Vol. **109** (2006) *ACTA PHYSICA POLONICA A* No. 1

Proceedings of the 2nd Workshop on Quantum Chaos and Localisation Phenomena,
Warsaw, Poland, May 19–22, 2005

Decoherence and Noise in Loschmidt Echo Experiments

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We discuss some recent results concerning the decoherence in controlled quantum open systems within the mathematical setting corresponding to motion reversal experiments (the Loschmidt echo). We compare the case of randomly chosen sequence of unitary dynamical maps with the case of a constant dynamics corresponding to a classically chaotic evolution. The interplay between chaos and decoherence is illustrated by the new numerical results on the quantum Arnold cat map perturbed by a measurement process. Open problems related to the simple operational characterization of the decoherence strength are discussed.

PACS numbers: 03.65.Yz

1. Introduction

In the last decade control of open quantum systems became an important topic in quantum information and quantum chemistry. The main obstacle to a coherent experimental control of increasingly large multi-body quantum systems is the decoherence phenomenon due to the interaction with an environment. Therefore the quantitative characterization of decoherence factor in terms of a single overall parameter and the operational methods of its determination are crucial. The idea of "Loschmidt echo" leading to experiments with reversed dynamics seems to be very useful in this context. For an isolated system and perfect realization of the dynamics we expect a complete revival of the initial state after reversal of motion. Any imperfections will produce the deviation of the final state from the initial one which can be measured, for instance, in terms of fidelity. Practically, we have to determine the residual population of the initial state.

In particular, the decay of fidelity for isolated quantum system with a perturbed Hamiltonian governing the reversed motion has been used to characterize quantum chaos of classically chaotic systems [1, 2]. The natural generalization

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of this idea is to study motion reversal for open systems [3, 4]. Here, the interaction with an environment produces a mixed state from the initially pure one. Once again one can consider either the decay of fidelity or the increasing deviation from the purity of the state characterized, for example, by entropic quantities. The Loschmidt echo experiments can be modified in the case of quantum devices capable of universal control. Then we can choose a smoothly time-dependent Hamiltonian or the sequence of quantum gates (kicked dynamics) which mimic a random choice with a reasonable accuracy.

We begin with a short presentation of the mathematical formalism used in the theory of controlled quantum open systems. Physical assumptions justifying this approach are also discussed. In Sect. 2 we briefly review the model with a random choice of unitary quantum gates and define the appropriate single parameter characterizing noise in a quantum device. The measures of decoherence and chaos based on the Renyi α-entropies are discussed in Sect. 3. In Sect. 4 we present the new results concerning decoherence for a certain model of open quantum system quantified in terms of the linear entropy. Its internal dynamics is given by a quantized automorphism of the torus ("quantum Arnold cat map") while the influence of the environment is described in terms of the coarse-grained von Neumann measurement with K orthogonal projections of equal dimension. The numerical results display different regimes of decoherence depending on the relations between two fundamental parameters: the Kolmogorov–Sinai entropy $h_{\rm KS}$ characterizing the corresponding classical Arnold cat map, and $\ln K$ which measures the strength of the noise. Such relations were already confirmed in a number of publications, however, for the present model we observe for the first time the crossover from the decoherence governed by $\ln K$ to decoherence determined by semiclassical chaos ($h_{\rm KS}$).

Section 4 is devoted to some open problems concerning the proper measure of the "decoherence strength" for a generic noise superoperator and the relations between fidelity decay and entropy production.

