Control and Cybernetics

vol. **49** (2020) No. 3

Existence and uniqueness of the solution to the optimal control problem with integral criterion over the entire domain for a nonlinear Schrödinger equation with a special gradient term *

by

Vuqar Salmanov

Department of Informatics, The Nakhchivan State University
Nakhchivan city, University campus, AZ7012
Azerbaijan Republic
vuqars69@mail.ru

Abstract: The paper concerns the optimal control problem with the full-range integral performance criterion for the nonlinear Schrödinger equation with the specific gradient summand and the complex potential when the performance criterion is the full-range integral. In this paper, the existence and uniqueness theorems regarding the solution of the optimal control problem under consideration are proven.

Keywords: Schrödinger equation, optimal control problem, the first variation of the functional, complex potential

1. Introduction

The optimal control problems for the linear and nonlinear Schrödinger equations often occur in quantum mechanics, nuclear physics, nonlinear optics, and other fields of modern physics and engineering, and the study of such problems has both theoretical and practical interests (see Butkovskii and Samoilenko, 1984, Vorontsov and Shmalgauzen, 1985, or Zhuravlev, 2001). One of such problems is the problem of motion of charged particles, in which the potential is unknown and is to be determined. It is known that if a charged particle in a constant uniform magnetic field moves and the direction of the magnetic field is chosen along the axis z, then the movement of such a particle occurs in the plane $(x, y) \in E_2$ and this movement is usually described by a two-dimensional linear Schrödinger equation with a specific gradient summand (see Butkovskii and Samoilenko, 1984).

*Submitted: July 2020; Accepted: October 2020

Similar optimal control problems for the linear Schrödinger equation with a specific gradient summand were previously studied in the papers by Akbaba (2011) or Yagubov, Toyoğlu and Subaşı (2012). Note also that the optimal control problems for linear and nonlinear non-stationary Schrödinger equations without a specific gradient summand were previously studied in detail in such papers as Baudouin, Kavian and Puel (2005), Iskenderov and Yagubov (1988, 1989, 2007), Iskenderov, Yagubov and Musaeva (2012), Yagubov and Musaeva (1997), and yet some other ones.

However, optimal control problems for a nonlinear Schrödinger equation with a specific gradient summand are most poorly studied. Such optimal control problems for a two-dimensional nonlinear non-stationary Schrödinger equation with a specific gradient summand and a real-valued potential, when the potential plays the control role and is searched for in the class of measurable bounded functions, and the coefficient in the nonlinear part of the equation is a purely imaginary number, have been studied in the papers by Iskenderov, Yagub and Zengin (2016) or Iskenderov, Yagub and Aksoy (2015). It should be noted that the optimal control problem for the three-dimensional nonlinear non-stationary Schrödinger equation with a gradient summand and a real-valued potential, when the potential depending on both spatial and temporal variables plays the control role and is searched for in the class of measurable bounded functions, and the coefficient in the nonlinear part of the equation is a complex number, has been first investigated by Yagubov and Musaeva (1997). Further, the optimal control problem for a three-dimensional nonlinear nonstationary Schrödinger equation with a specific gradient term and with a complex-valued potential depending on both spatial and temporal variables was previously studied in the work by Iskenderov, Yagub, Salmanov and Aktsoi (2019), for the case, when the quality criterion is final.

Therefore, this paper is devoted to the study of the optimal control problem for the three-dimensional nonlinear Schrödinger equation with a special gradient term and with a complex potential, when the controls are the real and the imaginary parts of the complex potential and are selected from the class of measurable bounded functions depending on a time variable, while the quality criterion is an integral throughout the region. This is not only an extension of the work here referred to, but, in itself, is of considerable scientific interest.

2. Statement of the problem

Let us assume that D is a bounded convex area of three-dimensional Euclidean space E_3, with the bound Γ, which is supposed to be fairly smooth, $x = (x_1, x_2, x_3)$ is an arbitrary point of the D_{area}, $T > 0$ is a predetermined number, $0 \leq t \leq T$, $\Omega_t = D \times (0, t)$, $\Omega = \Omega_T$, $S = \Gamma \times (0, T)$ is the lateral surface of Ω; $C^k([0, T], B)$ is the Banach space of functions, k is the number of times of continuous differentiability on the segment $[0,T]$ with the values in the Banach space B; $L_p(D)$ is the Lebesgue space of functions summed modularly with a degree of $p \geq 1$; $L_2(0,T;B)$ is the Banach space of functions defined and summo modulable with square on an interval $[0,T]$ with the values in the Banach space B; $L_\infty(0,T;B)$ is the Banach space of measurable bounded functions on the segment $(0,T)$ with the values in the Banach space B; Sobolev spaces $W_p^k(D), W_p^{k,m}(\Omega), p \geq 1,\ k \geq 0,\ m \geq 0$, are specified, for example, in Lions and Magenes (1972), Ladyzhenskaya (1973) or in Ladyzhenskaya, Solonnikov and Ural'tseva (1967). $\dot{W}_2^1(D)$ is the subspace of the space $W_2^1(D)$, where a dense set is the set of all smooth functions equal to zero near the boundary of the D_{area}; $\dot{W}_2^2(D) \equiv W_2^2(D) \cap \dot{W}_2^1(D)$.

