 s \right) + \frac{B}{\sqrt{x}}
\end{aligned}$$



$$\begin{aligned}
&= \frac{\pi^2 \log x}{12 \sqrt{x}} \left( \sum_{s \leq \sqrt{x}} \frac{\mu(s)}{s^3} + \sqrt{2^v} \sum_{s \leq \sqrt{x/2^v}} \frac{\mu(s)}{s^3} \right) + \frac{B}{\sqrt{x}} \\
&= \frac{\pi^2 \log x}{12 \sqrt{x}} (1 + \sqrt{2^v}) \left( \sum_{s=1}^{\infty} \frac{\mu(s)}{s^3} + \frac{B}{x} \right) + \frac{B}{\sqrt{x}} \\
(47) \quad &= \frac{\pi^2}{12 \zeta(3)} \left( 1 + 2^{\frac{v}{2}} \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}}.
\end{aligned}$$

$$(48) \quad S_0 = \frac{5\pi^2}{12} - \frac{\pi^2}{12 \zeta(3)} \left( 1 + 2^{\frac{v}{2}} \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}} \quad (45-47).$$

$$S_p = \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{\substack{m_p \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m_p, 2^n n) = r}} \frac{1}{m_p^2 n^2} \quad (43)$$

$$= \sum_{\substack{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x} \\ r=1}} r^2 \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n)=r, m=1}} \frac{1}{m^2 n^2} \quad (42)$$

(ich habe  $m_p, n, r$  durch  $2^{p-1}m, n, 2^{p-1}r$  ersetzt)

$$= \sum_{\substack{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x} \\ r=1}} \frac{1}{r^2} \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2 n^2}$$

$$= \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2 n^2} \sum_{\substack{r \leq \min \left( 2^{\frac{v}{2}-p+1} \sqrt{x}, \frac{\sqrt{x}}{n} \right) \\ r=1}} \frac{1}{r^2}$$

$$\begin{aligned}
(49) \quad &= \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2 n^2} \sum_{r=1}^{\infty} \frac{1}{r^2} \\
&\quad - \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2 n^2} \sum_{r > \sqrt{x} \min \left( 2^{\frac{v}{2}-p+1} \sqrt{x}, \frac{1}{n} \right)} \frac{1}{r^2}
\end{aligned}$$

$$(50) = S_{p1} - S_{p2}.$$

$$S_{p1} = \frac{\pi^2}{8} \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2 n^2} \quad (49, 50)$$

$$(51) \quad = \frac{\pi^2}{8} \sum_{\substack{m, n=1 \\ (m, n)=1, m=1}}^{\infty} \frac{1}{m^2 n^2} + \frac{B}{\sqrt{x}} = \frac{\pi^2}{4} + \frac{B}{\sqrt{x}} \quad (33).$$

$$S_{p2} = \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2 n^2} \frac{\text{Max} \left( 2^{\frac{v}{2}-p-1} m; n \right)}{2 \sqrt{x}}$$

$$+ \frac{B}{x} \sum_{m, n \leq \sqrt{2^v x}} \frac{1}{m^2 n^2} \text{Max} (m^2; n^2) \quad (49, 50)$$

$$= \frac{1}{2 \sqrt{x}} \sum_{\substack{m \leq 2^{\frac{v}{2}-1} \sqrt{x} \\ (m, n)=1, m=1}} \sum_{\substack{n \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2 n} + \frac{1}{2 \sqrt{x}} \sum_{\substack{n \leq 2^{\frac{v}{2}-1} \sqrt{x} \\ (m, n)=1, m=1}} \sum_{\substack{m \leq \sqrt{x} \\ (m, n)=1, m=1}} \frac{2^{\frac{v}{2}-p-1}}{m n^2} + \frac{B}{\sqrt{x}}$$

$$\begin{aligned}
&= \frac{1}{2 \sqrt{x}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{m^2} + \frac{2^{\frac{v}{2}-2}}{\sqrt{x}} \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x} \\ m=1}} \frac{1}{n} \sum_{\substack{m \leq 2^{\frac{v}{2}-p-1} \sqrt{x} \\ (m, n)=1, m=1}} \frac{1}{n} \\
&\quad + \frac{B}{\sqrt{x}}
\end{aligned}$$

$$= \frac{1}{2 \sqrt{x}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{\substack{m=1 \\ (m, n)=1, m=1}}^{\infty} \frac{1}{m^2} + \frac{2^{\frac{v}{2}-2}}{\sqrt{x}} \sum_{\substack{n \leq 2^{\frac{v}{2}-p+1} \sqrt{x} \\ n=1}} \frac{1}{n} \sum_{\substack{m=1 \\ (m, n)=1, m=1}}^{\infty} \frac{1}{m^2} + \frac{B}{\sqrt{x}}$$

$$= \frac{\pi^2}{12 \sqrt{x}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{s|n} \frac{\mu(s)}{s^2} + \frac{\pi^2}{16 \sqrt{x}} \sum_{n \leq \sqrt{x}} \frac{1}{n} \sum_{s|n} \frac{\mu(s)}{s^2}$$

$$\begin{aligned}
& + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24 \sqrt{x}} \sum_{\substack{n \leq \frac{\nu}{2} \\ n=1}} \frac{1}{n} \sum_{s|n} \frac{\mu(s)}{s^2} + \frac{B}{\sqrt{x}} \quad (35, 34) \\
& = \frac{\pi^2}{12 \sqrt{x}} \sum_{\substack{sn \leq \sqrt{x} \\ sn=0}} \frac{1}{n} \frac{\mu(s)}{s^3} + \frac{\pi^2}{16 \sqrt{x}} \sum_{\substack{sn \leq \sqrt{x} \\ sn=1}} \frac{1}{n} \frac{\mu(s)}{s^3} \\
& + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24 \sqrt{x}} \sum_{\substack{sn \leq 2^{\frac{\nu}{2}-\rho+1} \sqrt{x} \\ sn=1}} \frac{1}{n} \frac{\mu(s)}{s^3} + \frac{B}{\sqrt{x}} \\
& = \frac{\pi^2}{12 \sqrt{x}} \sum_{\substack{s \leq \sqrt{x} \\ s=1}} \frac{\mu(s)}{s^3} \sum_{\substack{n \leq \frac{\sqrt{x}}{s} \\ n=0}} \frac{1}{n} + \frac{\pi^2}{12 \sqrt{x}} \sum_{\substack{s \leq \sqrt{x} \\ s=0}} \frac{\mu(s)}{s^3} \sum_{\substack{n \leq \frac{\sqrt{x}}{s} \\ n=1}} \frac{1}{n} \\
& + \frac{\pi^2}{16 \sqrt{x}} \sum_{\substack{s \leq \sqrt{x} \\ s=1}} \frac{\mu(s)}{s^3} \sum_{\substack{n \leq \frac{\sqrt{x}}{s} \\ n=1}} \frac{1}{n} + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24 \sqrt{x}} \sum_{\substack{s \leq 2^{\frac{\nu}{2}-\rho+1} \sqrt{x} \\ s=1}} \frac{\mu(s)}{s^3} \sum_{\substack{n \leq 2^{\frac{\nu}{2}-\rho+1} \frac{\sqrt{x}}{s} \\ n=1}} \frac{1}{n} \\
& + \frac{B}{\sqrt{x}} \\
& = \frac{\pi^2}{12 \sqrt{x}} \sum_{\substack{s \leq \sqrt{x} \\ s=1}} \frac{\mu(s)}{s^3} \left( \frac{1}{2} \log \sqrt{x} + B \log 2s \right) \\
& + \frac{\pi^2}{12 \sqrt{x}} \sum_{\substack{s \leq \sqrt{x} \\ s=0}} \frac{\mu(s)}{s^3} \left( \log \sqrt{x} + B \log 2s \right) \\
& + \frac{\pi^2}{16 \sqrt{x}} \sum_{\substack{s \leq \sqrt{x} \\ s=1}} \frac{\mu(s)}{s^3} \left( \frac{1}{2} \log \sqrt{x} + B \log 2s \right) \\
& + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24 \sqrt{x}} \sum_{\substack{s \leq 2^{\frac{\nu}{2}-\rho+1} \sqrt{x} \\ s=1}} \frac{\mu(s)}{s^3} \left( \frac{1}{2} \log \sqrt{x} + B \log 2s \right) + \frac{B}{\sqrt{x}}
\end{aligned}$$

$$\begin{aligned}
& = \left( \frac{\pi^2}{48} \sum_{s=1}^{\infty} \frac{\mu(s)}{s^3} + \frac{\pi^2}{24} \sum_{s=0}^{\infty} \frac{\mu(s)}{s^3} + \frac{\pi^2}{64} \sum_{s=1}^{\infty} \frac{\mu(s)}{s^3} + \right. \\
& \quad \left. + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{96} \sum_{s=1}^{\infty} \frac{\mu(s)}{s^3} \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}} \\
(52) \quad & = \left( \frac{\pi^2}{28 \zeta(3)} + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{84 \zeta(3)} \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}} \quad (36, 37).
\end{aligned}$$

$$S_\rho = \frac{\pi^2}{4} - \left( \frac{\pi^2}{28 \zeta(3)} + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{84 \zeta(3)} \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}} \quad (50-52),$$

$$\begin{aligned}
(53) \quad \sum_{\rho=1}^{\nu} S_\rho &= \frac{\nu \pi^2}{4} - \left( \frac{\nu \pi^2}{28 \zeta(3)} + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{84 \zeta(3)} (2^{\nu+1} - 2) \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}} \\
&= \frac{\nu \pi^2}{4} - \frac{\pi^2}{\zeta(3)} \left( \frac{\nu}{28} + \frac{2^{\frac{\nu}{2}+1} - 2^{1-\frac{\nu}{2}}}{84} \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}}.
\end{aligned}$$

(41) folgt aus (43), (48) und (53).

Hilfssatz 11:

$$(54) \quad \sum_{\substack{m \leq \sqrt{2^\nu x} \\ n \leq \sqrt{x}}} \frac{(m, 2^\nu n)^2}{m^2} = \left( \frac{\pi^2}{12 \zeta(3)} + \frac{\nu \pi^2}{28 \zeta(3)} \right) \sqrt{x} \log x + B \sqrt{x},$$

$$(55) \quad \sum_{\substack{m \leq \sqrt{2^\nu x} \\ n \leq \sqrt{x}}} \frac{(m, 2^\nu n)^2}{n^2} = \left( \frac{2^{\frac{3}{2}\nu} \cdot 3 \pi^2}{28 \zeta(3)} - \frac{2^{\frac{\nu}{2}} \pi^2}{42 \zeta(3)} \right) \sqrt{x} \log x + B \sqrt{x}.$$

Beweis:

$$\sum_{\substack{m \leq \sqrt{2^\nu x} \\ n \leq \sqrt{x}}} \frac{(m, 2^\nu n)^2}{m^2} = \sum_{r \leq \sqrt{2^\nu x}} r^2 \sum_{\substack{m \leq \sqrt{2^\nu x} \\ (m, 2^\nu n)=r}} \frac{1}{m^2}$$

$$(56) \quad = \sum_{\mu=0}^v \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{\substack{m, n \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r}} \frac{1}{m^2} = \sum_{\mu=0}^v S'_{\mu} \quad (42).$$

$$S'_0 = \sum_{\substack{r \leq \sqrt{2^v x} \\ r \equiv 0 \pmod{2^v}}} r^2 \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r, m \equiv 0 \pmod{2^v}}} \frac{1}{m^2} \quad (56, 42)$$

$$= \sum_{r \leq \sqrt{x/2^v}} 2^{2v} r^2 \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n) = r}} \frac{1}{2^{2v} m^2} = \sum_{r \leq \sqrt{x/2^v}} \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n) = r}} \frac{1}{m^2}$$

$$= \sum_{r \leq \sqrt{x/2^v}} \sum_{m \leq \frac{1}{r} \sqrt{x/2^v}} \frac{1}{m^2} \sum_{\substack{n \leq \frac{\sqrt{x}}{r} \\ (n, m) = 1}} 1$$

$$= \sum_{r \leq \sqrt{x/2^v}} \sum_{m \leq \frac{1}{r} \sqrt{x/2^v}} \frac{1}{m^2} \frac{\sqrt{x}}{r m} \varphi(m) + B \sum_{r \leq \sqrt{x}} \sum_{m \leq \frac{\sqrt{x}}{r}} \frac{1}{m^2} m$$