2. Controlled quantum open systems

We restrict ourselves to models of N-level quantum systems with N-dimensional Hilbert space $\boldsymbol{C}^N$. Such finite systems are frequently used in the context of quantum information processing and quantum control. Typically, they are decomposed into "qubits" which can correspond either to well-defined physical subsystems (localized spins) or some approximations of real systems (ground and first excited state of atom, ion or quantum dot; vacuum and 1-photon state in a cavity, two directions of current in superconducting devices, etc.). We assume that the system can be controlled, i.e., its ideal evolution is programmable and for mathematical simplicity described by a sequence of unitary matrices $U_1, U_2, \ldots$ from the group $\boldsymbol{U}(N)$ of all $N \times N$ unitary matrices. The influence of the environment and the (random) errors in preparation of the initial state and the implementation of the controlled dynamics imply the use of density matrices ρ (positive, trace $= 1$, $N \times N$ matrices) representing mixed states of the system. To describe the most general irreversible evolution we use dynamical maps treated as linear completely positive and trace preserving "superoperators" acting on the space $\boldsymbol{M}_N$ of all complex $N \times N$ matrices. Such maps can be always represented in a standard form in terms of the matrices A_j:

$$\rho \to \hat{\Lambda}\rho = \sum_{k=1}^{K} A_k \rho A_k^{\dagger}, \qquad \sum_{k=1}^{K} A_k^{\dagger} A_k = \mathbb{1}. \tag{1}$$

Two special cases will be used frequently:

1) a reversible unitary evolution in the superoperator form

$$\rho \to \hat{\mathcal{U}}\rho = U\rho U^{\dagger}, \qquad UU^{\dagger} = U^{\dagger}U = \mathbb{1}, \tag{2}$$

2) a dynamical map corresponding to a von Neumann projective measurement (without recording the outcome)

$$\rho \to \hat{\mathcal{E}}\rho = \sum_{k=1}^{K} P_k \rho P_k, \qquad P_k = P_k^2 = P_k^{\dagger}, \qquad \sum_{k=1}^{K} P_k = \mathbb{1}. \tag{3}$$

In the following we use a picture of the system observed in discrete time steps $n = 0, 1, 2, \ldots$ with the corresponding sequence of states $\rho(n)$:

$$\rho(n) = \hat{\Lambda}_n \hat{\mathcal{U}}_n \cdots \hat{\Lambda}_2 \hat{\mathcal{U}}_2 \hat{\Lambda}_1 \hat{\mathcal{U}}_1 \rho(0), \tag{4}$$

where $\hat{\mathcal{U}}_k$ represents the perfect controlled unitary evolution while $\hat{\Lambda}_k$ describes the cumulative influence of noise during k-th evolution step. The product structure of the total dynamics reflects the Markovian assumption behind this model. One should remember that although the Markovian approximation is justified for the weak coupling with an environment and constant system Hamiltonian, in the case of fast quantum gates U_k the non-Markovian effects can be relevant. Another, often used assumption is the independence of the noise from the unitary dynamics which implies

$$\hat{\Lambda}_n \equiv \hat{\Lambda} \qquad \text{for all } n = 1, 2, 3 \ldots \tag{5}$$

This is again an idealization which cannot be universally justified. Indeed, in the case of "thermal noise" one should expect that the system is driven to a thermal equilibrium given by a Gibbs state depending on the temporal Hamiltonian and hence noise must be correlated with the unitary dynamics. However, very often the main source of noise are various macroscopic imperfections in the devices controlling the system and then the assumption (5) can be a proper one.

3. Fidelity in Loschmidt echo experiment

We consider now a discrete-time model of Loschmidt echo experiment [3] for a quantum open system with the dynamics of the type (4) with the condition (5). The final state for the reversal motion experiment is given by

$$\rho(2n) = \hat{\Lambda}\hat{\mathcal{U}}_1^{-1}\hat{\Lambda}\hat{\mathcal{U}}_2^{-1} \cdots \hat{\Lambda}\hat{\mathcal{U}}_n^{-1}\hat{\Lambda}\hat{\mathcal{U}}_n \cdots \hat{\Lambda}\hat{\mathcal{U}}_2\hat{\Lambda}\hat{\mathcal{U}}_1\rho(0). \tag{6}$$