Let us consider the functional minimization problem for:

$$J_\alpha(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} + \alpha \|v - \omega\|^2_H \tag{1}$$

on the set:

$$V = \Big\{v = v(t) = (v_0(t), v_1(t)) : \ v_m \in W_2^1(0,T), |v_m(t)| \leq \\ \leq b_m, \left|\tfrac{dv_m(t)}{dt}\right| \leq d_m, m = 0,1, \dot{\forall} t \in (0,T)\Big\}$$

subject to the following conditions:

$$i\frac{\partial \psi_p}{\partial t} + a_0 \Delta \psi_p + i a_1(x,t) \nabla \psi_p - a(x)\psi_p + v_0(t)\psi_p + i v_1(t)\psi_p + \\ + a_2 |\psi_p|^2 \psi_p = f_p(x,t), p = 1,2, \quad (x,t) \in \Omega \tag{2}$$

$$\psi_p(x,0) = \varphi_p(x), \ \text{p=1,2}, \ x \in D, \tag{3}$$

$$\psi_1|_S = 0, \left.\frac{\partial \psi_2}{\partial v}\right|_S = 0, \tag{4}$$

where $i = \sqrt{-1}$; $T > 0,\ b_m > 0,\ d_m > 0,\ m = 0,1\ \ a_0 > 0,\ \alpha \geq 0$ are predetermined numbers, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ is the nabla operator; ν is the external normal of the boundary Γ of the area D; the complex number a_2 satisfies the condition:

$$a_2 = Rea_2 + iIma_2,\ Rea_2 < 0,\ Ima_2 > 0,\ Ima_2 \geq 2|Rea_2|; \tag{5}$$

$a(x)$ is a measurable bounded function that satisfies the condition:

$$\mu_0 \leq a(x) \leq \mu_1, \quad \overset{o}{\forall} x \in D,\ \mu_0, \mu_1 = const > 0; \tag{6}$$

(the symbol $\overset{o}{\forall}$ meaning "almost all");

$a_1(x,t) = (a_{11}(x,t), a_{12}(x,t), a_{13}(x,t))$ is the specified function vector with the components that satisfy the conditions:

$$\begin{aligned} &|a_1(x,t)| \le \mu_2, \quad \left|\tfrac{\partial a_{1j}(x,t)}{\partial x_k}\right| \le \mu_3, \left|\tfrac{\partial a_{1j}(x,t)}{\partial t}\right| \le \mu_4, j,k = 1,2,3 \\ &\overset{o}{\forall}(x,t) \in \Omega,\ a_{1j}|_S = 0,\ j = 1,...,n,\ \mu_2,\mu_3,\mu_4 = const > 0; \end{aligned} \tag{7}$$

$\varphi_p(x), f_p(x,t), p = 1,2$ are the complex-valued functions that satisfy the conditions:

$$\varphi_1 \in \dot{W}_2^2(D), \varphi_2 \in W_2^2(D), \left.\frac{\partial \varphi_2}{\partial \nu}\right|_\Gamma = 0; \tag{8}$$

$$f_p \in W_2^{0,1}(\Omega), p = 1,2; \tag{9}$$

$\omega \in H$ is a specified element, where $H \equiv W_2^1(0,T) \times W_2^1(0,T)$.

The problem of determining functions $\psi_p = \psi_p(x,t) \equiv \psi_p(x,t;v), p = 1,2$ from the conditions (2)-(4) for every $v \in V$ will be referred to as a *reduced problem*. It is obvious that a reduced problem consists of two, first and second, initial boundary value problems for a nonlinear Schrödinger equation with a specific gradient summand and a complex potential.

Definition 1 *Under the solution of the reduced problems (2)-(4) we mean the functions*

$$\psi_1 \in B_1 \equiv C^0\left([0,T], \dot{W}_2^2(D)\right) \cap C^1([0,T], L_2(D)),$$

$$\psi_2 \in B_2 \equiv C^0([0,T], W_2^2(D)) \cap C^1([0,T], L_2(D)),$$

satisfying the equations (2) for any $t \in [0,T]$ and almost all $x \in D$, the initial conditions (3) for almost all $x \in D$ and the boundary conditions (4) for almost all $(\xi,t) \in S$.