(jedes Intervall der Länge  $m$  enthält  $\varphi(m)$  zu  $m$  teilerfremde  $n$ )

$$= \sqrt{x} \sum_{r \leq \sqrt{x/2^v}} \frac{1}{r} \sum_{m=1}^{\infty} \frac{\varphi(m)}{m^3} + B \sqrt{x} \sum_{r \leq \sqrt{x}} \frac{1}{r} \sum_{m > \frac{1}{r} \sqrt{x/2^v}} \frac{\varphi(m)}{m^3} \\ + B \sum_{m \leq \sqrt{x}} \frac{1}{m} \sum_{r \leq \frac{\sqrt{x}}{m}} 1$$

$$= \frac{\pi^2}{6 \zeta(3)} \sqrt{x} \sum_{r \leq \sqrt{x/2^v}} \frac{1}{r} + B \sqrt{x} \sum_{r \leq \sqrt{x}} \frac{1}{r} \frac{r}{\sqrt{x}} + B \sqrt{x} \quad (38)$$

$$(57) \quad = \frac{\pi^2}{12 \zeta(3)} \sqrt{x} \log x + B \sqrt{x}.$$

$$S'_\mu = \sum_{r=1}^{\frac{v}{2} - \mu + 1} r^2 \sum_{\substack{m \leq 2^{\frac{v}{2} - \mu + 1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n) = r, m \equiv 1 \pmod{2}}} \frac{1}{m^2} \quad (56, 42)$$

$$= \sum_{r=1}^{\frac{v}{2} - \mu + 1} \sum_{\substack{m \leq 2^{\frac{v}{2} - \mu + 1} \sqrt{x}, n \leq \frac{\sqrt{x}}{r} \\ (m, n) = 1, m \equiv 1 \pmod{2}}} \frac{1}{m^2}$$

$$= \sum_{r=1}^{\frac{v}{2} - \mu + 1} \sum_{m=1}^{\frac{v}{2} - \mu + 1} \frac{1}{m^2} \sum_{\substack{n \leq \frac{\sqrt{x}}{r} \\ (n, m) = 1}} 1$$

$$= \sum_{r=1}^{\frac{v}{2} - \mu + 1} \sum_{m=1}^{\frac{v}{2} - \mu + 1} \frac{1}{m^2} \left( \frac{\sqrt{x}}{r m} \varphi(m) + B m \right)$$

$$= \sqrt{x} \sum_{r=1}^{\frac{v}{2} - \mu + 1} \frac{1}{r} \sum_{m=1}^{\infty} \frac{\varphi(m)}{m^3} + B \sqrt{x} \sum_{r \leq \sqrt{2^v x}} \frac{1}{r} \frac{r}{\sqrt{x}}$$

$$+ B \sum_{r \leq \sqrt{2^v x}} \sum_{m \leq \frac{\sqrt{x}}{r}} \frac{1}{m}$$

$$= \frac{\pi^2}{7 \zeta(3)} \sqrt{x} \sum_{r=1}^{\frac{v}{2} - \mu + 1} \frac{1}{r} + B \sqrt{x} \quad (39)$$

$$(58) \quad = \frac{\pi^2}{28 \zeta(3)} \sqrt{x} \log x + B \sqrt{x}.$$

(54) folgt aus (56) — (58).

$$\sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{(m, 2^v n)^2}{n^2} = \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r}} \frac{1}{n^2}$$

$$(59) \quad = \sum_{\mu=0}^v \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{\substack{m, n \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r}} \frac{1}{n^2} = \sum_{\mu=0}^v S''_{\mu} \quad (42).$$

$$S_0'' = \sum_{\substack{r \leq \sqrt{2^v x} \\ r \equiv 0 \pmod{2^v}}} r^2 \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{x} \\ (m, 2^v n) = r, m \equiv 0 \pmod{2^v}}} \frac{1}{n^2} \quad (59, 42)$$

$$= \sum_{r \leq \sqrt{x/2^v}} 2^{2v} r^2 \sum_{\substack{m \leq \sqrt{x/2^v}, n \leq \sqrt{x} \\ (m, n) = r}} \frac{1}{n^2} = 2^{2v} \sum_{r \leq \sqrt{x/2^v}} \sum_{\substack{m \leq \frac{1}{r} \sqrt{x/2^v}, n \leq \frac{\sqrt{x}}{r} \\ (m, n) = 1}} \frac{1}{n^2}$$

$$= 2^{2v} \sum_{r \leq \sqrt{x/2^v}} \sum_{n \leq \frac{\sqrt{x}}{r}} \frac{1}{n^2} \sum_{\substack{m \leq \frac{1}{r} \sqrt{x/2^v} \\ (m, n) = 1}} 1$$

$$= 2^{2v} \sum_{r \leq \sqrt{x/2^v}} \sum_{n \leq \frac{\sqrt{x}}{r}} \frac{1}{n^2} \left( \frac{1}{r} \sqrt{\frac{x}{2^v}} \varphi(n) + Bn \right)$$

$$= 2^{\frac{3}{2}v} \sqrt{x} \sum_{r \leq \sqrt{x/2^v}} \frac{1}{r} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} + B \sqrt{x} \sum_{r \leq \sqrt{x}} \frac{1}{r} \frac{r}{\sqrt{x}} + B \sqrt{x}$$

$$= \frac{2^{\frac{3}{2}v} \pi^2}{6 \zeta(3)} \sqrt{x} \sum_{r \leq \sqrt{x/2^v}} \frac{1}{r} + B \sqrt{x} \quad (38)$$

$$(60) = \frac{2^{\frac{3}{2}v} \pi^2}{12 \zeta(3)} \sqrt{x} \log x + B \sqrt{x}.$$

$$S_p'' = \sum_{\substack{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x} \\ r=1}} 2^{2p-2} r^2 \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \sqrt{x}, n \leq \sqrt{x} \\ (m, n) = r, m \equiv 1 \pmod{2}}} \frac{1}{n^2} \quad (59, 42)$$

$$= 2^{2p-2} \sum_{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x}} \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \frac{\sqrt{x}}{r}, n \leq \frac{\sqrt{x}}{r} \\ (m, n) = 1, m \equiv 1 \pmod{2}}} \frac{1}{n^2}$$

$$= 2^{2p-2} \sum_{\substack{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x} \\ r=1}} \sum_{\substack{n \leq \frac{\sqrt{x}}{r} \\ (m, 2n) = 1}} \frac{1}{n^2} \sum_{\substack{m \leq 2^{\frac{v}{2}-p+1} \frac{\sqrt{x}}{r} \\ (m, 2n) = 1}} 1$$

$$= 2^{2p-2} \sum_{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x}} \sum_{n \leq \frac{\sqrt{x}}{r}} \frac{1}{n^2} \left( 2^{\frac{v}{2}-p+1} \frac{\sqrt{x}}{2rn} \varphi(2n) + Bn \right)$$

$$= 2^{p+\frac{v}{2}-2} \sqrt{x} \sum_{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x}} \frac{1}{r} \sum_{n=1}^{\infty} \frac{\varphi(2n)}{n^3} + B \sqrt{x}$$

$$= \frac{2^{p+\frac{v}{2}} \pi^2}{21 \zeta(3)} \sqrt{x} \sum_{r \leq 2^{\frac{v}{2}-p+1} \sqrt{x}} \frac{1}{r} + B \sqrt{x} \quad (40)$$

$$(61) = \frac{2^{p+\frac{v}{2}} \pi^2}{84 \zeta(3)} \sqrt{x} \log x + B \sqrt{x}.$$

$$\sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{(m, 2^v n)^2}{n^2} = \left( \frac{2^{\frac{3}{2}v} \pi^2}{12 \zeta(3)} + \frac{2^{\frac{v}{2}} \pi^2}{84 \zeta(3)} (2^{v+1} - 2) \right) \sqrt{x} \log x + B \sqrt{x} \quad (59-61)$$

$$(55) = \left( \frac{2^{\frac{3}{2}v} \cdot 3 \pi^2}{28 \zeta(3)} - \frac{2^{\frac{v}{2}} \pi^2}{42 \zeta(3)} \right) \sqrt{x} \log x + B \sqrt{x}.$$

Hilfssatz 12:

$$(62) \sum_{\substack{a, b \leq x^2 \\ an = bm}} \frac{1}{ab} = \frac{\pi^2}{6} \frac{(m, n)^2}{mn} + \frac{B}{x^2}.$$

Beweis: Das ist  $T_{III}$  (18).

Hilfssatz 13:

$$(63) \quad \frac{2^v}{2\pi^2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{1}{mn} \sum_{\substack{a, b \leq x^2 \\ 2^v an = bm}} \frac{1}{ab} \int_{r_v}^x u^2 du$$

$$(64) = \frac{(5+3v)\pi^2}{432} x^3 - \frac{\pi^2}{30\zeta(3)} \left( \frac{1+2^{\frac{v}{2}}}{12} + \frac{v}{28} + \frac{2^{\frac{v}{2}} - 2^{-\frac{v}{2}}}{42} \right) x^5 \log x + Bx^{\frac{5}{2}},$$

Beweis: Der Ausdruck (63) ist

$$= \frac{2^v}{2\pi^2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{1}{mn} \left( \frac{\pi^2 (m, 2^v n)^2}{6 \cdot 2^v mn} + \frac{B}{x^2} \right) \int_{r_v}^x u^2 du \quad (62, 11)$$

$$= \frac{1}{12} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{(m, 2^v n)^2}{m^2 n^2} \int_{r_v}^x u^2 du + \frac{B}{x^2} \sum_{m, n \leq \sqrt{2^v x}} \frac{1}{mn} x^3$$

$$= \frac{1}{12} \int_1^x u^2 \sum_{\substack{m \leq \sqrt{2^v u} \\ n \leq \sqrt{u}}} \frac{(m, 2^v n)^2}{m^2 n^2} + Bx \log^2 x \quad (11)$$

$$= \frac{1}{12} \left( \frac{5\pi^2}{12} + \frac{v\pi^2}{4} \right) \int_3^x u^2 du - \frac{\pi^2}{12\zeta(3)} \left( \frac{1+2^{\frac{v}{2}}}{12} + \frac{v}{28} + \frac{2^{\frac{v}{2}} - 2^{-\frac{v}{2}}}{42} \right) \times \int_3^x u^{\frac{3}{2}} \log u du + Bx^{\frac{5}{2}} \quad (41),$$

genügt also der Abschätzung (64).

Hilfssatz 14:

$$(65) \quad \frac{2^v}{2\pi^2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{n}{m} \sum_{\substack{a, b \leq x^2 \\ 2^v an = bm}} \frac{1}{ab} \int_{r_v}^x u du$$

$$(66) = \frac{\pi^2}{30\zeta(3)} \left( \frac{1}{12} + \frac{v}{28} \right) x^5 \log x + Bx^{\frac{5}{2}}.$$

Beweis: Der Ausdruck (65) ist

$$= \frac{2^v}{2\pi^2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{n}{m} \left( \frac{\pi^2 (m, 2^v n)^2}{6 \cdot 2^v mn} + \frac{B}{x^2} \right) \int_{r_v}^x u du \quad (62, 11)$$

$$= \frac{1}{12} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{(m, 2^v n)^2}{m^2} \int_{r_v}^x u du + \frac{B}{x^2} \sum_{m, n \leq \sqrt{2^v x}} \frac{n}{m} x^2$$

$$= \frac{1}{12} \int_1^x u \sum_{\substack{m \leq \sqrt{2^v u} \\ n \leq \sqrt{u}}} \frac{(m, 2^v n)^2}{m^2} du + Bx \log x \quad (11)$$

$$= \frac{\pi^2}{12\zeta(3)} \left( \frac{1}{12} + \frac{v}{28} \right) \int_3^x u^{\frac{3}{2}} \log u du + Bx^{\frac{5}{2}} \quad (54),$$

genügt also der Abschätzung (66).

Hilfssatz 15:

$$(67) \quad \frac{1}{2\pi^2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{m}{n} \sum_{\substack{a, b \leq x^2 \\ 2^v an = bm}} \frac{1}{ab} \int_{r_v}^x u du$$

$$(68) = \frac{\pi^2}{30\zeta(3)} \left( \frac{2^{\frac{v}{2}} \cdot 3}{28} - \frac{2^{-\frac{v}{2}}}{42} \right) x^{\frac{5}{2}} \log x + Bx^{\frac{5}{2}}.$$

Beweis: Der Ausdruck (67) ist

$$= \frac{1}{2\pi^2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{m}{n} \left( \frac{\pi^2 (m, 2^v n)^2}{6 \cdot 2^v mn} + \frac{B}{x^2} \right) \int_{r_v}^x u du \quad (62, 11)$$

$$= \frac{2^{-v}}{12} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} \frac{(m, 2^v n)^2}{n^2} \int_{r_v}^x u du + \frac{B}{x^2} \sum_{m, n \leq \sqrt{2^v x}} \frac{m}{n} x^2$$

$$= \frac{2^{-v}}{12} \int_1^x u \sum_{\substack{m \leq \sqrt{2^v u} \\ n \leq \sqrt{u}}} \frac{(m, 2^v n)^2}{n^2} du + B x \log x \quad (11)$$

$$= \frac{\pi^2}{12 \zeta(3)} \left( \frac{2^{\frac{v}{2}} \cdot 3}{28} - \frac{2^{-\frac{v}{2}}}{42} \right) \int_3^x u^2 \log u du + B x^{\frac{5}{2}} \quad (55),$$

genügt also der Abschätzung (68).