A natural measure of a noise strength could be a fidelity of the final state with respect to the initial one, which is assumed to be pure, i.e. $\rho(0) = |\psi\rangle\langle\psi|$, given by the following formula:

$$F(2n) = \langle \psi, \rho(2n)\psi \rangle$$
$$= \mathrm{Tr}\left[\rho(0)\hat{\Lambda}\hat{\mathcal{U}}_1^{-1}\hat{\Lambda}\hat{\mathcal{U}}_2^{-1}\cdots\hat{\Lambda}\hat{\mathcal{U}}_n^{-1}\hat{\Lambda}\hat{\mathcal{U}}_n\cdots\hat{\Lambda}\hat{\mathcal{U}}_2\hat{\Lambda}\hat{\mathcal{U}}_1\rho(0)\right]. \tag{7}$$

Obviously, the fidelity (7) generally depends on the choice of unitary gates and the initial state and therefore a proper averaging is necessary to obtain a universal behavior. We can imagine that we are able to implement a set of unitary maps $\{U_k\}$ randomly and independently chosen from the group $\boldsymbol{U}(N)$. But even then we are not able to compute the average of (7) because of statistical correlation between the reversed path of the evolution and the initial one. However, we can compute a simpler object, the fidelity of the "interaction picture"

$$F^{\mathrm{int}}(n) = \langle \psi, \rho(2n)\psi \rangle$$
$$= \mathrm{Tr}\left[\rho(0)\hat{\mathcal{U}}_1^{-1}\hat{\mathcal{U}}_2^{-1}\cdots\hat{\mathcal{U}}_{\mathrm{n}}^{-1}\hat{\Lambda}\hat{\mathcal{U}}_{\mathrm{n}}\cdots\hat{\Lambda}\hat{\mathcal{U}}_2\hat{\Lambda}\hat{\mathcal{U}}_1\rho(0)\right]. \tag{8}$$

Now we can average the fidelity (8) by multiple integration over the normalized Haar measure $\mathrm{d}U$ on $\boldsymbol{U}(N)$:

$$\overline{F^{\mathrm{int}}}(n) = \langle \psi, \rho(2n)\psi \rangle$$
$$= \int \mathrm{d}U_1 \ldots \int \mathrm{d}U_n \mathrm{Tr}\left[(\rho(0)\hat{\mathcal{U}}_1^{-1}\cdots\hat{\mathcal{U}}_n^{-1}\hat{\Lambda}\hat{\mathcal{U}}_n\cdots\hat{\Lambda}\hat{\mathcal{U}}_1\rho(0)\right]. \tag{9}$$

We use the identity

$$\mathrm{Tr}\left[\rho(0)\hat{\mathcal{U}}_1^{-1}\cdots\hat{\mathcal{U}}_n^{-1}\hat{\Lambda}\hat{\mathcal{U}}_n\cdots\hat{\Lambda}\hat{\mathcal{U}}_1\rho(0)\right] = \mathrm{Tr}\left[\rho(0)\hat{\Lambda}_n\cdots\hat{\Lambda}_1\rho(0)\right], \tag{10}$$

where $\hat{\Lambda}_k = \hat{\mathcal{U}}_1^{-1}\cdots\hat{\mathcal{U}}_k^{-1}\hat{\Lambda}\hat{\mathcal{U}}_n\cdots\hat{\mathcal{U}}_1$ and the invariance of Haar measure with respect to the group action to obtain

$$\overline{F^{\mathrm{int}}}(n) = \mathrm{Tr}\left[\rho(0)(\hat{\Lambda}^{\mathrm{av}})^n\rho(0)\right]. \tag{11}$$

In the formula (11) $\hat{\Lambda}^{\mathrm{av}}$ denotes the averaged superoperator

$$\hat{\Lambda}^{\mathrm{av}} \equiv \int \mathrm{d}U\hat{\mathcal{U}}^{-1}\hat{\Lambda}\hat{\mathcal{U}}. \tag{12}$$