The reduced problems, i.e. the initial-boundary value problems for linear stationary Schrödinger equations with a special gradient summand and a real-valued potential, were previously studied by Akbaba (2011) and Yagubov, Toyoğlu and Subaşı (2012), and the initial-boundary value problems for a nonlinear non-stationary Schrödinger equation with a special gradient summand were previously studied by Yagub, Ibrahimov and Zengin (2015, 2018), Yagub, İbrahimov and Aksoy (2016), Yagub, Ibragimov, Musaeva and Zenghin (2017), and Yagubov, Salmanov, Yagubov and Zenghin (2017), for the situations, when the Schrödinger equation is a two-dimensional equation or a three-dimensional equation and the potential is a real-valued measurable bounded function or a quadratically summable function that depends only on a spatial variable, and the coefficient in the nonlinear part of the equation is a purely imaginary number or a complex number. It should be noted that the reduced problem of the type (2)-(4), that is, the first and second initial-boundary value problems (2)-(4) were previously studied in the paper by Iskenderov, Yagub and Salmanov (2018). Based on the results of this paper the following statement can be formulated:

Theorem 1 *Assume that the complex number a_2 satisfies the condition (5), and the functions $a(x), a_1(x,t), \varphi_p(x), f_p(x,t), p=1,2$ satisfy the conditions (6)-(9). Then, the reduced problems (2)-(4) upon every $v \in V$ have the single solution $\psi_1 \in B_1, \psi_2 \in B_2$, and the following statements are valid for such solution:*

$$\|\psi_1(\cdot,t)\|^2_{\dot{W}_2^2(D)} + \left\|\frac{\partial\psi_1(\cdot,t)}{\partial t}\right\|^2_{L_2(D)}$$
$$\leq c_1\left(\|\varphi_1\|^2_{\dot{W}_2^2(D)} + \|f_1\|^2_{W_2^{0,1}(\Omega)} + \|\varphi_1\|^6_{\dot{W}_2^1(D)}\right), \quad \forall t \in [0,T], \tag{10}$$

$$\|\psi_2(\cdot,t)\|^2_{\dot{W}_2^2(D)} + \left\|\frac{\partial\psi_2(\cdot,t)}{\partial t}\right\|^2_{L_2(D)}$$
$$\leq c_2\left(\|\varphi_2\|^2_{W_2^2(D)} + \|f_2\|^2_{W_2^{0,1}(\Omega)} + \|\varphi_2\|^6_{W_2^1(D)}\right), \quad \forall t \in [0,T], \tag{11}$$

where $C_p > 0, p=1,2$ are the constants not depending on t.

According to the theorem, the functional (1) is meaningful for the reduced problem solution class under consideration.

3. Existence and uniqueness of the optimal control problem solution

This section deals with the existence and uniqueness of the solution to the optimal control problem (1)-(4). Therefore, at first, the result concerning the existence of a single solution to the problem is to be established. For this purpose, a well-known theorem on the existence and uniqueness of a non-convex optimization solution is to be specified.

Theorem 2 *(Goebel, 1979) Suppose that $\tilde{X}$ is a uniformly convex space, U is a closed bounded set of $\tilde{X}$, the functional $I(v)$ on U is semi-continuous and limited at the bottom, $\alpha > 0, \beta \geq 1$ are the predetermined numbers. Then, a dense subset G of the space $\tilde{X}$ exists such that for any $\omega \in G$ the functional:*

$$J_\alpha(v) = I(v) + \alpha\|v-\omega\|^\beta_{\tilde{X}}$$

has the least value on U. If $\beta > 1$, the minimum value of the functional $J_\alpha(v)$ on U is attained on a single element.

This theorem is useful in proving the following statement:

Theorem 3 *Suppose that the number* a_2 *and the functions* $a(x), a_1(x,t), \varphi_p(x)$, $f_p(x,t)$, $p=1,2$ *satisfy the conditions (5)-(9). Also, suppose that* $\omega \in H$. *Then, there exists an everywhere dense subset* G *of the space* H, *such that for any* $\omega \in G$ *and any* $\alpha > 0$ *the optimal control problem (1)-(4) has a unique solution.*