Hilfssatz 16:

$$(69) \quad \frac{1}{2 \pi^2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m n \sum_{\substack{a, b \leq x^2 \\ 2^v a n = b m}} \frac{1}{a b} \int_{r_v}^x du = B x^{\frac{5}{2}}.$$

Beweis: Die linke Seite von (69) ist

$$= B \sum_{m, n \leq \sqrt{2^v x}} m n \left( \frac{(m, n)^2}{m n} + \frac{1}{x^2} \right) x = B x \sum_{m, n \leq \sqrt{2^v x}} (m, n)^2 \quad (62, 11)$$

$$= B x \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{\substack{m, n \leq \sqrt{2^v x} \\ (m, n) = r}} 1 = B x \sum_{r \leq \sqrt{2^v x}} r^2 \sum_{m, n \leq \frac{\sqrt{2^v x}}{r}} 1$$

$$= B x \sum_{r \leq \sqrt{2^v x}} r^2 \left( \frac{\sqrt{x}}{r} \right)^2 = B x^{\frac{5}{2}}.$$

Hilfssatz 17:

$$(70) \quad \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \sum_{a, b \leq x^2} \frac{1}{a b} \int_{r_v}^x u^{2-E_1-E_2} \cos \left\{ 2 \pi u \left( \frac{2^v a}{m} + \frac{b}{n} \right) \right\} da db$$

$$(71) \quad = B x^{\frac{5}{2}}.$$

Beweis: Der Ausdruck (70) ist

$$= B \sum_{E_1, E_2=0}^1 \sum_{m, n \leq \sqrt{2^v x}} m^{2E_1-1} n^{2E_2-1} \sum_{a, b \leq x^2} \frac{1}{a b} x^{2-E_1-E_2} \frac{1}{\frac{a}{m} + \frac{b}{n}}$$

$$= B \sum_{E_1, E_2=0}^1 x^{2-E_1-E_2} \sum_{m, n \leq \sqrt{2^v x}} m^{2E_1-1} n^{2E_2-1} \sum_{a, b \leq x^2} \frac{1}{a b} \frac{1}{\sqrt{\frac{a}{m} + \frac{b}{n}}}$$

$$= B \sum_{E_1, E_2=0}^1 x^{2-E_1-E_2} \sum_{m, n \leq \sqrt{2^v x}} m^{2E_1-\frac{1}{2}} n^{2E_2-\frac{1}{2}} \sum_{a, b=1}^{\infty} (a b)^{-\frac{3}{2}} = B x^{\frac{5}{2}}.$$

Hilfssatz 18: Mit

$$(72) \quad \mathfrak{S}(x) = \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1}$$

$$\times \sum_{\substack{a, b \leq x^2 \\ 2^v a n \neq b m}} \frac{1}{a b} \int_{r_v}^x u^{2-E_1-E_2} \cos \left\{ 2 \pi u \left( \frac{2^v a}{m} + \frac{b}{n} \right) \right\} du$$

ist

$$(73) \quad V_v(x) = \left( \frac{2^v}{12} + \frac{(5+3v)\pi^2}{432} \right) x^3 + \frac{1}{2\pi^2} \mathfrak{S}(x) + B x^{\frac{5}{2}}.$$

Beweis: Für reelle  $v$  sei (nur für den vorliegenden Beweis)

$$(74) \quad \langle v \rangle = \text{Min}(v - [v]; [v] + 1 - v)$$

(die positiv genommene Entfernung von  $v$  zur nächstliegenden ganzen Zahl). Dann gilt bekanntlich für jedes natürliche  $k$ , falls  $\text{Min}\left(1; \frac{1}{0}\right)$

Eins bedeutet,

$$(75) \quad \sum_{q=k+1}^{\infty} \frac{\sin 2 \pi q v}{q} = B \text{Min}\left(1; \frac{1}{k \langle v \rangle}\right),$$

$$(76) \quad \phi(v) = -\frac{1}{\pi} \sum_{q=1}^{\infty} \frac{\sin 2 \pi q v}{q} \quad (v \text{ nicht ganz}).$$

((76) wurde schon bei (18) und (19) benutzt.)

Es sei ferner

$$(77) \quad k = [x^2],$$

$$(78) \quad s_k(u) = -\frac{1}{\pi} \sum_{a=1}^h \frac{1}{a} \sin \frac{2 \pi a 2^v u}{m}, t_k(u) = -\frac{1}{\pi} \sum_{b=1}^k \frac{1}{b} \sin \frac{2 \pi b u}{n}.$$

$$(79) \quad s_k(u) = B, \quad t_k(u) = B \quad (78, 75, 76, 2),$$

$$(80) \quad \psi\left(\frac{2^\nu u}{m}\right) - s_k(u) = B \operatorname{Min}\left(1; x^{-2}\left(\frac{2^\nu u}{m}\right)^{-1}\right) \quad (75-78),$$

$$(81) \quad \psi\left(\frac{u}{n}\right) - t_k(u) = B \operatorname{Min}\left(1; x^{-2}\left(\frac{u}{n}\right)^{-1}\right) \quad (75-78).$$

$$\int_{r_\nu}^x \psi\left(\frac{2^\nu u}{m}\right) \psi\left(\frac{u}{n}\right) u^{2-E_1-E_2} du - \int_{r_\nu}^x s_k(u) t_k(u) u^{2-E_1-E_2} du$$

$$= \int_{r_\nu}^x \left(\psi\left(\frac{2^\nu u}{m}\right) - s_k(u)\right) t_k(u) u^{2-E_1-E_2} du + \int_{r_\nu}^x \left(\psi\left(\frac{u}{n}\right) - t_k(u)\right) s_k(u) u^{2-E_1-E_2} du$$

$$+ \int_{r_\nu}^x \left(\psi\left(\frac{2^\nu u}{m}\right) - s_k(u)\right) \left(\psi\left(\frac{u}{n}\right) - t_k(u)\right) u^{2-E_1-E_2} du$$

$$(82) \quad = B x^{2-E_1-E_2} \left( \int_{r_\nu}^x \operatorname{Min}\left(1; x^{-2}\left(\frac{2^\nu u}{m}\right)^{-1}\right) du + \right.$$

$$\left. + \int_{r_\nu}^x \operatorname{Min}\left(1; x^{-2}\left(\frac{u}{n}\right)^{-1}\right) du \right) \quad (79-81).$$

$$\int_{r_\nu}^x \operatorname{Min}\left(1; x^{-2}\left(\frac{2^\nu u}{m}\right)^{-1}\right) du = \frac{m}{2^\nu} \int_{\frac{2^\nu r_\nu}{m}}^{\frac{2^\nu x}{m}} \operatorname{Min}\left(1; x^{-2}(v)^{-1}\right) dv$$

$$(83) = B m \left[ \int_{\substack{(v) \leq \frac{1}{x} \\ 1 \leq \frac{2^\nu x}{m}}}^{\frac{2^\nu x}{m}} dv + x^{-2+1} \int_{\substack{(v) \geq \frac{1}{x} \\ 1 \leq \frac{2^\nu x}{m}}}^{\frac{2^\nu x}{m}} dv \right] = B m \left( \frac{x}{m} \frac{1}{x} + \frac{1}{x} \frac{x}{m} \right) = B \quad (11, 74),$$

$$(84) \quad \int_{r_\nu}^x \operatorname{Min}\left(1; x^{-2}\left(\frac{u}{n}\right)^{-1}\right) du = B \quad (11, 74).$$

$$(85) \quad \int_{r_\nu}^x \psi\left(\frac{2^\nu u}{m}\right) \psi\left(\frac{u}{n}\right) u^{2-E_1-E_2} du - \int_{r_\nu}^x s_k(u) t_k(u) u^{2-E_1-E_2} du = B x^{2-E_1-E_2} \quad (82-84).$$

$$\sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^\nu x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \int_{r_\nu}^x \psi\left(\frac{2^\nu u}{m}\right) \psi\left(\frac{u}{n}\right) u^{2-E_1-E_2} du$$

$$- \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^\nu x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \int_{r_\nu}^x s_k(u) t_k(u) u^{2-E_1-E_2} du$$

$$(86) \quad = B \sum_{E_1, E_2=0}^1 x^{2-E_1-E_2} \sum_{m, n \leq \sqrt{2^\nu x}} m^{2E_1-1} n^{2E_2-1} = B x^{\frac{5}{2}} \quad (85).$$

$$\int_{r_\nu}^x s_k(u) t_k(u) u^{2-E_1-E_2} du = \frac{1}{\pi^2} \sum_{a, b \leq x^2} \frac{1}{a b} \int_{r_\nu}^x u^{2-E_1-E_2} \sin \frac{2\pi a 2^\nu u}{m} \sin \frac{2\pi b u}{n} du$$

$$(11, 77, 78)$$

$$(87) \quad = \frac{1}{2\pi^2} \sum_{\substack{a, b \leq x^2 \\ 2^\nu a n \neq b m}} \frac{1}{a b} \int_{r_\nu}^x u^{2-E_1-E_2} du +$$

$$+ \frac{1}{2\pi^2} \sum_{a, b \leq x^2} \frac{1}{a b} \int_{r_\nu}^x u^{2-E_1-E_2} \cos \left\{ 2\pi u \left( \frac{2^\nu a}{m} - \frac{b}{n} \right) \right\} du$$

$$- \frac{1}{2\pi^2} \sum_{a, b \leq x^2} \frac{1}{a b} \int_{r_\nu}^x u^{2-E_1-E_2} \cos \left\{ 2\pi u \left( \frac{2^\nu a}{m} + \frac{b}{n} \right) \right\} du.$$

$$V_\nu(x) = \frac{2^\nu}{12} x^3 + \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^\nu x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \int_{r_\nu}^x s_k(u) t_k(u) u^{2-E_1-E_2} du$$

$$+ B x^{\frac{5}{2}} \quad (31, 86)$$

$$\begin{aligned}
&= \frac{2^v}{12} x^3 + \frac{1}{2\pi^2} \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \sum_{\substack{a, b \leq x^2 \\ 2^v a n = b m}} \frac{1}{a b} \int_{r_{\nu}}^x u^{2-E_1-E_2} du \\
&\quad + \frac{1}{2\pi^2} \zeta(x) - \frac{1}{2\pi^2} \sum_{E_1, E_2=0}^1 c_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{x}}} m^{2E_1-1} n^{2E_2-1} \\
&\quad \times \sum_{\substack{a, b \leq x^2 \\ 2^v a n < 2^v b m}} \frac{1}{a b} \int_{r_{\nu}}^x u^{2-E_1-E_2} \cos \left\{ 2\pi u \left( \frac{2^v a}{m} + \frac{b}{n} \right) \right\} du + B x^{\frac{5}{2}} \quad (87, 72)
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{2^v}{12} + \frac{(5+3v)\pi^2}{432} \right) x^3 - \frac{\pi^2}{30\zeta(3)} \left( \frac{1+2^{\frac{v}{2}}}{12} + \frac{v}{28} + \frac{2^{\frac{v}{2}}-2^{-\frac{v}{2}}}{42} - \frac{1}{12} - \frac{v}{28} \right. \\
&\quad \left. - \frac{2^{\frac{v}{2}} \cdot 3}{28} + \frac{2}{42} \right) x^{\frac{5}{2}} \log x + \frac{1}{2\pi^2} \zeta(x) + B x^{\frac{5}{2}} \quad (10, 64, 66, 68, 69, 71)
\end{aligned}$$

$$(73) \quad = \left( \frac{2^v}{12} + \frac{(5+3v)\pi^2}{432} \right) x^3 + \frac{1}{2\pi^2} \zeta(x) + B x^{\frac{5}{2}}.$$

Es sei  $\nu$  eine der Zahlen 0, 1, 2, 3, 4; ferner

$$(88) \quad r = r_{\nu, \nu} = \text{Max} \left( \frac{m^2}{2^v}; \frac{n^2}{2^v} \right),$$

$$(89) \quad \zeta_{E_1, E_2} = \zeta_{E_1, E_2, \nu, \nu} = c_{E_1, E_1, \nu} c_{E_1, E_2, \nu},$$

$$\begin{aligned}
(90) \quad \zeta_{\nu, \nu}(x) &= \sum_{E_1, E_2=0}^1 \zeta_{E_1, E_2, \nu, \nu} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{2^v x}}} m^{2E_1-1} n^{2E_2-1} \\
&\quad \times \sum_{\substack{a, b \leq x^2 \\ 2^v a n > 2^v b m}} \frac{1}{a b} \int_{r_{\nu, \nu}}^x u^{2-E_1-E_2} \cos \left\{ 2\pi u \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} du.
\end{aligned}$$

Hilfssatz 19:

$$(91) \quad V_{\nu}(x) = \left( \frac{2^v}{12} + \frac{(5+3v)\pi^2}{432} \right) x^3 + \frac{1}{2\pi^2} (\zeta_{\nu, 0}(x) + \zeta_{0, \nu}(x)) + B x^{\frac{5}{2}}.$$

Beweis: Ich ersetze in (90)

$$\nu, \nu, E_1, E_2, m, n, a, b \quad \text{durch} \quad \nu, \nu, E_2, E_1, n, m, b, a.$$

Dann folgt aus (90) und (88)

$$\begin{aligned}
(92) \quad \zeta_{\nu, \nu}(x) &= \sum_{E_1, E_2=0}^1 \zeta_{E_2, E_1, \nu, \nu} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{2^v x}}} m^{2E_1-1} n^{2E_2-1} \\
&\quad \times \sum_{\substack{a, b \leq x^2 \\ 2^v a n < 2^v b m}} \frac{1}{a b} \int_{r_{\nu, \nu}}^x u^{2-E_1-E_2} \cos \left\{ 2\pi u \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} du.
\end{aligned}$$

$$(93) \quad \zeta_{E_1, E_2, 0, \nu} = \zeta_{E_2, E_1, \nu, 0} = c_{E_1, E_2} \quad (89, 10),$$

$$(94) \quad r_{0, \nu} = r_{\nu} \quad (88, 11).$$

(91) folgt aus (73), (72), (90) und (92) — (94).

Hilfssatz 20:  $a, b, c, m, n$  seien natürliche Zahlen. Zu jedem vorgegebenen Tripel  $a, b, c$  gibt es höchstens

$$B \left( \frac{(a, b)}{\sqrt{a b}} \sqrt{x} + 1 \right)$$

Paare  $m, n$  mit

$$m \leq \sqrt{x}, n \leq \sqrt{x}, a n - b m = c.$$

Beweis: Das ist  $T_{III}$ , Hilfssatz 9.