The averaging procedure has been discussed in [3] and always leads to the so-called depolarizing channel of the form

$$\hat{\Lambda}^{\mathrm{av}}\rho = p\rho + (1-p)\mathrm{Tr}(\rho)\frac{\mathbb{1}}{N}, \tag{13}$$

where

$$p = \frac{\sum_k |\mathrm{Tr}(A_k)|^2 - 1}{N^2 - 1}. \tag{14}$$

Combining (11), (13) and (14) we obtain the exponential decay of fidelity to the value $1/N$:

$$\overline{F^{\text{int}}}(n) = p^n + \frac{1-p^n}{N}. \tag{15}$$

For large n and N we can expect that the statistical dependence present in the formula (7) becomes practically irrelevant and one can approximately use the averaged interaction picture fidelity (8) to analyze the Loschmidt echo experiment putting

$$\overline{F}(2n) \simeq \overline{F^{\text{int}}}(2n). \tag{16}$$

Therefore, at least in principle we have a single parameter characterizing the level of noise in a quantum device which can be directly measured.

The main difficulty in implementation of the above scheme is the practical realization of the random and statistically independent choice of quantum gates U_k. As random evolution provides a generic model of quantum chaos the natural question arises: Can we replace a sequence of random unitaries $\{U_k\}$ by a single "chaotic enough" U? A partial answer is known in the case of the Loschmidt echo with unitary perturbation [1, 2]. If the perturbation is large enough the exponential decay of fidelity $F^{\text{int}}(n)$ for short times is governed by the Kolmogorov–Sinai entropy of the corresponding classical unperturbed dynamics. Therefore, at least in this regime we do not obtain any information about the noise level but only about the chaotic properties of the unperturbed evolution U.

In the next sections we shall investigate the similar problem for open systems (i.e., irreversible nonunitary perturbation) using a different entropic measure of decoherence.

4. Entropic measures of decoherence and chaos

Applying an irreversible dynamical map $\hat{\Lambda}$ to an initial pure state $|\psi\rangle\langle\psi|$ we generally obtain a mixed state $\rho = \hat{\Lambda}(|\psi\rangle\langle\psi|)$. Its Renyi α-entropy defined as

$$I_\alpha(\rho) = \frac{1}{1-\alpha} \ln \mathrm{Tr}\rho^\alpha \le \ln N \tag{17}$$

is a natural measure of decoherence. In practice two cases are most frequently used: the von Neumann entropy $S(\rho) = \lim_{\alpha\to 1} I_\alpha(\rho)$ and the "linear entropy" $I_2(\rho)$ directly related to the "purity" $\mathrm{Tr}\rho^2$. In the scheme similar to that of the previous section we can define the time-dependent entropy

$$I_\alpha[n,\psi] = I_\alpha\left[[\hat{\Lambda}\hat{\mathcal{U}}]^n(|\psi\rangle\langle\psi|)\right], \tag{18}$$

which describes a loss of quantum coherence due to a combined effect of the unitary evolution and the environmental influence. To extract a measure of decoherence independent of the initial state we have to perform certain averaging with respect to ψ. An interesting and important construction involves an "ancillary" N-level system with a trivial dynamics. Starting with an initial state for the composed system (system+ancilla) chosen as a maximally entangled state,

$$|\Psi_{\max}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |e_j\rangle \otimes |e'_j\rangle, \tag{19}$$

where $\{e_j\}$ and $\{e'_j\}$ are orthonormal bases for the system and ancilla, respectively, we can define a time-dependent entropy

$$I_\alpha[n] = I_\alpha([\hat{\Lambda}\hat{\mathcal{U}} \otimes \mathbb{1}_N]^n(|\Psi_{\max}\rangle\langle\Psi_{\max}|)) \leq 2\ln N. \tag{20}$$