Proof First, the continuity of the functional $J_0(v)$ in the set V should be proven:

$$J_0(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)}. \tag{12}$$

Suppose that $\delta v \in B \equiv W^1_\infty(0,T) \times W^1_\infty(0,T)$ is the increment of any control $v \in V$ so that $v + \delta v \in V$ and

$$\delta\psi_p = \delta\psi_p(x,t) \equiv \psi_p(x,t;v+\delta v) - \psi_p(x,t;v),\ p=1,2,$$

where $\psi_p(x,t;v), p=1,2$ is the solution of the reduced problems (2)-(4) for $v \in V$. According to the conditions (2)-(4), the functions $\delta\psi_p = \delta\psi_p(x,t),\ p=1,2$ are the solution of the following initial boundary value problem system:

$$\begin{aligned} & i\tfrac{\partial\delta\psi_p}{\partial t} + a_0\Delta\psi_p + ia_1(x,t)\nabla\delta\psi_p - a(x)\delta\psi_p + \\ & \qquad + (v_0(t)+\delta v_0(t))\delta\psi_p + i(v_1(t)+\delta v_1(t))\delta\psi_p = \\ & = -\delta v_0(t)\psi_p - i\delta v_1(t)\psi_p - a_2\left(|\psi_{p\delta}|^2\psi_{p\delta} - |\psi_p|^2\psi_p\right), \\ & p=1,2,(x,t)\in\Omega, \end{aligned} \tag{13}$$

$$\delta\psi_p(x,0)=0, p=1,2, \quad x\in D, \quad \delta\left.\psi_1\right|_S = 0, \left.\frac{\partial\delta\psi_2}{\partial\nu}\right|_S = 0, \tag{14}$$

where $\psi_{p\delta} = \psi_{p\delta}(x,t) \equiv \psi_p(x,t;v+\delta v),\ p=1,2$ is the solution of the reduced problems (2)-(4) provided that $v+\delta v \in V$, $\delta v \in B$.

Let us establish an estimate for the solution of the system of initial-boundary value problems (13), (14). For this purpose, both parts of equations (13) are multiplied by the functions $\delta\bar{\psi}_p(x,t), p=1,2$ and the equalities obtained are to be integrated over the range Ω_t. Then, the following is specified using the integration by parts formula and the boundary conditions as per (14):

$$\int\limits_{\Omega_t}\left(i\frac{\partial\delta\psi_p}{\partial t}\delta\bar{\psi}_p-a_0\left|\nabla\delta\psi_p\right|^2+ia_1\left(x,\tau\right)\nabla\delta\psi_p\delta\bar{\psi}_p-a\left(x\right)\left|\delta\psi_p\right|^2\right.$$

$$+\left(v_0\left(\tau\right)+\delta v_0\left(\tau\right)\right)\left|\delta\psi_p\right|^2dxd\tau+i\int\limits_{\Omega_t}\left(v_1\left(t\right)+\delta v_1\left(t\right)\right)\left|\delta\psi_p\right|^2dxd\tau$$

$$=-\int\limits_{\Omega_t}\delta v_0\left(\tau\right)\psi_p\delta\bar{\psi}_pdxd\tau-i\int\limits_{\Omega_t}\delta v_1\left(\tau\right)\psi_p\delta\bar{\psi}_p$$

$$-\int\limits_{\Omega_t}a_2\left[\left(\left|\psi_{p\delta}\right|^2+\left|\psi_p\right|^2\right)\delta\psi_p+\psi_{p\delta}\psi_p\delta\bar{\psi}_p\right]\delta\bar{\psi}_pdxd\tau, p=1,2,\forall t\in\left[0,T\right].$$

Deducting from these equalities their complex conjugations and applying the Cauchy-Bunyakovsky inequality, using the initial and boundary conditions from (14), as well as the conditions on the function $a_1\left(x,t\right)$, the validity of the following inequalities is obtained:

$$\left\|\delta\psi_p\left(.,t\right)\right\|^2_{L_2(D)}+2Ima_2\int\limits_{\Omega_t}\left(\left|\psi_{p\delta}\right|^2+\left|\psi_p\right|^2\right)\left|\delta\psi_p\right|^2dxd\tau$$

$$\leq\left|a_2\right|\int\limits_{\Omega_t}\left|\psi_{p\delta}\right|\left|\psi_p\right|\left|\delta\psi_p\right|^2dxd\tau+$$

$$+\left(3\mu_3+2\right)\int\limits_0^t\left\|\delta\psi_p\left(.,\tau\right)\right\|^2_{L_2(D)}d\tau+\int\limits_{\Omega_t}\left|\delta v_0\left(\tau\right)\right|^2\left|\psi_p\right|^2dxd\tau+$$

$$+\int\limits_{\Omega_t}\left|\delta v_1\left(t\right)\right|^2\left|\psi_p\right|^2dxd\tau,\ p=1,2,\ \forall t\in\left[0,T\right].$$

Hence, by virtue of the estimates (10), (11) and condition (5) for a complex number using Gronwall's Lemma, the following value is obtained:

$$\left\|\delta\psi_p\left(.,t\right)\right\|^2_{L_2(D)}+\tfrac{Ima_2}{2}\int\limits_{\Omega_t}\left(\left|\psi_{p\delta}\right|^2+\left|\psi_p\right|^2\right)\left|\delta\psi_p\right|^2dxd\tau\leq c_3\left\|\delta v\right\|^2_B, \quad p=1,2,\ \forall t\in\left[0,T\right]. \tag{15}$$

where $c_3>0$ is a constant not depending on δv.