Hilfssatz 21:

$$\begin{aligned}
(95) \quad \zeta_{\nu, \nu}(x) &= \frac{1}{2\pi} \sum_{E_1, E_2=0}^1 \zeta_{E_1, E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{2^v x}}} m^{2E_1} n^{2E_2} \sum_{\substack{a, b \leq x^2 \\ 2^v a n > 2^v b m}} (a b (2^v a n - 2^v b m))^{-1} \\
&\quad \times \left( x^{2-E_1-E_2} \sin \left\{ 2\pi x \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} - r^{2-E_1-E_2} \sin \left\{ 2\pi r \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} \right) + B x^{\frac{5}{2}},
\end{aligned}$$



Beweis:

$$(96) \int_r^x u^{2-E_1-E_2} \cos \left\{ 2\pi u \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} du = \frac{u^{2-E_1-E_2} \sin \left\{ 2\pi u \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\}}{2\pi \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right)} \Bigg|_r^x$$

$$- \frac{2-E_1-E_2}{2\pi \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right)} \int_r^x u^{1-E_1-E_2} \sin \left\{ 2\pi u \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} du$$

$$(97) = \frac{1}{2\pi} \frac{mn}{(2^v a n - 2^v b m)} \left( x^{2-E_1-E_2} \sin \left\{ 2\pi x \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} - r^{2-E_1-E_2} \sin \left\{ 2\pi r \left( \frac{2^v a}{m} - \frac{2^v b}{n} \right) \right\} \right) + B x^{1-E_1-E_2} m^2 n^2 (2^v a n - 2^v b m)^{-2}$$

(für  $E_1 = E_2 = 1$  ist  $2 - E_1 - E_2 = 0$  in (96)).

$$\sum_{E_1, E_2=0}^1 \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{2^v x}}} m^{2E_1-1} n^{2E_2-1} \sum_{\substack{a, b \leq x^2 \\ 2^v a n > 2^v b m}} (ab)^{-1} x^{1-E_1-E_2} m^2 n^2 (2^v a n - 2^v b m)^{-2}$$

$$= \sum_{E_1, E_2=0}^1 x^{1-E_1-E_2} \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{2^v x}}} m^{2E_1+1} n^{2E_2+1} \sum_{\substack{a, b \leq x^2 \\ 2^v a n > 2^v b m}} (ab)^{-1} (2^v a n - 2^v b m)^{-2}$$

$$= B x^2 \sum_{\substack{m \leq \sqrt{2^v x} \\ n \leq \sqrt{2^v x}}} \sum_{\substack{a, b \leq x^2 \\ 2^v a n > 2^v b m}} (ab)^{-1} (2^v a n - 2^v b m)^{-2}$$

$$(98) = B x^2 \sum_{a, b \leq x^2} \frac{1}{ab} \sum_{c=1}^{\infty} \frac{1}{c^2} \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{2^v x} \\ 2^v a n - 2^v b m = c}} 1$$

$$(99) = B x^2 \sum_{a, b \leq x^2} \frac{1}{ab} \sum_{c=1}^{\infty} \frac{1}{c^2} \left( \frac{(a, b)}{\sqrt{ab}} \sqrt{x} + 1 \right)$$

(Hilfssatz 20 mit  $2^{v+1}x, 2^v a, 2^v b, c, m, n$  statt  $x, a, b, c, m, n$ )

$$= B x^{\frac{5}{2}} \sum_{a, b=1}^{\infty} \frac{(a, b)}{(ab)^{\frac{3}{2}}} + B x^2 \sum_{a, b \leq x^2} \frac{1}{ab}$$

$$= B x^{\frac{5}{2}} \sum_{d=1}^{\infty} d \sum_{\substack{a, b=1 \\ (a, b)=d}}^{\infty} (ab)^{-\frac{3}{2}} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \sum_{d=1}^{\infty} d^{-2} \sum_{a, b=1}^{\infty} (ab)^{-\frac{3}{2}}$$

$$(100) + B x^{\frac{5}{2}} = B x^{\frac{5}{2}}.$$

(95) folgt aus (90), (89), (88), (97) und (100).

Hilfssatz 22:

$$(101) \quad \mathfrak{S}_{\mu, \nu}(x) = \frac{1}{2\pi} \sum_{E_1, E_2=0}^1 \mathfrak{G}_{E_1, E_2} \sum_{a, b \leq x^2} \frac{1}{ab} \sum_{\substack{c \leq 2^{v+1}x \\ (2^v a, 2^v b) \mid x}} \frac{1}{c}$$

$$\times \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{2^v x} \\ 2^v a n - 2^v b m = c}} m^{2E_1} n^{2E_2} \left( x^{2-E_1-E_2} \sin \frac{2\pi x c}{m n} - r^{2-E_1-E_2} \sin \frac{2\pi r c}{m n} \right) + B x^{\frac{5}{2}}.$$

Beweis: In (95) ist

$$(102) \quad 2^v a n - 2^v b m \leq 2^v a n \leq 2^v x^2 \sqrt{2^v x} \leq 2^{v+1} x^3.$$

$$(103) \quad \mathfrak{S}_{\mu, \nu}(x) = \frac{1}{2\pi} \sum_{E_1, E_2=0}^1 \mathfrak{G}_{E_1, E_2} \sum_{a, b \leq x^2} \frac{1}{ab} \sum_{\substack{c \leq 2^{v+1}x^3 \\ c \mid x}} \frac{1}{c} \times \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{2^v x} \\ 2^v a n - 2^v b m = c}} m^{2E_1} n^{2E_2} \left( x^{2-E_1-E_2} \sin \frac{2\pi x c}{m n} - r^{2-E_1-E_2} \sin \frac{2\pi r c}{m n} \right) = B x^{\frac{5}{2}} \quad (95, 102).$$

Behalte ich in (103) nur die  $a, b \leq x^2$  bei, so ist der Fehler

$$= B \sum_{E_1, E_2=0}^1 \left( \sum_{\substack{x^2 < a \leq x^3 \\ b \leq x^2}} + \sum_{\substack{x^2 < b \leq x^3 \\ a \leq x^2}} \right) \frac{1}{ab} \sum_{\substack{c \leq 2^{v+1}x^3 \\ 2^v a n - 2^v b m = c}} \sum_{\substack{m \leq \sqrt{2^v x}, n \leq \sqrt{2^v x}}} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \quad (88)$$

$$= B x^2 \left( \sum_{\substack{x^b < a \leq x^2 \\ b \leq x^2}} + \sum_{\substack{x^b < b \leq x^2 \\ a \leq x^2}} \right) \frac{1}{ab} \sum_{c \leq 2^{n+\nu} x^3} \frac{1}{c} \sum_{\substack{m \leq \sqrt{2^\nu x}, n \leq \sqrt{2^\nu x} \\ 2^\nu a n - 2^\nu b m = c}} 1$$

$$= B x^2 \left( \sum_{\substack{x^b < a \leq x^2 \\ b \leq x^2}} + \sum_{\substack{x^b < b \leq x^2 \\ a \leq x^2}} \right) \frac{1}{ab} \sum_{c \leq 2^{n+\nu} x^3} \frac{1}{c} \left( \frac{(a, b)}{\sqrt{ab}} \sqrt{x} + 1 \right)$$

(Hilfssatz 20, wie beim Übergang von (98) zu (99))

$$= B x^{\frac{5}{2}} \sum_{\substack{x^b < a \leq x^2 \\ b \leq x^2}} \frac{(a, b)}{(ab)^{\frac{3}{2}}} \sum_{c \leq 2^{n+\nu} x^3} \frac{1}{c} + B x^{2+\eta}$$

$$= B x^{\frac{5+\eta}{2}} \sum_{\substack{x^b < a \leq x^2 \\ b \leq x^2}} \frac{(a, b)}{(ab)^{\frac{3}{2}}} + B x^{\frac{5}{2}} \quad (3)$$

$$= B x^{\frac{5+\eta}{2}} \sum_{d=1}^{\infty} d \sum_{\substack{x^b < a \\ a > \frac{x^b}{d}, b \leq 1}} d^{-3} a^{-\frac{3}{2}} b^{-\frac{3}{2}} + B x^{\frac{5}{2}}$$

$$= B x^{\frac{5+\eta}{2}} \sum_{d=1}^{\infty} d^{-2} \sum_{a > \frac{x^b}{d}} a^{-\frac{3}{2}} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}}.$$

Also ist

$$\mathfrak{C}_{n, \nu}(x) = \frac{1}{2\pi} \sum_{E_1, E_2=0}^1 \mathfrak{E}_{E_1, E_2} \sum_{a, b \leq x^2} \frac{1}{ab} \sum_{c \leq 2^{n+\nu} x^3} \frac{1}{c}$$

$$\times \sum_{\substack{m \leq \sqrt{2^\nu x}, n \leq \sqrt{2^\nu x} \\ 2^\nu a n - 2^\nu b m = c}} m^{2E_1} n^{2E_2} \left( x^{2-E_1-E_2} \sin \frac{2\pi x c}{m n} - r^{2-E_1-E_2} \sin \frac{2\pi r c}{m n} \right) + B x^{\frac{5}{2}},$$

und hierin ist wegen

$$c = 2^\nu a n - 2^\nu b m \leq 2^\nu a n \leq 2^\nu x^3 \sqrt{2^\nu x} \leq 2^{n+\nu} x^{\frac{1}{2}+\eta}$$

die  $m$ - $n$ -Summe für  $c > 2^{n+\nu} x^{\frac{1}{2}+\eta}$  leer.

Bevor ich weitergehe, schicke ich einige einfache Bemerkungen voraus, die sich auf die in (101) auftretenden  $a, b, c, m, n$  und  $r$  beziehen.

Es sei

$$(104) \quad (2^\nu a, 2^\nu b) = g, \quad 2^\nu a = \alpha g, \quad 2^\nu b = \beta g.$$

Ist die  $m$ - $n$ -Summe in (101) nicht leer, so muss  $g$  in  $c$  aufgehen:

$$(105) \quad c = \gamma g.$$

Die Lösungen von

$$(106) \quad 2^\nu a n - 2^\nu b m = c,$$

d. h. von

$$(107) \quad \alpha n - \beta m = \gamma$$

sind

$$(108) \quad m = m_h = m' + \alpha h, \quad n = n_h = n' + \beta h \quad (h = 0, 1, 2, \dots),$$

wobei  $m = m', n = n'$  die Lösung mit kleinstem  $m' > 0$  bedeutet. Es ist also

$$(109) \quad \alpha n' - \beta m' = \gamma.$$

$$(110) \quad 0 < m' \leq \alpha, \quad 0 < n' = \frac{\gamma + \beta m'}{\alpha} \leq \frac{\gamma}{\alpha} + \beta.$$

Von jetzt ab bedeuten  $m, n$  die durch (108) gegebenen Funktionen von  $h$ ; damit hängt auch  $r$  nach (88) von  $h$  ab.

Für das folgende wird es nötig sein,  $m$  und  $n$  als Funktionen einer stetigen reellen Veränderlichen zu betrachten. Um aber Missverständnissen vorzubeugen, will ich dem  $h$  seine Bedeutung als ganze Zahl lassen und lieber

$$(111) \quad m_u = m' + \alpha u, \quad n_u = n' + \beta u$$

schreiben.

Hilfssatz 23: Für  $a \leq 2^{\frac{n-\nu}{2}} b$  ist  $m \leq 2^{\frac{n-\nu}{2}} n$ . Für  $a > 2^{\frac{n-\nu}{2}} b$  ist

$$m \leq 2^{\frac{n-\nu}{2}} n \text{ dann und nur dann, wenn } h \leq \frac{2^{\frac{n-\nu}{2}} n' - m'}{\alpha - 2^{\frac{n-\nu}{2}} \beta} \text{ und nur dann, wenn}$$

$$h \leq 4^{n+\nu+1} (\gamma + \alpha b) \text{ ist.}$$

Beweis: Für  $a \leq 2^{\frac{n-\nu}{2}} b$  ist  $c \leq 2^{\frac{n+\nu}{2}} b \left( n - 2^{\frac{n-\nu}{2}} m \right)$  (106), also sogar  $m < 2^{\frac{n-\nu}{2}} n$ . Für  $a > 2^{\frac{n-\nu}{2}} b$ , d. h.  $a > 2^{\frac{n-\nu}{2}} \beta$  (104), ist  $m \leq 2^{\frac{n-\nu}{2}} n$  mit

$h \leq \frac{2^{\frac{\nu-1}{2}} n' - m'}{a - 2^{\frac{\nu-1}{2}} \beta}$  gleichbedeutend (108), und hierin ist

$$\frac{2^{\frac{\nu-1}{2}} n' - m'}{a - 2^{\frac{\nu-1}{2}} \beta} \leq \frac{2^{\frac{\nu-1}{2}} n' (a + 2^{\frac{\nu-1}{2}} \beta)}{a^2 - 2^{\nu-1} \beta^2} = \frac{2^{\frac{\nu-1}{2}} n' (a + 2^{\frac{\nu-1}{2}} \beta)}{2^\nu a^2 - 2^\nu \beta^2} \\ \leq 2^{\frac{\nu-1}{2} + 1} n' a \leq 2^{\nu+1} (\gamma + a \beta) \leq 4^{\nu+1} (\gamma + a b) \quad (110, 104).$$

Ich setze jetzt zur Abkürzung

$$(112) \quad \text{Min} \left( \frac{\sqrt{2^\nu x} - m'}{a}, \frac{\sqrt{2^\nu x} - n'}{\beta} \right) = X.$$

Dann ist

$$(113) \quad X = B \frac{\sqrt{x}}{\beta^{1-j}} \quad (j \text{ eine beliebige Zahl des Intervalls } 0 \leq j \leq 1).$$

Hilfssatz 24:

$$(114) \quad \mathfrak{S}_{\nu, \nu}(x) = \frac{1}{2\pi} \sum_{E_1, E_2=0}^1 \mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^\beta} \frac{1}{a b g} \sum_{\gamma \leq 2^{\nu+\nu} x^{\frac{1}{2} + \frac{1}{2}}} \frac{1}{\gamma} \\ \times \sum_{\substack{x^{\frac{4}{9}} \leq h \leq X}} m^{2E_1} n^{2E_2} \left( x^{2-E_1-E_2} \sin \frac{2\pi x \gamma g}{m n} - r^{2-E_1-E_2} \sin \frac{2\pi r \gamma g}{m n} \right) + B x^{\frac{5}{2}}.$$

Beweis:

$$(115) \quad \mathfrak{S}_{\nu, \nu}(x) = \frac{1}{2\pi} \sum_{E_1, E_2=0}^1 \mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^\beta} \frac{1}{a b g} \sum_{\gamma \leq 2^{\nu+\nu} x^{\frac{1}{2} + \frac{1}{2}}} \frac{1}{\gamma} \\ \times \sum_{0 \leq h \leq X} m^{2E_1} n^{2E_2} \left( x^{2-E_1-E_2} \sin \frac{2\pi x \gamma g}{m n} - r^{2-E_1-E_2} \sin \frac{2\pi r \gamma g}{m n} \right) \\ + B x^{\frac{5}{2}} \quad (101, 105, 104, 108, 112).$$

Lässt man in (115) die  $h < x^{\frac{4}{9}}$  weg, so ist der Fehler

$$= B \sum_{E_1, E_2=0}^1 \sum_{a, b \leq x^\beta} \frac{1}{a b} \sum_{\gamma \leq 2^{\nu+\nu} x^{\frac{1}{2} + \frac{1}{2}}} \frac{1}{\gamma} \sum_{0 \leq h < x^{\frac{4}{9}}} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \quad (88)$$

$$= B \sum_{E_1, E_2=0}^1 \sum_{a, b, \gamma \leq 2^{\nu+\nu} x^{\frac{1}{2} + \frac{1}{2}}} \frac{1}{a b \gamma} \sum_{0 \leq h < x^{\frac{4}{9}}} x^{E_1+E_2+2-E_1-E_2} \\ = B x^{2+\frac{4}{9}+\frac{1}{2}} = B x^{\frac{5}{2}} \quad (3).$$

Hilfssatz 25: Es sei  $p - \frac{1}{2}$  und  $q - \frac{1}{2}$  ganz,  $0 \leq p < q$ ;  $f(u)$  im Intervall  $p \leq u \leq q$  reell und zweimal differenzierbar;  $f''(u)$  dort beständig  $\geq \omega$  und  $= B \omega$ , wo  $\omega$  eine von  $u$  unabhängige positive Zahl bedeutet;  $F(u)$  dort positiv,  $F'(u)$  vorhanden und stetig.