It is not difficult to show that the entropy $I_\alpha[n]$ does not depend on the choice of these two bases. One can find also an upper bound in terms of the number of matrices in the Kraus representation of $\hat{\Lambda}$ (see (1))

$$I_\alpha[n] \leq n \ln K. \tag{21}$$

The interesting feature of the definition (20) is its relation to the Kolmogorov–Sinai dynamical entropy h_{KS} in the case of $\alpha = 1,\ I_1 \equiv S$. The extensive discussion of this topic can be found in [4, 5]. Here we notice only that for the quantum open system possessing a classical counterpart the von Neumann entropy $S[n]$ converges in the classical limit ($N \to \infty$) to the quantity which coincides with the entropy used in the definition of h_{KS} [6]. Therefore for large enough N and the choice of the map $\hat{\Lambda}$ such that $\ln K > h_{\mathrm{KS}}$ we can expect a linear increase in $S[n]$ with the slope approximately equal to h_{KS} up to the saturation point $2\ln N$. Indeed such a short time semiclassical behavior has been observed in [7, 4] for the von Neumann entropy.

5. Quantum cat map subjected to measurements

The automorphism of the torus called Arnold cat map defined by

$$\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x}|\mathrm{mod}1, \qquad \boldsymbol{x} = (x_1, x_2), \qquad x_k \in [0,1),$$

$$\boldsymbol{T} = [t_{kl}], \qquad t_{kl} - \text{integer}, \qquad \det\boldsymbol{T} = 1, \qquad \mathrm{Tr}\boldsymbol{T} > 2 \tag{22}$$

is the simplest example of a smooth conservative dynamical system with highest ergodic properties. It is ergodic, mixing and even Anosov dynamical system. The last strongest property means that we can globally decompose the (2-dimensional) phase space into expanding and contracting components characterized by a positive and negative Lapunov exponents λ_+ and λ_-, respectively. The Lapunov exponents are given by the logarithms of the eigenvalues of $\boldsymbol{T}$ and satisfy $\lambda_+ + \lambda_- = 0$. Due to the Pesin theorem the Kolmogorov–Sinai entropy is equal to the sum of positive Lapunov exponents and hence for the cat map $h_{\mathrm{KS}} = \lambda_+$. The different types of quantization of the cat map were discussed extensively in the literature [8–10]. We use here the simplest version based on the N-dimensional Hilbert space with a basis $\{|m\rangle;\ m = 1, 2, \ldots, N\}$ and under the assumption $t_{21} \equiv 1$. Then the unitary matrix U_T describing the quantization of the evolution map (22) has the following matrix elements:

$$\langle m|U_T|n\rangle = \frac{1}{\sqrt{N}} \exp\left[-\frac{\pi \mathrm{i}}{N t_{21}}(t_{11}m^2 + t_{22}n^2 - 2mn)\right]. \tag{23}$$

We assume that after any evolution step governed by the unitary map (23) a von Neumann measurement $\hat{\mathcal{E}}_K$ is performed. It is given in terms of K-projections with respect to the canonical basis $\{|m\rangle\}$:

$$P_k = \sum_{r(k-1)<m\leq rk} |m\rangle\langle m|, \qquad k = 1, 2, \ldots, K, \qquad N = Kr. \tag{24}$$

We have computed the values of linear entropy $I_2[n]$ for different $N = 16$, 32, 64 and $K = 2$, 8 with two choices of $\boldsymbol{T}$ and corresponding values of $h_{\mathrm{KS}} = \lambda_+$:

$$\boldsymbol{T}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad h_{\mathrm{KS}}(\boldsymbol{T}_1) = 0.96, \tag{25}$$

$$\boldsymbol{T}_2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \qquad h_{\mathrm{KS}}(\boldsymbol{T}_2) = 1.32. \tag{26}$$

The results confirm the expectation that the behavior of I_2 should be essentially the same as for the von Neumann entropy used previously in different examples [4, 7]. This expectation is based on the Shannon−McMillan–Breiman theorem for classical systems [11] which translated to the quantum domain implies that for large N the density matrix $[\hat{\mathcal{E}}\hat{\mathcal{U}} \otimes \mathbb{1}_N]^n(|\Psi_{\max}\rangle\langle\Psi_{\max}|)$ satisfies "microcanonical condition", i.e., all eigenvalues are practically equal (in logarithmic scale) or vanish. Then such a density matrix is roughly proportional to a projector ($\simeq P/(\mathrm{Tr}P)$) and therefore all entropies ($I_\alpha(P/(\mathrm{Tr}P)) = \ln(\mathrm{Tr}P)$) coincide.