Now, consider the increment of the functional $J_0\left(v\right)$ in any element $v\in V$.

According to formula (12), the following is found:

$$\delta J_0(v) = J_0(v+\delta v) - J_0(v) =$$
$$2\int_\Omega Re\left[\left(\psi_1(x,t) - \psi_2(x,t)\right)\left(\delta\bar{\psi}_1(x,t) - \delta\bar{\psi}_2(x,t)\right)\right]dx +$$
$$+\|\delta\psi_1\|^2_{L_2(\Omega)} + \|\delta\psi_2\|^2_{L_2(\Omega)} - 2\int_\Omega Re\left(\delta\psi_1(x,t)\,\delta\bar{\psi}_2(x,t)\right)dxdt. \quad (16)$$

According to this formula, applying the Cauchy-Bunyakovsky inequality and using values (10), (11) and (15), the validity of the following inequalities is obtained:

$$|\delta J_0(v)| \le c_4\left(\|\delta v\|_B + \|\delta v\|_B^2\right), \forall v \in V,$$

where $c_4 > 0$ is a constant not depending on δv. Based on this inequality, the continuity of the functional $J_0(v)$ in the set V is established. The set V is a closed bounded and convex set of the space B. It is not difficult to prove that it is a closed bounded and convex set of uniform convex space H (Yosida, 1967). Then, according to Theorem 2 a dense subset G from the space H exists such that for any $\omega \in G$ and subject to any $\alpha > 0$ the optimal control problem (1)-(4) has a single solution. So, Theorem 3 is proven. □

Now, let us prove that with $\alpha \ge 0$ and for any $\omega \in H$ the optimal control problem (1)-(4) has at least one solution.

Theorem 4 *Suppose that the conditions of Theorem 3 are satisfied. Then, for $\alpha \ge 0$ and subject to any $\omega \in H$ the optimal control problem (1)-(4) has, at least, one solution.*

Proof Let us look at any minimizing sequence $\{v^k\} \subset V$:

$$\lim_{k\to\infty} J_\alpha\left(v^k\right) = J_{\alpha *} = \inf_{v\in V} J_\alpha(v).$$

Suppose that $\psi_{pk} = \psi_{pk}(x,t) \equiv \psi_p\left(x,t;v^k\right),\ p = 1,2,\ k = 1,2,....$ According to Theorem 1, for every $v^k \in V$ the reduced problems (2)-(4) have the single solution $\psi_{pk} \in B_p,\ p = 1,2,$ and the following estimates are correct for such solution:

$$\|\psi_{1k}(\cdot,t)\|^2_{\dot{W}_2^2(D)} + \left\|\frac{\partial\psi_{1k}(\cdot,t)}{\partial t}\right\|^2_{L_2(D)}$$
$$\le c_1\left(\|\varphi_1\|^2_{\dot{W}_2^2(D)} + \|f_1\|^2_{W_2^{0,1}(\Omega)} + \|\varphi_1\|^6_{\dot{W}_2^1(D)}\right)$$
$$k = 1,2,... \quad (17)$$

$$\|\psi_{2k}(\cdot,t)\|^2_{W_2^2(D)} + \left\|\frac{\partial\psi_{2k}(\cdot,t)}{\partial t}\right\|^2_{L_2(D)}$$
$$\le c_2\left(\|\varphi_2\|^2_{W_2^2(D)} + \|f_2\|^2_{W_2^{0,1}(\Omega)} + \|\varphi_2\|^6_{W_2^1(D)}\right),$$
$$k = 1,2,... \tag{18}$$

for $\forall t \in [0,T]$, where the right hand sides of the estimates do not depend on k.

As the set V is a limited set of the Banach space $B \equiv W^1_\infty(0,T) \times W^1_\infty(0,T)$, then, the sequence $\{v^k\} \subset V$ may provide such a subsequence $\{v^{k_l}\}$, to be noted as $\{v^k\}$ to simplify the explanation, that

$$v^k \to v(*) \text{ weakly in } L_\infty(0,T) \times L_\infty(0,T), \tag{19}$$

$$\frac{dv^k}{dt} \to \frac{dv}{dt}(*) \text{ weakly in } L_\infty(0,T) \times L_\infty(0,T) \tag{20}$$

provided that $k \to \infty$. Also, V is the bounded convex set from B. Therefore, V is the (*) weakly closed convex set. Then, it is true that $v \in V$.