Dann ist

$$\sum_{p \leq h \leq q} F(h) \sin \{2\pi f(h)\} = B \left( F(p) + \int_p^q |F'(u)| du \right) \left( q \sqrt{\omega} + \frac{1}{\sqrt{\omega}} \right),$$

also, falls noch  $F'(u) > 0$  vorausgesetzt wird,

$$(116) \quad \sum_{p \leq h \leq q} F(h) \sin \{2\pi f(h)\} = B F(q) \left( q \sqrt{\omega} + \frac{1}{\sqrt{\omega}} \right).$$

Beweis: Das ist  $T_{III}$ , Hilfssatz 15, der eine leichte Folgerung aus dem van der Corputschen Hilfssatz 14 von  $T_{III}$  ist.

Hilfssatz 26:

$$(117) \quad \mathfrak{S}_1(x) = \sum_{E_1, E_2=0}^1 \mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^\beta} \frac{1}{a b g} \sum_{\gamma \leq 2^{\nu+\nu} x^{\frac{1}{2} + \frac{1}{2}}} \frac{1}{\gamma} \\ \times \sum_{\substack{x^{\frac{4}{9}} \leq h \leq X}} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \sin \frac{2\pi x \gamma g}{m n} \\ = B x^{\frac{5}{2}}. \quad (118)$$

Beweis: Es sei

$$(119) \quad x^{\frac{4}{9}} \leq H \leq H' \leq 2H \leq 2X, \quad H \text{ und } H' \text{ ganz,}$$

$$(120) \quad p = H - \frac{1}{2}, \quad q = H' + \frac{1}{2}.$$

Aus (111), (120), (119), (112) und (110) folgt

$$(121) \quad m_p \leq m_H \leq m_q = m' + \alpha \left( H' + \frac{1}{2} \right) \leq 3(m' + \alpha H) = 3 m_H = B \sqrt{x},$$

$$(122) \quad n_p \leq n_H \leq n_q \leq 3(n' + \beta H) = 3 n_H = B \sqrt{x},$$

$$(123) \quad m_p = m' + \alpha \left( H - \frac{1}{2} \right) \geq \frac{1}{2}(m' + \alpha H) \geq \frac{1}{4}(m' + \alpha H) \geq \frac{1}{8} \left( m' + \alpha \left( H' + \frac{1}{2} \right) \right) = \frac{1}{8} m_q,$$

$$(124) \quad n_p \geq \frac{1}{2}(n' + \beta H) \geq \frac{1}{8} n_q,$$

$$(125) \quad \frac{1}{m_q} = \frac{1}{m' + \alpha q} \leq \frac{1}{\alpha H'} \leq \frac{1}{\alpha x^{\frac{4}{9}}},$$

$$(126) \quad \frac{1}{n_q} \leq \frac{1}{\beta x^{\frac{4}{9}}},$$

$$(127) \quad q \leq m_q^{\frac{1}{2}} n_q^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}}.$$

Es sei  $u$  eine beliebige Zahl des Intervalls  $p \leq u \leq q$ .

$$(128) \quad f_1(u) = \frac{x \gamma g}{m_u n_u}, \quad F_1(u) = m_u^{2E_1} n_u^{2E_2},$$

$$(129) \quad \omega_1 = f_1''(q).$$

Dann gilt

$$(130) \quad f_1'(u) = -\frac{x \gamma g}{(m_u n_u)^2} (\alpha n_u + \beta m_u) \quad (128, 111),$$

$$f_1''(u) = \frac{2 x \gamma g}{(m_u n_u)^3} ((\alpha n_u + \beta m_u)^2 - \alpha \beta m_u n_u) \quad (130, 111)$$

$$= \frac{2 x \gamma g}{(m_u n_u)^3} ((\alpha n_u - \beta m_u)^2 + 3 \alpha \beta m_u n_u)$$

$$(131) \quad = \frac{2 x \gamma g}{(m_u n_u)^3} (\gamma^2 + 3 \alpha \beta m_u n_u) \quad (111, 109).$$

Mit wachsendem  $u$  nimmt  $f_1''(u)$  ab (131), da  $m_u$  und  $n_u$  zunehmen (111). Also ist

$$\omega_1 = f_1''(q) \leq f_1''(u) \leq f_1''(p) = B f_1''(q) = B \omega_1 \quad (129, 131, 123, 124).$$

$F_1(u)$  ist stetig und positiv (128, 111, 3);  $p - \frac{1}{2}$ ,  $q - \frac{1}{2}$  sind ganz,  $0 \leq p < q$  (120, 119). Also darf (116) auf die durch (120) (128), (129) gegebenen  $p$ ,  $q$ ,  $f_1$ ,  $F_1$ ,  $\omega_1$  angewandt werden:

$$(132) \quad \sum_{p \leq h < q} F_1(h) \sin \{2 \pi f_1(h)\} = B F_1(q) \left( q \sqrt{\omega_1} + \frac{1}{\sqrt{\omega_1}} \right),$$

$$(133) \quad \sum_{h=H}^{H'} m^{2E_1} n^{2E_2} \sin \frac{2 \pi x \gamma g}{m n} = B m_q^{2E_1} n_q^{2E_2} \left( q \sqrt{\omega_1} + \frac{1}{\sqrt{\omega_1}} \right) \quad (132, 120, 128, 108).$$

$$m_q^{2E_1} n_q^{2E_2} q \sqrt{\omega_1} = B m_q^{2E_1} n_q^{2E_2} m_q^{\frac{1}{2}} n_q^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} x^{\frac{1}{2}} \gamma^{\frac{1}{2}} g^{\frac{1}{2}} m_q^{-\frac{3}{2}} n_q^{-\frac{3}{2}} \times \left( \gamma + \alpha^{\frac{1}{2}} \beta^{\frac{1}{2}} m_q^{\frac{1}{2}} n_q^{\frac{1}{2}} \right) \quad (127, 129, 131)$$

$$= B x^{\frac{1}{2}} m_q^{2E_1-1} n_q^{2E_2-1} \gamma^{\frac{3}{2}} g^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} + B x^{\frac{1}{2}} m_q^{2E_1-\frac{1}{2}} n_q^{2E_2-\frac{1}{2}} \gamma^{\frac{1}{2}} g^{\frac{1}{2}} \\ = B x^{\frac{1}{2}+E_1-\frac{4}{9}+E_2-\frac{4}{9}} \gamma^{\frac{3}{2}} + B x^{\frac{1}{2}+E_1-\frac{2}{9}+E_2-\frac{2}{9}} \gamma^{\frac{1}{2}} g \quad (121, 122, 125, 126)$$

$$(134) \quad = B x^{\frac{E_1+E_2-7}{18}} g \gamma^{\frac{3}{2}} + B x^{\frac{E_1+E_2+1}{18}} g \gamma^{\frac{1}{2}}.$$

$$m_q^{2E_1} n_q^{2E_2} \frac{1}{\sqrt{\omega_1}} = B m_q^{2E_1} n_q^{2E_2} m_q^{\frac{3}{2}} n_q^{\frac{3}{2}} x^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} g^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} m_q^{-\frac{1}{2}} n_q^{-\frac{1}{2}} \quad (129, 131)$$

$$= B x^{-\frac{1}{2}} m_q^{2E_1+1} n_q^{2E_2+1} \gamma^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}}$$

$$= B x^{-\frac{1}{2}} m_q x^{E_1} x^{E_2+\frac{1}{2}} \gamma^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} = B x^{E_1+E_2} \alpha H \gamma^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} \quad (121, 122, 110)$$

$$(135) \quad = B x^{E_1+E_2} \alpha^{\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} H.$$

$$(136) \quad \sum_{h=H}^{H'} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \sin \frac{2 \pi x \gamma g}{m n} = B x^{2-\frac{7}{18}} g \gamma^{\frac{3}{2}}$$

$$+ B x^{2+\frac{1}{18}} g \gamma^{\frac{1}{2}} + B x^2 a^{\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} H \quad (133 - 135).$$

Ein bekannter Kungstgriff ergibt jetzt

$$\sum_{\substack{4 \\ x^9 \leq h \leq X}} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \sin \frac{2\pi x \gamma g}{m n} = B x^{2-\frac{7}{18}+\delta} g \gamma^{\frac{3}{2}} \\ + B x^{2+\frac{1}{18}+\delta} g \gamma^{\frac{1}{2}} + B x^2 a^{\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} x^{\frac{1}{2}} a^{-1} \quad (119, 136, 113) \\ (137) = B x^{2-\frac{7}{18}+\delta} g \gamma^{\frac{3}{2}} + B x^{2+\frac{1}{18}+\delta} g \gamma^{\frac{1}{2}} + B x^2 a^{\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} x^{\frac{1}{2}} a^{-1}.$$

Setze ich (137) in (117) ein, so folgt

$$\mathfrak{S}_1(x) = B x^{2-\frac{7}{18}+\delta} \sum_{a,b \leq x^{\frac{1}{2}}} (ab)^{-1} \sum_{\substack{\gamma \leq 2 \\ \gamma \leq \frac{1}{2}+\delta}} \gamma^{\frac{1}{2}} \\ + B x^{2+\frac{1}{18}+\delta} \sum_{a,b \leq x^{\frac{1}{2}}} (ab)^{-1} \sum_{\substack{\gamma \leq 2 \\ \gamma \leq \frac{1}{2}+\delta}} \gamma^{-\frac{1}{2}} + B x^{\frac{5}{2}} \sum_{a,b=1}^{\infty} (ab)^{-\frac{3}{2}} \sum_{\gamma=1}^{\infty} \gamma^{-\frac{3}{2}} \quad (104) \\ = B x^{2-\frac{7}{18}+\delta+\delta+\frac{3}{4}+2\delta} + B x^{2+\frac{1}{18}+\delta+\delta+\frac{1}{4}+\delta} + B x^{\frac{5}{2}} \\ (118) = B x^{\frac{5}{2}} \quad (3).$$

Hilfssatz 27:

$$(138) \quad \mathfrak{S}_2(x) = \sum_{E_1, E_2=0} \mathfrak{S}_{E_1, E_2} \sum_{a,b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{\frac{1}{\alpha} \leq \frac{1}{\beta} \\ \alpha \leq 2^{\frac{1}{2}} b}} \gamma \\ \times \sum_{\substack{4 \\ x^9 \leq h \leq X}} m^{2E_1} n^{2E_2} r^{2-E_1-E_2} \sin \frac{2\pi r \gamma g}{m n} \\ (139) = B x^{\frac{5}{2}}.$$

Beweis:

$$(140) \quad \left( a \leq 2^{\frac{1}{2}} b \right) m \leq 2^{\frac{1}{2}} n, r = \frac{n^2}{2^k} \quad (\text{Hilfssatz 23, 88}).$$

$$(141) \quad \mathfrak{S}_2(x) = \sum_{E_1, E_2=0} 2^{k(E_1+E_2-2)} \mathfrak{S}_{E_1, E_2} \sum_{a,b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{\frac{1}{\alpha} \leq \frac{1}{\beta} \\ \alpha \leq 2^{\frac{1}{2}} b}} \gamma \\ \times \sum_{\substack{4 \\ x^9 \leq h \leq X}} n^4 \left( \frac{m}{n} \right)^{2E_1} \sin \frac{2\pi n \gamma g}{2^k m} \quad (138, 140).$$

$$(142) \quad (a \leq 2^{\frac{1}{2}} b) \quad \alpha \leq 2^{\frac{1}{2}} \beta \quad (104).$$

$$(143) \quad \frac{n}{m} - \frac{\beta}{\alpha} = \frac{\gamma}{\alpha m} = B \frac{\gamma}{\alpha^2 h}, \quad \frac{m}{n} - \frac{\alpha}{\beta} = -\frac{\gamma}{\beta n} = B \frac{\gamma}{\beta^2 h} \quad (107, 108, 110),$$

$$(144) \quad \left( \frac{m}{n} \right)^{2E_1} - \left( \frac{\alpha}{\beta} \right)^{2E_1} = B \left( \left( \frac{m}{n} \right)^2 - \left( \frac{\alpha}{\beta} \right)^2 \right) = B \left( \frac{m}{n} - \frac{\alpha}{\beta} \right) = B \frac{\gamma}{\beta^2 h} \quad (3, 140, 142, 143),$$

$$(145) \quad n^4 - \beta^4 h^4 = B (n - \beta h) n^3 = B n' x^{\frac{3}{2}} = B (\gamma + \beta) x^{\frac{3}{2}} \quad (108, 101, 110),$$

$$\frac{2\pi n \gamma g}{2^k m} - \frac{2\pi \gamma b}{\alpha} = B \frac{\gamma^2 g}{\alpha^2 h} \quad (143, 104),$$

$$(146) \quad \sin \frac{2\pi n \gamma g}{2^k m} - \sin \frac{2\pi \gamma b}{\alpha} = B \operatorname{Min} \left( \frac{\gamma^2 g}{\alpha^2 h}; 1 \right) = B \frac{\gamma}{\alpha} \sqrt{\frac{g}{h}}.$$

$$n^4 \left( \frac{m}{n} \right)^{2E_1} \sin \frac{2\pi n \gamma g}{2^k m} - \beta^4 h^4 \left( \frac{\alpha}{\beta} \right)^{2E_1} \sin \frac{2\pi \gamma b}{\alpha}$$

$$= \beta^4 h^4 \left( \frac{\alpha}{\beta} \right)^{2E_1} \left( \sin \frac{2\pi n \gamma g}{2^k m} - \sin \frac{2\pi \gamma b}{\alpha} \right)$$

$$+ (n^4 - \beta^4 h^4) \left( \frac{\alpha}{\beta} \right)^{2E_1} \sin \frac{2\pi n \gamma g}{2^k m} + n^4 \left( \left( \frac{m}{n} \right)^{2E_1} - \left( \frac{\alpha}{\beta} \right)^{2E_1} \right) \sin \frac{2\pi n \gamma g}{2^k m}$$

$$(147) = B \beta^4 h^{\frac{7}{2}} \gamma g^{\frac{1}{2}} \alpha^{-1} + B (\gamma + \beta) x^{\frac{3}{2}} + B x^3 \frac{\gamma}{\beta^2 h} \quad (142, 146, 145, 101, 144).$$