Figures 1 and 2 show that for $\ln K < h_{\mathrm{KS}}$ the entropy $I_2[n]$ grows linearly with the slope equal to the maximal admissible by (21) value $\ln K$. This confirms the previous result obtained for different systems and von Neumann entropy. The case $\ln K > h_{\mathrm{KS}}$ is illustrated by Figs. 3 and 4. We observe for the first time a clear crossover between two regimes. In the first interval, independent of the Hilbert space dimension, the slope of the plot $I_2[n]$ is equal to $\ln K$ then the chaotic behavior dominates and the slope is given by h_{KS}.

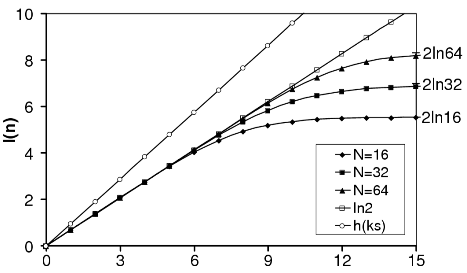

Fig. 1

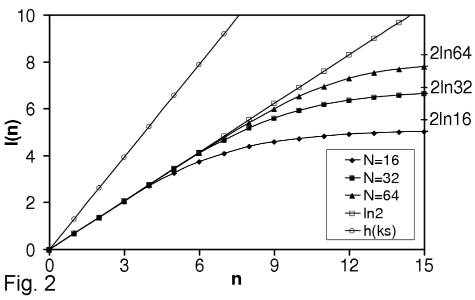

Fig. 2

Fig. 1. The linear entropy evolution governed by $\boldsymbol{T}_1$ matrix for different Hilbert space dimensions $N = 16$, 32 and 64 and 2-projections ($K = 2$) vs. the step number. The slope of the initial evolution is equal to $\ln 2 \approx 0.693$ and the slope given by $h_{\mathrm{KS}}(\boldsymbol{T}_1)$ equals 0.96.

Fig. 2. The linear entropy evolution governed by $\boldsymbol{T}_2$ matrix for different Hilbert space dimensions $N = 16$, 32 and 64 and 2-projections ($K = 2$) vs. the step number. The slope of the initial evolution is equal to $\ln 2 \approx 0.693$ and the slope given by $h_{\mathrm{KS}}(\boldsymbol{T}_2)$ equals 1.32.

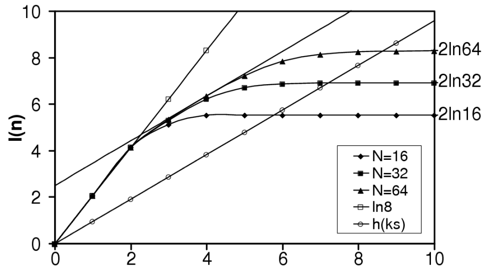

Fig. 3

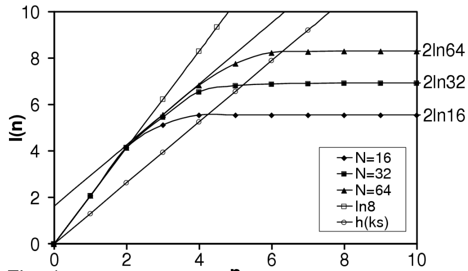

Fig. 4

Fig. 3. The linear entropy evolution governed by $\boldsymbol{T}_1$ matrix for different Hilbert space dimensions $N = 16$, 32 and 64 and 8-projections ($K = 8$) vs. the step number. The slope of the initial evolution is equal to $\ln 8 \approx 2.08$ and the slope determined by $h_{\mathrm{KS}}(\boldsymbol{T}_1)$ equals 0.96. The continuous line with the slope given by h_{KS} demonstrates the slope of the linear evolution in the second interval.