According to the estimates (17), (18), it is established that the sequences $\{\psi_{pk}(x,t)\}, p = 1,2$ are equally bounded within the norm of the spaces B_1 and B_2. Then, from these sequences one can extract such subsequences $\{\psi_{pk_l}(x,t)\}$, $p = 1,2$, to be denoted as $\{\psi_{pk}(x,t)\}$ to again simplify the explanation, that

$$\psi_{pk}(\cdot,t) \to \psi_p(\cdot,t), p = 1,2 \text{ weakly in } W_2^2(D); \tag{21}$$

$$\frac{\partial\psi_{pk}(\cdot,t)}{\partial t} \to \frac{\partial\psi_p(\cdot,t)}{\partial t}, p = 1,2 \text{ weakly in } L_2(D) \tag{22}$$

for every $t \in [0,T]$, provided that $k \to \infty$.

It is obvious that every element $\{\psi_{pk}(x,t)\}$, $p = 1,2$, from B_1 and B_2 satisfies the equalities:

$$\int_D \left(i\frac{\partial\psi_{pk}(x,t)}{\partial t} - a_0\Delta\psi_{pk}(x,t) + ia_1(x,t)\nabla\psi_{pk}(x,t)\right.$$
$$-a(x)\psi_{pk}(x,t) + v_0^k(t)\psi_{pk}(x,t) +$$
$$\left. +iv_1^k(t)\psi_{pk}(x,t) + a_2|\psi_{pk}(x,t)|^2\psi_{pk}(x,t) - f(x,t)\right)\bar{\eta}_p(x)\,dx$$
$$= 0,\ \ p = 1,2, \forall t \in [0,T], k = 1,2,..., \tag{23}$$

for any function $\eta_p = \eta_p(x), p = 1,2$ from $L_2(D)$, it satisfies the initial conditions:

$$\psi_{pk}(x,0) = \varphi_p(x), p = 1,2, \quad \forall x \in D, \quad k = 1,2,... \tag{24}$$

and the boundary conditions:

$$\psi_{1k}|_S = 0, \left.\frac{\partial \psi_{2k}}{\partial \nu}\right|_S = 0, k = 1,2,.... \tag{25}$$

Due to the compactness of the embedding spaces B_1, B_2 in $C^0([0,T], L_2(D))$ the following is true:

$$\|\psi_{pk}(\cdot,t) - \psi_p(\cdot,t)\|_{L_2(D)} \to 0, p = 1,2 \tag{26}$$

relative to $t \in [0,T]$, when $k \to \infty$. Also, based on the limit relations (19), (20), the validity of the limit relation may be specified as follows:

$$v^k \to v \text{ strongly in } C[0,T] \times C[0,T] \tag{27}$$

when $k \to \infty$. This limit relation means that the subsequence $\{v^k(t)\}$ converges to the element $v(t)$ uniformly in $t \in [0,T]$. Using this finding and the limit relations (27), we can establish the validity of the relations:

$$\begin{gathered}\int_D v_m^k(t)\psi_{pk}(x,t)\,\bar{\eta}_p(x)\,dx \to \int_D v_m(t)\psi_p(x,t)\,\bar{\eta}_p(x)\,dx, \\ p = 1,2, m = 0,1,\end{gathered} \tag{28}$$

$$\begin{gathered}\int_D |\psi_{pk}(x,t)|^2\psi_{pk}(x,t)\,\bar{\eta}_p(x)\,dx \to \int_D |\psi_p(x,t)|^2\psi_p(x,t)\,\bar{\eta}_p(x)\,dx, \\ p = 1,2\end{gathered} \tag{29}$$

for every $t \in [0,T]$, and for any functions $\eta_p \in L_2(D)$, $p = 1,2$, when $k \to \infty$. Via the limit relations (21), (22) and (28), (29), upon passing to the limit in the integral identities (23), the validity of the following equalities will be obtained:

$$\begin{aligned}\int_D \bigg(& i\frac{\partial \psi_p(x,t)}{\partial t} - a_0\Delta\psi_p(x,t) + ia_1(x,t)\nabla\psi_p(x,t) - a(x)\psi_p(x,t) + \\ & + v_0(t)\psi_p(x,t) + iv_1(t)\psi_p(x,t) + \\ & + a_2|\psi_p(x,t)|^2\psi_p(x,t) - f_p(x,t)\bigg)\bar{\eta}_p(x)\,dx = 0, \quad p = 1,2\end{aligned} \tag{30}$$