Ersetzt man in der  $h$ -Summe von (141)

$$n^4 \left( \frac{m}{n} \right)^{2E_1} \sin \frac{2\pi n \gamma g}{2^k m} \text{ durch } \beta^4 h^4 \left( \frac{\alpha}{\beta} \right)^{2E_1} \sin \frac{2\pi \gamma b}{\alpha},$$

so ändert sich nach (147)  $\mathfrak{S}_2(x)$  um

$$\begin{aligned}
 & B \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \frac{1}{\gamma} \sum_{h \leq X} \left( \beta^4 h^{\frac{7}{2}} \gamma g^{\frac{1}{2}} \alpha^{-1} + (\gamma + \beta) x^{\frac{3}{2}} + x^2 \frac{\gamma}{\beta^2 h} \right) \\
 &= B \sum_{a, b \leq x^{\frac{1}{2}}} \frac{\beta^4}{a b \alpha \sqrt{g}} \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} 1 \cdot \sum_{h \leq \frac{\sqrt{2^{\frac{1}{2}} x}}{\beta}} h^{\frac{7}{2}} + B x^{\frac{3}{2}} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} 1 \cdot \sum_{h \leq \frac{\sqrt{2^{\frac{1}{2}} x}}{\beta}} 1 \\
 &+ B x^{\frac{3}{2}} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{\beta}{a b} \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \frac{1}{\gamma} \sum_{h \leq \frac{\sqrt{2^{\frac{1}{2}} x}}{\beta}} 1 + B x^2 \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} 1 \cdot \sum_{h \leq \frac{\sqrt{2^{\frac{1}{2}} x}}{\beta}} \frac{1}{h} \quad (112) \\
 &= B x^{\frac{5}{2}} \sum_{a, b=1}^{\infty} \left( a b \alpha^{\frac{1}{4}} \beta^{\frac{1}{4}} g^{\frac{1}{2}} \right)^{-1} + B x^{\frac{9}{4}} + B x^{2+\frac{1}{2}} + B x^{\frac{9}{4}+\frac{1}{2}}
 \end{aligned}$$

$$(148) \quad = B x^{\frac{5}{2}} \sum_{a, b=1}^{\infty} (a b)^{-\frac{5}{4}} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \quad (104, 3).$$

$$\begin{aligned}
 \mathfrak{S}_2(x) &= \sum_{E_1, E_2=0}^1 2^{n(E_1+E_2-2)} (\mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \frac{1}{\gamma} \\
 &\quad \times \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \beta^4 h^4 \left( \frac{\alpha}{\beta} \right)^{2E_1} \sin \frac{2\pi \gamma b}{\alpha} + B x^{\frac{5}{2}} \quad (141, 148) \\
 &\quad \times \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \beta^4 h^4 \left( \frac{\alpha}{\beta} \right)^{2E_1} \sin \frac{2\pi \gamma b}{\alpha} + B x^{\frac{5}{2}}
 \end{aligned}$$

$$\begin{aligned}
 (149) \quad &= \sum_{E_1, E_2=0}^1 2^{n(E_1+E_2-2)} (\mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \beta^4 \left( \frac{\alpha}{\beta} \right)^{2E_1} \\
 &\quad \times \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \frac{1}{\gamma} \sin \frac{2\pi \gamma b}{\alpha} \sum_{\substack{x^{\frac{1}{2}} \leq h \leq X \\ \gamma \leq \frac{1}{4}}} h^4 + B x^{\frac{5}{2}}.
 \end{aligned}$$

$$(150) \quad \frac{\sqrt{2^{\frac{1}{2}} x} - m'}{\alpha} - \frac{\sqrt{2^{\frac{1}{2}} x} - n'}{\beta} = \frac{(\sqrt{2^{\frac{1}{2}} \beta} - \sqrt{2^{\frac{1}{2}} \alpha}) \sqrt{x} + \gamma}{\alpha \beta} \geq \frac{\gamma}{\alpha \beta} > 0 \quad (109, 142).$$

$$(151) \quad X = \frac{\sqrt{2^{\frac{1}{2}} x} - n'}{\beta} \quad (151, 150),$$

$$(152) \quad \sum_{\substack{x^{\frac{1}{2}} \leq h \leq X \\ \gamma \leq \frac{1}{4}}} h^4 = \sum_{\substack{x^{\frac{1}{2}} \leq h \leq \frac{\sqrt{2^{\frac{1}{2}} x}}{\beta}}} h^4 + B \gamma \frac{x^2}{\beta^4} \quad (151, 110).$$

$$\begin{aligned}
 \mathfrak{S}_2(x) &= \sum_{E_1, E_2=0}^1 2^{n(E_1+E_2-2)} (\mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \beta^4 \left( \frac{\alpha}{\beta} \right)^{2E_1} \\
 &\quad \times \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \frac{1}{\gamma} \sin \frac{2\pi \gamma b}{\alpha} \cdot \sum_{\substack{x^{\frac{1}{2}} \leq h \leq \frac{\sqrt{2^{\frac{1}{2}} x}}{\beta}}} h^4 \\
 &\quad + B x^2 \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \left( \frac{\alpha}{\beta} \right)^{2E_1} \sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} 1 + B x^{\frac{5}{2}} \quad (149, 152) \\
 &= B \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \beta^4 \frac{x^2}{\beta^5} + B x^{\frac{9}{4}} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} + B x^{\frac{5}{2}} \quad (142)
 \end{aligned}$$

(man beachte, dass die letzte  $h$ -Summe nicht von  $\gamma$  abhängt und

$$\begin{aligned}
 &\sum_{\substack{\gamma \leq x \\ \gamma \leq \frac{1}{4}}} \frac{1}{\gamma} \sin \frac{2\pi \gamma b}{\alpha} = B \quad (\text{z. B. 75, 76, 2}) \\
 &= B x^{\frac{5}{2}} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b^2} + B x^{\frac{5}{2}} \quad (104, 3) \\
 &= B x^{\frac{5}{2}} \sum_{a, b=1}^{\infty} (a b)^{-\frac{3}{2}} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}}.
 \end{aligned}$$

Hilfssatz 28:

$$(153) \quad \mathfrak{S}_3(x) = \sum_{E_1, E_2=0}^1 \mathfrak{S}_{E_1, E_2} \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ b < 2^{\frac{1}{2}} a}} \frac{1}{a b g} \sum_{\substack{\gamma \leq x^{\frac{1}{4}} \\ \gamma \equiv 1 \pmod{4}}} \frac{1}{\gamma} \\ \times \sum_{\substack{x^{\frac{1}{9}} \leq h \leq X}} m^{2E_1} n^{2E_2} r^{2-E_1-E_2} \sin \frac{2\pi r g}{m n}$$

$$(154) \quad = B x^{\frac{5}{2}}.$$

Beweis: Für  $b < 2^{\frac{1}{2}} a, h > 4^{v+v+1} (\gamma + ab)$  ist  $m > 2^{\frac{1}{2}} n$  (Hilfssatz 23), also  $r = \frac{m^2}{2^v}$  (88). Ersetzt man daher in (153)  $r$  durch  $\frac{m^2}{2^v}$ , so ist der Fehler

$$= B \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\substack{\gamma \leq x^{\frac{1}{4}} \\ \gamma \equiv 1 \pmod{4}}} \frac{1}{\gamma} \sum_{h \leq 4^{v+v+1} (\gamma + ab)} x^2 \quad (101, 88)$$

$$= B x^2 \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\gamma \leq x^{\frac{1}{4}}} 1 + B x^2 \sum_{a, b \leq x^{\frac{1}{2}}} 1 \cdot \sum_{\gamma \leq x^{\frac{1}{4}}} \frac{1}{\gamma}$$

$$(155) \quad = B x^{\frac{9}{4} + \frac{5}{2}} + B x^{2 + \frac{5}{2}} = B x^{\frac{5}{2}} \quad (3).$$

$$(156) \quad \mathfrak{S}_3(x) = \sum_{E_1, E_2=0}^1 2^{v(E_1+E_2-2)} \mathfrak{S}_{E_1, E_2} \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ b < 2^{\frac{1}{2}} a}} \frac{1}{a b g} \sum_{\substack{\gamma \leq x^{\frac{1}{4}} \\ \gamma \equiv 1 \pmod{4}}} \frac{1}{\gamma} \\ \times \sum_{\substack{x^{\frac{1}{9}} \leq h \leq X}} m^4 \left( \frac{n}{m} \right)^{2E_2} \sin \frac{2\pi m \gamma g}{2^v n} + B x^{\frac{5}{2}} \quad (153, 155).$$

$$(157) \quad \left( b < 2^{\frac{1}{2}} a \right) \quad \beta < 2^{\frac{1}{2}} a \quad (104).$$

$$(158) \quad \frac{n}{m} = B \left( \frac{\beta}{a} + \frac{\gamma}{a^2 h} \right) = B \left( \beta + \frac{\gamma}{h} \right) = B \left( \beta + x^{\frac{1}{4} - \frac{4}{9}} \right) = B \beta \quad (143, 153).$$

$$(159) \quad \left( \frac{n}{m} \right)^{2E_2} - \left( \frac{\beta}{a} \right)^{2E_2} = B \left( \left( \frac{n}{m} \right)^2 - \left( \frac{\beta}{a} \right)^2 \right) = B \frac{\beta \gamma}{a^2 h} = B \frac{\gamma}{a h} \quad (3, 143, 158, 157),$$

$$(160) \quad m^4 - a^4 h^4 = B (m - a h) m^3 = B m' x^{\frac{3}{2}} = B a x^{\frac{3}{2}} \quad (108, 101, 110),$$

$$\frac{2\pi m \gamma g}{2^v n} - \frac{2\pi \gamma a}{\beta} = B \frac{\gamma^2 g}{\beta^2 h} \quad (143, 104),$$

$$(161) \quad \sin \frac{2\pi m \gamma g}{2^v n} - \sin \frac{2\pi \gamma a}{\beta} = B \operatorname{Min} \left( \frac{\gamma^2 g}{\beta^2 h}; 1 \right) = B \frac{\gamma}{\beta} \sqrt{\frac{g}{h}}.$$

$$m^4 \left( \frac{n}{m} \right)^{2E_2} \sin \frac{2\pi m \gamma g}{2^v n} - a^4 h^4 \left( \frac{\beta}{a} \right)^{2E_2} \sin \frac{2\pi \gamma a}{\beta} \\ = a^4 h^4 \left( \frac{\beta}{a} \right)^{2E_2} \left( \sin \frac{2\pi m \gamma g}{2^v n} - \sin \frac{2\pi \gamma a}{\beta} \right) \\ + (m^4 - a^4 h^4) \left( \frac{\beta}{a} \right)^{2E_2} \sin \frac{2\pi m \gamma g}{2^v n} + m^4 \left( \left( \frac{n}{m} \right)^{2E_2} - \left( \frac{\beta}{a} \right)^{2E_2} \right) \sin \frac{2\pi m \gamma g}{2^v n}$$

$$(162) \quad = B a^4 h^{\frac{7}{2}} \gamma g^{\frac{1}{2}} \beta^{-1} + B a x^{\frac{3}{2}} + B x^2 \frac{\gamma}{a h} \quad (157, 161, 160, 101, 159).$$