Fig. 4. The linear entropy evolution governed by $\boldsymbol{T}_2$ matrix for different Hilbert space dimensions $N = 16$, 32 and 64 and 8-projections ($K = 8$) vs. the step number. The slope of the initial evolution is equal to $\ln 8 \approx 2.08$ and the slope determined by $h_{KS}(\boldsymbol{T}_2)$ equals 1.32. The continuous line with the slope given by h_{KS} demonstrates the slope of the linear evolution in the second interval.

6. Discussion

The presented results provide a partial answer to the question: How to characterize by a single measurable parameter the averaged level of noise in a quantum device with a controlled dynamics?

The case of a device capable of universal control seems to be much simpler. Applying a sufficiently random sequence of unitary gates one can characterize the level of noise by the parameter $\gamma = -\ln p$ with p given by (13)$-$(14) which can be determined from the exponential decay of the fidelity.

In a more realistic case of restricted control the results of Sect. 5 and of the previous publications [12, 4, 1] show that we have to use a dynamical map with a high degree of chaos in comparison with the expected degree of noise. Otherwise, the observed decoherence rate depends rather on the internal dynamics of the device and not on the interaction with an environment — the source of noise.

In this context several open problems have to be discussed. First of all one should confirm that the fidelity decay, which is a much easier parameter to be determined experimentally than the entropic quantities, displays also the different regimes of decoherence. Secondly, it is not obvious which single parameter associated with the noise map $\hat{\Lambda}$ characterizes the decay rate of fidelity. There are at least three candidates:

1) $\ln K$, where K is a minimal number of terms in the Kraus decomposition (1),

2) the Renyi entropy $I_\alpha[\hat{\Lambda}] \equiv I_\alpha\left(\hat{\Lambda} \otimes \mathbb{1}_N(|\Psi_{\max}\rangle\langle\Psi_{\max}|)\right) \leq \ln K$,

3) $\gamma = \ln(N^2 - 1) - \ln(\sum_k |\mathrm{Tr}(A_k)|^2 - 1)$.

In the case of a simple von Neumann measurement $\hat{\mathcal{E}}_K$ (24) with a uniform choice of projection ranges all these parameters practically coincide

$$I_\alpha[\hat{\mathcal{E}}_K] = \ln K, \qquad \gamma \simeq \ln K \qquad (\text{for } N \gg K), \tag{27}$$

and therefore one should study more generic examples of $\hat{\Lambda}$.

Acknowledgments

We acknowledge financial support by the Polish Ministry of Science and Information Technology — grant no. 2P03 B 08425.

References

[1] R.A. Jalabert, H.M. Pastawski, *Phys. Rev. Lett.* **86**, 2490 (2001).

[2] J. Emerson, Y.W. Weinstein, S. Lloyd, D.G. Cory, *Phys. Rev. Lett.* **89**, 284102 (2002).

[3] J. Emerson, R. Alicki, K. Życzkowski, arXiv: quant-ph/053243.

[4] R. Alicki, A. Łoziński, P. Pakoński, K. Życzkowski, *J. Phys. A* **37**, 5157 (2004).

[5] R. Alicki, M. Fannes, *Quantum Dynamical Systems*, Oxford University Press, Oxford 2001.

[6] F. Benatti, V. Cappellini, M. De Cock, M. Fannes, D. Vanpeteghem, *Rev. Math. Phys.* **15**, 847 (2003).

[7] R. Alicki, D. Makowiec, W. Miklaszewski, *Phys. Rev. Lett.* **77**, 838 (1996).

[8] J.H. Hannay, M.V. Berry, *Physica D* **1**, 267 (1980).

[9] M.D. Esposti, S. Graffi, S. Isola, *Commun. Math. Phys.* **167**, 471 (1995).

[10] S. De Biévre, M.D. Esposti, R. Giachetti, *Commun. Math. Phys.* **176**, 73 (1996).

[11] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, *Ergodic Theory*, Springer-Verlag, Berlin 1982.

[12] M. Žnidarič, T. Prosen, *J. Phys. A* **36**, 2463 (2003).