for every $t \in [0,T]$ and for any functions $\eta_p = \eta_p(x)$, $p = 1,2$ from $L_2(D)$. Hence, the limit functions $\psi_p(x,t)$, $p = 1,2$ for every $t \in [0,T]$ and for almost all $x \in D$ satisfy the equations (2). Satisfaction of the initial conditions follows from the limiting relation (26) at $t=0$, the initial conditions (24), and from the following inequalities:

$$\left\|\psi_p(\cdot,0)-\varphi_p\right\|_{L_2(D)}\leq\left\|\psi_p(\cdot,0)-\psi_{pk}(\cdot,0)\right\|_{L_2(D)}+\left\|\psi_{pk}(\cdot,0)-\varphi_p\right\|_{L_2(D)},$$
$$p=1,2.$$

Indeed, taking into account the limit relations (26) for $t=0$ and the initial conditions (24), if passing to the limit in the last inequalities is assumed, then, subject to $k\to\infty$, the validity of the following relations is obtained:

$$\left\|\psi_p(.,0)-\varphi_p\right\|_{L_2(D)}=0,\ p=1,2.$$

According to these relations it is found that the limit functions $\psi_p(x,t)$, $p=1,2$ satisfy $\psi_p(x,0)=\varphi_p(x)$, $p=1,2$, $\forall x\in D$, that is, the initial conditions (3) are satisfied.

Finally, let us prove that the limit function satisfies the boundary conditions (4). First, we prove that the limit function $\psi_1(x,t)$ satisfies the first boundary condition from (4). Indeed, due to the compactness of the embedding B_1 in the space $L_2(S)$, we have:

$$\left\|\psi_{1k}-\psi_1\right\|_{L_2(S)}\to 0$$

for $k\to\infty$. Then, using this finding and the first boundary condition from (25), from the inequality:

$$\left\|\psi_1\right\|_{L_2(S)}\leq\left\|\psi_1-\psi_{1k}\right\|_{L_2(S)}+\left\|\psi_{1k}\right\|_{L_2(S)},$$

upon passing to the limit, we obtain the validity of the boundary condition:

$$\psi(\xi,t)=0,\quad \overset{0}{\forall}(\xi,t)\in S.$$

Now, let us prove that the limit function $\psi_2(x,t)$ satisfies the second boundary conditions from the ones appearing in (4). Indeed, in view of Lemma 3.4 from Ladyzhenskaya, Solonnikov and Ural'tseva (1967), as well as satisfaction of the condition that the subsequence $\{\psi_{2k}(x,t)\}$ belongs to the space $B_2\subset W_2^{2,1}(\Omega)$, the validity of the limit relations may be proven:

$$\left.\frac{\partial\psi_{2k}}{\partial\nu}\right|_S\to\left.\frac{\partial\psi_2}{\partial\nu}\right|_S,\ \text{weakly in } L_2(S)\ \text{ for } k\to\infty.$$

Then, using these limit relations and the second boundary conditions from (25), according to the equality:

$$\int\limits_S\frac{\partial\psi_2(\xi,t)}{\partial\nu}\bar{\eta}_2(\xi,t)\,d\xi dt=\int\limits_S\left(\frac{\partial\psi_2(\xi,t)}{\partial\nu}-\frac{\partial\psi_{2k}(\xi,t)}{\partial\nu}\right)\bar{\eta}_2(\xi,t)\,d\xi dt+$$

$$+\int\limits_{S}\frac{\partial\psi_k(\xi,t)}{\partial\nu}\bar{\eta}_2(\xi,t)\,d\xi dt,\forall\eta_2\in L_2(S),$$

by passing to the limit with $k\to\infty$, the validity of the following condition is obtained:

$$\int\limits_{S}\frac{\partial\psi_2(\xi,t)}{\partial\nu}\bar{\eta}_2(\xi,t)\,d\xi dt=0,\forall\eta_2\in L_2(S),$$

and, therefore, the second boundary condition is established:

$$\frac{\partial\psi_2(\xi,t)}{\partial\nu}=0,\ \overset{0}{\forall}(\xi,t)\in S.$$