Ersetzt man in der  $h$ -Summe von (156)

$$m^4 \left( \frac{n}{m} \right)^{2E_2} \sin \frac{2\pi m \gamma g}{2^v n} \text{ durch } a^4 h^4 \left( \frac{\beta}{a} \right)^{2E_2} \sin \frac{2\pi \gamma a}{\beta},$$

so ändert sich  $\mathfrak{S}_3(x)$  nach (162) um

$$B \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{\gamma \leq x^{\frac{1}{4}} \\ \gamma \equiv 1 \pmod{4}}} \frac{1}{\gamma} \sum_{h \leq X} \left( a^4 h^{\frac{7}{2}} \gamma g^{\frac{1}{2}} \beta^{-1} + a x^{\frac{3}{2}} + x^2 \frac{\gamma}{a h} \right) \\ = B \sum_{a, b \leq x^{\frac{1}{2}}} \frac{a^4}{a b \beta \sqrt{g}} \sum_{\substack{\gamma \leq x^{\frac{1}{4}} \\ \gamma \equiv 1 \pmod{4}}} 1 \cdot \sum_{\substack{h \leq \frac{\sqrt{2^v x}}{a} \\ h \equiv 1 \pmod{2^v}}} h^{\frac{7}{2}} + B x^{\frac{3}{2}} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{a}{a b} \sum_{\gamma \leq x^{\frac{1}{4}}} \frac{1}{\gamma} \sum_{\substack{h \leq \frac{\sqrt{2^v x}}{a} \\ h \equiv 1 \pmod{2^v}}} 1 \\ + B x^2 \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\gamma \leq x^{\frac{1}{4}}} 1 \cdot \sum_{h \leq \sqrt{2^v x}} \frac{1}{h} \quad (112)$$

$$= B x^{\frac{5}{2}} \sum_{a, b=1}^{\infty} \left( a b \alpha^{\frac{1}{4}} \beta^{\frac{1}{4}} g^{\frac{1}{2}} \right)^{-1} + B x^{2+\frac{5}{2}} + B x^{\frac{9}{4}+\frac{5}{2}}$$

$$(163) \quad = B x^{\frac{5}{2}} \sum_{a, b=1}^{\infty} (a b)^{-\frac{5}{4}} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \quad (104, 3).$$

$$\begin{aligned} \mathfrak{S}_3(x) &= \sum_{E_1, E_2=0}^1 2^{v(E_1+E_2-2)} \mathfrak{S}_{E_1, E_2} \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ \frac{v-1}{2} \leq a \\ b < \frac{2}{2} a}} \frac{1}{a b g} \sum_{\substack{\gamma \leq x^{\frac{1}{4}} \\ \gamma < \frac{1}{4} a}} \frac{1}{\gamma} \\ &\times \sum_{\substack{\frac{4}{9} \leq h \leq X \\ x^{\frac{9}{9}} \leq h \leq X}} \alpha^4 h^4 \left( \frac{\beta}{\alpha} \right)^{2E_2} \sin \frac{2\pi \gamma a}{\beta} + B x^{\frac{5}{2}} \quad (156, 163) \end{aligned}$$

$$(164) \quad = \sum_{E_1, E_2=0}^1 2^{v(E_1+E_2-2)} \mathfrak{S}_{E_1, E_2} \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ \frac{v-1}{2} \leq a \\ b < \frac{2}{2} a}} \frac{1}{a b g} \alpha^4 \left( \frac{\beta}{\alpha} \right)^{2E_2} \\ \times \sum_{\substack{\frac{1}{4} \leq \gamma \\ \gamma \leq x^{\frac{1}{4}}}} \frac{1}{\gamma} \sin \frac{2\pi \gamma a}{\beta} \sum_{\substack{\frac{4}{9} \leq h \leq X \\ x^{\frac{9}{9}} \leq h \leq X}} h^4 + B x^{\frac{5}{2}}.$$

Im weiteren Verlauf des Beweises von (154) soll

$$(165) \quad x > 2^{\frac{v+1}{4} - \frac{1}{2}} + 3$$

angenommen werden (rechts steht eine absolute Konstante).

$$\sqrt{2^v} \alpha - \sqrt{2^v} \beta = \frac{2^v \alpha^2 - 2^v \beta^2}{\sqrt{2^v} \alpha + \sqrt{2^v} \beta} \geq \frac{1}{\sqrt{2^v} \alpha + \sqrt{2^v} \beta} \quad (157)$$

$$(166) \quad \geq \frac{1}{2^{v+1}(\alpha + \beta)} \geq \frac{1}{2^{v+1+1} x^{\frac{5}{2}}} \quad (104, 164).$$

$$\left( \gamma \leq x^{\frac{1}{4}} \right) \frac{\sqrt{2^v} x - n'}{\beta} - \frac{\sqrt{2^v} x - m'}{\alpha} = \frac{(\sqrt{2^v} \alpha - \sqrt{2^v} \beta) \sqrt{x} - \gamma}{\alpha \beta}$$

$$\geq \frac{(\sqrt{2^v} \alpha - \sqrt{2^v} \beta) \sqrt{x} - \gamma}{\alpha \beta} \quad (109)$$

$$(167) \quad \geq \frac{1}{\alpha \beta} \left( 2^{-v-1} x^{\frac{1}{2} - \frac{5}{2}} - x^{\frac{1}{4}} \right) = \frac{x^{\frac{1}{4}}}{\alpha \beta} \left( 2^{-v-1} x^{\frac{1}{4} - \frac{5}{2}} - 1 \right) > 0 \quad (166, 165),$$

$$(168) \quad X = \frac{\sqrt{2^v} x - m'}{\alpha} \quad (112, 167),$$

$$(169) \quad \sum_{\substack{\frac{4}{9} \leq h \leq X \\ x^{\frac{9}{9}} \leq h \leq X}} h^4 = \sum_{\substack{\frac{4}{9} \leq h \leq X \\ x^{\frac{9}{9}} \leq h \leq \frac{\sqrt{2^v} x}{\alpha}}} h^4 + B \frac{x^2}{\alpha^4} \quad (168, 110).$$

$$\begin{aligned} \mathfrak{S}_3(x) &= \sum_{E_1, E_2=0}^1 2^{v(E_1+E_2-2)} \mathfrak{S}_{E_1, E_2} \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ \frac{v-1}{2} \leq a \\ b < \frac{2}{2} a}} \frac{1}{a b g} \alpha^4 \left( \frac{\beta}{\alpha} \right)^{2E_2} \\ &\times \sum_{\substack{\frac{1}{4} \leq \gamma \\ \gamma \leq x^{\frac{1}{4}}}} \frac{1}{\gamma} \sin \frac{2\pi \gamma a}{\beta} \cdot \sum_{\substack{\frac{4}{9} \leq h \leq \frac{\sqrt{2^v} x}{\alpha} \\ x^{\frac{9}{9}} \leq h \leq \frac{\sqrt{2^v} x}{\alpha}}} h^4 + B x^2 \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ \gamma \leq x^{\frac{1}{4}}}} \frac{1}{a b} \sum_{\substack{\frac{1}{4} \leq \gamma \\ \gamma \leq x^{\frac{1}{4}}}} \frac{1}{\gamma} \\ &+ B x^{\frac{5}{2}} \quad (164, 169, 157) \end{aligned}$$

$$= B \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ \frac{v-1}{2} \leq a \\ b < \frac{2}{2} a}} \frac{1}{a b g} \alpha^4 \frac{x^{\frac{5}{2}}}{\alpha^5} + B x^{\frac{5}{2}} \quad (75, 76, 2)$$

$$= B x^{\frac{5}{2}} \sum_{\substack{a, b \leq x^{\frac{1}{2}} \\ \frac{v-1}{2} \leq a \\ b < \frac{2}{2} a}} \frac{1}{a^2 b} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \sum_{a, b=1}^{\infty} (a b)^{-\frac{3}{2}} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \quad (104).$$

Hilfssatz 29:

$$(170) \quad \mathfrak{S}_4(x) = \sum_{E_1, E_2=0}^1 \mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{\frac{1}{4} \leq \gamma < \frac{v+1}{2} \\ x^{\frac{1}{4}} < \gamma < \frac{v+1}{2}}} \frac{1}{\gamma}$$



$$\begin{aligned}
 & \times \sum_{\substack{4 \\ x^9 \leq h \leq X \\ m \leq 2^{\frac{v-1}{2}} n}} m^{2E_1} n^{2E_2} r^{2-E_1-E_2} \sin \frac{2\pi r \gamma g}{m n} \\
 (171) \quad & = B x^{\frac{5}{2}}.
 \end{aligned}$$

Beweis: Für  $m \leq 2^{\frac{v-1}{2}} n$  ist  $r = \frac{n^2}{2^v}$  (88). Nach (170) ist also

$$\begin{aligned}
 (172) \quad \mathfrak{S}_4(x) &= \sum_{E_1, E_2=0}^1 2^{v(E_1+E_2-2)} \mathfrak{S}_{E_1, E_2} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{4 \\ x^9 \leq h \leq X \\ m \leq 2^{\frac{v-1}{2}} n}} \frac{1}{x^4 \leq \gamma \leq 2^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}}} \gamma \\
 & \times \sum_{\substack{4 \\ x^9 \leq h \leq X \\ m \leq 2^{\frac{v-1}{2}} n}} n^4 \left( \frac{m}{n} \right)^{2E_1} \sin \frac{2\pi n \gamma g}{2^v m}.
 \end{aligned}$$

Ich mache wieder die Annahmen (119) und (120), setze jetzt aber überdies voraus, dass

$$(173) \quad m \leq 2^{\frac{v-1}{2}} n \text{ für } H \leq h \leq H'$$

ist. Es sei  $u$  eine beliebige Zahl des Intervalls  $p \leq u \leq q$ .

$$(174) \quad f_2(u) = \frac{n_u \gamma g}{2^v m_u}, \quad F_2(u) = m_u^{2E_1} n_u^{4-2E_1},$$

$$(175) \quad \omega_2 = f_2''(q).$$

Dann gilt

$$(176) \quad f_2'(u) = \frac{(\beta m_u - \alpha n_u) \gamma g}{2^v m_u^2} = -\frac{\gamma^3 g}{2^v m_u^3} \quad (174, 111, 109),$$

$$(177) \quad f_2''(u) = \frac{2\gamma^2 \alpha g}{2^v m_u^3} \quad (176, 111).$$

Mit wachsendem  $u$  nimmt  $f_2''(u)$  ab (177), da  $m_u$  wächst (111). Also ist

$$\omega_2 = f_2''(q) \leq f_2''(u) \leq f_2''(p) = B f_2''(g) = B \omega_2 \quad (175, 177, 123).$$

$F_2'(u)$  ist stetig und positiv (174, 111, 3);  $p - \frac{1}{2}$ ,  $q - \frac{1}{2}$  sind ganz,  $0 \leq p < q$  (120, 119). Also darf (116) auf die durch (120), (174), (175) gegebenen  $p, q, f_2, F_2, \omega$  angewandt werden:

$$(178) \quad \sum_{p \leq h \leq q} F_2(h) \sin \{2\pi f_2(h)\} = B F_2(q) \left( q \sqrt{\omega_2} + \frac{1}{\sqrt{\omega_2}} \right).$$

$$\sum_{h=1}^{H'} n^4 \left( \frac{m}{n} \right)^{2E_1} \sin \frac{2\pi n \gamma g}{2^v m} = B n_q^4 \left( \frac{m_q}{n_q} \right)^{2E_1} \left( q \sqrt{\omega_2} + \frac{1}{\sqrt{\omega_2}} \right) \quad (178, 120, 174, 108)$$

$$= B x^2 \left( \frac{m_H}{n_H} \right)^{2E_1} \left( H \gamma \alpha^{\frac{1}{2}} g^{\frac{1}{2}} m_q^{-\frac{3}{2}} + m_q^{\frac{1}{2}} m_q \gamma^{-1} \alpha^{-\frac{1}{2}} g^{-\frac{1}{2}} \right) \quad (121, 122, 119, 120, 175, 177)$$

$$= B x^2 \left( H \gamma \alpha^{\frac{1}{2}} g \alpha^{-\frac{3}{2}} x^{-\frac{2}{3}} + x^{\frac{1}{4}} \alpha H \gamma^{-1} \alpha^{-\frac{1}{2}} \right) \quad (173, 108, 125, 121, 110)$$

$$(179) \quad = B x^{\frac{4}{3}} g \gamma H + B x^{\frac{9}{4}} \alpha^{\frac{1}{2}} \gamma^{-1} H.$$

Ist für ein gegebenes Tripel  $a, b, \gamma$  die  $h$ -Summe in (172) nicht leer, so läuft  $h$  über ein Intervall (Hilfssatz 23), Zerlege ich dieses in Teilintervalle  $H \leq h \leq H'$  mit (119), (173) und wende (179) an, so folgt

$$\sum_{\substack{4 \\ x^9 \leq h \leq X \\ m \leq 2^{\frac{v-1}{2}} n}} n^4 \left( \frac{m}{n} \right)^{2E_1} \sin \frac{2\pi n \gamma g}{2^v m} = B x^{\frac{4}{3}} g \gamma X + B x^{\frac{9}{4}} \alpha^{\frac{1}{2}} \gamma^{-1} X$$

$$= B x^{\frac{4}{3}} g \gamma x^{\frac{1}{2}} + B x^{\frac{9}{4}} \alpha^{\frac{1}{2}} \gamma^{-1} x^{\frac{1}{2}} \alpha^{-\frac{3}{4}} \beta^{-\frac{1}{4}} \quad (113)$$

$$(180) \quad = B x^{\frac{11}{6}} g \gamma + B x^{\frac{11}{4}} \alpha^{-\frac{1}{4}} \beta^{-\frac{1}{4}} \gamma^{-1}.$$

$$\mathfrak{S}_4(x) = B x^{\frac{11}{6}} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\substack{4 \\ \gamma \leq 2^{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} x^{\frac{1}{2} + \frac{1}{2}}}} 1 + B x^{\frac{11}{4}} \sum_{a, b=1}^{\infty} (a b)^{-\frac{5}{4}} \sum_{\gamma > x^{\frac{1}{4}}} \gamma^{-2} \quad (172, 180, 104)$$

$$= B x^{\frac{5}{2} - \frac{1}{6} + 2\beta} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \quad (3).$$

Hilfssatz 30:

$$(181) \quad \mathfrak{C}_5(x) = \sum_{E_1, E_2=0}^1 \mathfrak{C}_{E_1, E_2} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{x^{\frac{1}{4}} < \gamma \leq 2^{\frac{1}{2}+\nu} x^{\frac{1}{2}+\delta} \\ \gamma}} \frac{1}{\gamma} \\ \times \sum_{\substack{4 \\ x^{\frac{1}{9}} \leq h \leq X \\ m > 2^{\frac{\nu-1}{2}} n}} m^{2E_1} n^{2E_2} r^{2-E_1-E_2} \sin \frac{2\pi r \gamma g}{m n} \\ = B x^{\frac{5}{2}}.$$

Beweis: Für  $m > 2^{\frac{\nu-1}{2}} n$  ist  $r = \frac{m^2}{2^\nu}$  (88). Nach (181) ist also

$$(183) \quad \mathfrak{C}_5(x) = \sum_{E_1, E_2=0}^1 2^{\nu(E_1+E_2-2)} \mathfrak{C}_{E_1, E_2} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b g} \sum_{\substack{x^{\frac{1}{4}} < \gamma \leq 2^{\frac{1}{2}+\nu} x^{\frac{1}{2}+\delta} \\ \gamma}} \frac{1}{\gamma} \\ \times \sum_{\substack{4 \\ x^{\frac{1}{9}} \leq h \leq X \\ m > 2^{\frac{\nu-1}{2}} n}} m^4 \left( \frac{n}{m} \right)^{2E_2} \sin \frac{2\pi m \gamma g}{2^\nu n}.$$

Ich mache wieder die Annahmen (119) und (120), setze jetzt aber überdies voraus, dass

$$(184) \quad m > 2^{\frac{\nu-1}{2}} n \text{ für } H \leq h \leq H'$$

ist. Es sei  $u$  eine beliebige Zahl des Intervalls  $p \leq u \leq q$ ,

$$(185) \quad f_3(u) = -\frac{m_u \gamma g}{2^\nu n_u}, \quad F_3(u) = n_u^{2E_2} m_u^{4-2E_2},$$

$$(186) \quad \omega_3 = f_3''(q).$$

Dann gilt

$$(187) \quad f_3'(u) = \frac{(\beta m_u - \alpha n_u) \gamma g}{2^\nu n_u^2} = -\frac{\gamma^2 g}{2^\nu n_u^2} \quad (185, 111, 109),$$

$$(188) \quad f_3''(u) = \frac{2\gamma^2 \beta g}{2^\nu n_u^3} \quad (187, 111).$$

Mit wachsendem  $u$  nimmt  $f_3''(u)$  ab (188), da  $n_u$  wächst (111). Also ist

$$\omega_3 = f_3''(q) \leq f_3''(u) \leq f_3''(p) = B f_3''(q) = B \omega_3 \quad (186, 188, 124).$$

$F_3'(u)$  ist stetig und positiv (185, 111, 3);  $p - \frac{1}{2}$ ,  $q - \frac{1}{2}$  sind ganz,  $0 \leq p < q$  (120, 119). Also darf (116) auf die durch (120), (185), (186) gegebenen  $p, q, f_3, F_3, \omega_3$  angewandt werden:

$$(189) \quad \sum_{p \leq h \leq q} F_3(h) \sin \{2\pi f_3(h)\} = B F_3(q) \left( q \sqrt{\omega_3} + \frac{1}{\sqrt{\omega_3}} \right).$$

$$\sum_{h=H}^{H'} m^4 \left( \frac{n}{m} \right)^{2E_2} \sin \frac{2\pi m \gamma g}{2^\nu n} = B m_q^4 \left( \frac{n_q}{m_q} \right)^{2E_2} \left( q \sqrt{\omega_3} + \frac{1}{\sqrt{\omega_3}} \right) \quad (189, 120, 185, 108)$$

$$= B x^5 \left( \frac{n_H}{m_H} \right)^{2E_2} \left( H \gamma \beta^{\frac{1}{2}} g^{\frac{1}{2}} n_q^{-\frac{3}{2}} + n_q^{\frac{1}{2}} n_q \gamma^{-1} \beta^{-\frac{1}{2}} g^{-\frac{1}{2}} \right) \quad (121, 122, 119, 120, 186, 188)$$

$$= B x^5 \left( H \gamma \beta^{\frac{1}{2}} g \beta^{-\frac{3}{2}} x^{-\frac{2}{3}} + x^{\frac{1}{4}} (\gamma + \beta H) \gamma^{-1} \beta^{-\frac{1}{2}} \right) \quad (184, 108, 126, 122, 110)$$

$$(190) \quad = B x^{\frac{4}{3}} g \gamma H + B x^{\frac{9}{4}} + B x^{\frac{9}{4}} \beta^{\frac{1}{2}} \gamma^{-1} H.$$

Ist für ein gegebenes Tripel  $a, b, \gamma$  die  $h$ -Summe in (183) nicht leer, so läuft  $h$  über ein Intervall (Hilfssatz 23). Zerlege ich dieses in Teilintervalle  $H \leq h \leq H'$  mit (119), (184) und wende (190) an, so folgt

$$\sum_{\substack{4 \\ x^{\frac{1}{9}} \leq h \leq X \\ m > 2^{\frac{\nu-1}{2}} n}} m^4 \left( \frac{n}{m} \right)^{2E_2} \sin \frac{2\pi m \gamma g}{2^\nu n} = B x^{\frac{4}{3}} g \gamma X + B x^{\frac{9}{4}+\delta} + B x^{\frac{9}{4}} \beta^{\frac{1}{2}} \gamma^{-1} X \\ = B x^{\frac{4}{3}} g \gamma x^{\frac{1}{2}} + B x^{\frac{9}{4}+\delta} + B x^{\frac{9}{4}} \beta^{\frac{1}{2}} \gamma^{-1} x^{\frac{1}{2}} \alpha^{-\frac{1}{4}} \beta^{-\frac{3}{4}} \quad (113)$$

$$(191) \quad = B x^{\frac{11}{6}} g \gamma + B x^{\frac{9}{4}+\delta} + B x^{\frac{11}{4}} \alpha^{-\frac{1}{4}} \beta^{-\frac{1}{4}} \gamma^{-1}.$$

$$\mathfrak{C}_5(x) = B x^6 \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\substack{\gamma \leq 2^{\frac{1}{2}+\nu} x^{\frac{1}{2}+\delta} \\ \gamma}} 1 + B x^{\frac{9}{4}+\delta} \sum_{a, b \leq x^{\frac{1}{2}}} \frac{1}{a b} \sum_{\substack{\gamma \leq 2^{\frac{1}{2}+\nu} x^{\frac{1}{2}+\delta} \\ \gamma}} \frac{1}{\gamma}$$

$$+ B x^{\frac{11}{4}} \sum_{a,b=1}^{\infty} (ab)^{-\frac{5}{4}} \sum_{\substack{\gamma > x^{\frac{1}{4}} \\ \gamma^{-2}}} \gamma^{-2} \quad (183, 191, 104)$$

$$= B x^{\frac{5}{2} - \frac{1}{6} + 2\delta} + B x^{\frac{9}{4} + 2\delta} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \quad (3).$$

Beweis von (VI): (91), (114), (118), (139), (154), (171), (182).

Beweis von (II): (VI) mit  $v=0$  und (1).

Beweis von (I): Zur Abkürzung werde gesetzt

$$(192) \quad \lambda = \frac{1}{2} (C + \log 2\pi - 1), \mu = \frac{36 + 5\pi^2}{432}.$$

$$\int_0^x T_0^2(u) du = \int_1^x T_0^2(u) du + B$$

$$= \int_1^x \left( \frac{T(u)}{u} - \lambda \right)^2 du + \int_1^x \left( T_0^2(u) - \left( \frac{T(u)}{u} - \lambda \right)^2 \right) du + B$$

$$(193) = \int_1^x \frac{T^2(u)}{u^2} du - 2\lambda \int_1^x \frac{T(u)}{u} du + \lambda^2 x +$$

$$+ B \int_1^x \left| T_0(u) - \frac{T(u)}{u} + \lambda \right| \left| T_0(u) + \frac{T(u)}{u} - \lambda \right| du + B.$$

$$(194) \quad \int_1^x \frac{T^2(u)}{u^2} du = \frac{1}{x^2} \int_1^x T^2(u) du + 2 \int_1^x \frac{dv}{v^3} \int_1^v T^2(u) du = 3\mu x + B x^{\frac{1}{2}} \quad (II, 193).$$

$$(u \geq 1) - \frac{T(u)}{u} = \frac{1}{2} + \sum_{n \leq \sqrt{u}} \frac{1}{n} \phi\left(\frac{u}{n}\right) + \frac{1}{u} \sum_{n \leq \sqrt{u}} n \phi\left(\frac{u}{n}\right) + B u^{-\frac{1}{2}} \quad (6),$$

$$- \int_1^x \frac{T(u)}{u} du = \frac{x}{2} + \sum_{n \leq \sqrt{x}} \frac{1}{n} \int_{n^2}^x \phi\left(\frac{u}{n}\right) du + \sum_{n \leq \sqrt{x}} n \int_{n^2}^x \phi\left(\frac{u}{n}\right) \frac{du}{u} + B x^{\frac{1}{2}}$$

$$(195) \quad = \frac{x}{2} + \sum_{n \leq \sqrt{x}} \frac{1}{n} n + \sum_{n \leq \sqrt{x}} n \frac{1}{n^2} \mu + B x^{\frac{1}{2}} = \frac{x}{2} + B x^{\frac{1}{2}} \quad (2).$$

$$\int_1^x \left| T_0(u) - \frac{T(u)}{u} + \lambda \right| \left| T_0(u) + \frac{T(u)}{u} - \lambda \right| du = B \int_1^x u^{-\frac{2}{3}} \log u du =$$

$$(196) \quad = B x^{\frac{1}{2}} \quad (4, 192).$$

$$\int_0^x T_0^2(u) du = (3\mu + \lambda + \lambda^2)x + B x^{\frac{1}{2}} \quad (193-196)$$

$$(I) \quad = \left( \left( \frac{C + \log 2\pi}{2} \right)^2 + \frac{5\pi^2}{144} \right) x + B x^{\frac{1}{2}} \quad (192).$$

Hilfssatz 31:

$$(197) \quad P_0(x) = 8T(x) - 32T\left(\frac{x}{4}\right),$$

$$(198) \quad P_1(x) = 4T(x) - 4T\left(\frac{x}{2}\right) + 8T\left(\frac{x}{4}\right) - 32T\left(\frac{x}{8}\right).$$

$$(199) \quad P_2(x) = 2T(x) - 2T\left(\frac{x}{2}\right) + 8T\left(\frac{x}{8}\right) - 32T\left(\frac{x}{16}\right).$$

Beweis: (197) ist  $T_{II}$  (106), (198) ist  $E_V$  (219), (199) ist  $E_V$  (220).

Beweis von (III) - (V):

$$(200) \quad \int_0^x P_0^2(u) du = 2^6 V_0(x) + 2^{12} V_0\left(\frac{x}{4}\right) - 2^{11} V_2\left(\frac{x}{4}\right) + B \quad (197, 1),$$

$$(201) \quad \int_0^x P_1^2(u) du = 2^4 V_0(x) + 2^5 V_0\left(\frac{x}{2}\right) + 2^8 V_0\left(\frac{x}{4}\right) + 2^{13} V_0\left(\frac{x}{8}\right) \\ - 2^6 V_1\left(\frac{x}{2}\right) - 2^8 V_1\left(\frac{x}{4}\right) - 2^{12} V_1\left(\frac{x}{8}\right) + 2^8 V_2\left(\frac{x}{4}\right) \\ + 2^{11} V_2\left(\frac{x}{8}\right) - 2^{11} V_3\left(\frac{x}{8}\right) + B \quad (198, 1),$$

$$\begin{aligned}
 (202) \quad \int_0^x P_2^2(u) du &= 2^2 V_0(x) + 2^3 V_0\left(\frac{x}{2}\right) + 2^9 V_0\left(\frac{x}{8}\right) + 2^{14} V_0\left(\frac{x}{16}\right) \\
 &- 2^4 V_1\left(\frac{x}{2}\right) - 2^{13} V_1\left(\frac{x}{16}\right) - 2^8 V_2\left(\frac{x}{8}\right) + 2^8 V_3\left(\frac{x}{8}\right) \\
 &+ 2^{11} V_3\left(\frac{x}{16}\right) - 2^{11} V_4\left(\frac{x}{16}\right) + B \quad (199, 1).
 \end{aligned}$$

(III) — (V) folgen aus (200) — (202) und (VI).

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Tadeusz Rakowiecki.

## Détermination des orbites des étoiles doubles spectroscopiques d'après les positions opposées.

Nombre d'étoiles doubles forment des couples si serrés, qu'aucun télescope ne peut séparer leurs composantes. C'est à l'aide des observations spectroscopiques qu'on démontre leur duplicité. Les lignes du spectre de telles étoiles se montrent déplacées de leurs positions normales. Ce déplacement pourtant ne reste pas constant, mais il varie régulièrement pendant une période déterminée. Les raies du spectre font des oscillations de part et d'autre d'une position moyenne. Conformément au principe de Doppler-Fizeaux, on explique ces déviations périodiques des raies par les *variables vitesses radiales* des composantes du couple dans leur mouvement autour du centre de gravité commun.

L'intervalle de temps pendant lequel les raies font une oscillation complète, c'est la durée de la révolution des étoiles ou la *période* de leur mouvement orbital. Quand la distance entre l'étoile et l'observatoire augmente, les ondes de la lumière émise par l'astre, deviennent plus longues, quand elle diminue, celles-ci deviennent plus courtes que dans le cas d'une distance invariable. Il en suit que dans le premier cas les raies du spectre sont déviées vers le rouge, dans le second, vers le violet. La grandeur du déplacement est proportionnelle à la vitesse radiale du mouvement relatif de l'étoile par rapport à la terre. Entre la vitesse radiale  $W$  d'un astre et la déviation observée de la raie du spectre il existe la relation suivante:

$$W = 299860 \cdot \frac{\lambda' - \lambda}{\lambda} \text{ kilomètres par seconde} \quad (1)$$