So, it has been proven that the limit functions $\psi_p(x,t),p=1,2$ are the solution for the reduced problems (2)-(4), satisfying the limit function $v\in V$, that is $\psi_p=\psi_p(x,t)\equiv\psi_p(x,t;v),\ p=1,2$. In addition, the estimates (10), (11) are valid for such functions, arising directly from the estimates (17), (18) through the passage to the lower limit as per weakly convergent subsequences $\{\psi_{pk}(x,t)\},\ p=1,2$. According to Theorem 1, such functions $\psi_p(x,t),p=1,2$ belong to the spaces $B_p,p=1,2$, respectively, and they are the single solutions for the reduced problems subject to $v\in V$. Using the weak lower semi-continuity of the norms of the spaces $L_2(\Omega)$, H, as well as the limit relations (19), (20) and the following limit relations:

$$\psi_{pk}\to\psi_p,p=1,2\text{ weakly in }L_2(\Omega)\ \text{ for }k\to\infty$$

for $\forall\alpha\geq 0$ and $\forall\omega\in L_2(D)$, the following is found:

$$J_{\alpha*}\leq J_\alpha(v)\leq\lim_{k\to\infty}J_\alpha\left(v^k\right)=\inf_{v\in V}J_\alpha(v)=J_{\alpha*}.$$

Hence, $v\in V$ is the solution of the optimal control problems (1)-(4) provided that $\alpha\geq 0$ and $\forall\omega\in H$. So, Theorem 4 is proven. □

4. Conclusions

The resolvability theorems as proven above, and the reported expression for the first variation of the quality criterion, as well as the developed essential extremum condition make it possible to apply the numerical methods for the solution of incorrect and inverse problems, including the optimal control problems with crude data, having arisen from studying the movement process of charged particles in the constant uniform magnetic field, where the complex potential is unknown and is to be determined.

References

Akbaba, G.D. (2011) The optimal control problem with the Lions functional for the Schrödinger equation including virtual coefficient gradient. Master's thesis, Kars (in Turkish).

Baudouin, L., Kavian, O. and Puel, J.P. (2005) Regularity for a Schrödinger equation with singular potentials and application to bilinear optimal control. *J. Differential Equations*, 216, 188-222.

Butkovskii, A. G. and Samoilenko, Yu. I. (1984) *Control of quantum mechanical processes*. Moscow, Nauka (in Russian).

Goebel, M. (1979) On existence of optimal control. *Math. Nachr.*, **93**, 67-73.

Iskenderov, A. D. and Yagubov, G. Ya. (1988) Variational method of solving the inverse problem of determining the quantum mechanical potential. *Doklady AN SSSR*, **303**(5), 1044-1048 (in Russian).

Iskenderov, A. D. and Yagubov, G. Ya. (1989) Optimal control of the nonlinear quantum mechanical systems. *Avtomatika i telemekhanika*, 12, 27-38 (in Russian).

Iskenderov, A. and Yagubov, G. (2007) Optimal control of unbounded potential in the multidimensional nonlinear non-stationary Schrödinger equation. *Vestnik Lenkoranskogo Gosudarstvennogo Universiteta. Seriya Estestvennykh Nauk.* Lenkoran', 3-56 (in Russian).

Iskenderov, A. D., Yagubov, G. Ya. and Musaeva, M. A. (2012) *Identification of quantum potentials*. Baku, Chashyoglu (in Russian).

Iskenderov, A.D., Yagub, G., Zengin, M. (2016) Optimal control problem for nonlinear Schrödinger equation with special gradient terms. *Abstracts of the XXVII International Conference: Problems of Decision Making under Uncertainties (PDMU-2016)*, Tbilisi-Batumi, Georgia, May 23-27, 79-80.

Iskenderov, A.D., Yagub, G. and Aksoy, N. Y. (2015) An optimal control problem for a two-dimensional nonlinear Schrödinger equation with a special gradient term. *Abstracts of the XXV International Conference: Problems of Decision Making under Uncertainties (PDMU-2015)*, Skhidnytsia, Ukraine, May 11-15, 27-28.

Iskenderov, A., Yagub, G. and Salmanov, V. (2018) Solvability of the initial-boundary problem for the nonlinear Schrödinger equation with a special gradient term and complex potential. *Nauchnyie Trudy Nakhiche–vanskogo Gosudarstvennogo Universiteta. Seriya fiziko-matematicheskikh i tekhnicheskikh nauk*, **4**(93), 28-43 (in Russian).

Iskenderov, A. D., Yagub, G., Salmanov, V. and Aktsoi, N. Y. (2019) Optimal control problem for the nonlinear Schrödinger equation with special gradient term and complex potential. *Nauchnyie Trudy Nakhichevanskogo Gosudarstvennogo Universiteta. Seriya fiziko-matematicheskikh i tekhnicheskikh nauk*, **4**(101), 32-44 (in Russian).

Ladyzhenskaya, O. A., Solonnikov, V. A. and Ural'tseva, N. N. (1967) *Linear and quasi-linear equations of parabolic type.* Moscow, Nauka (in Russian).

Ladyzhenskaya, O. A. (1973) *Boundary problems of mathematical physics.* Moscow, Nauka (in Russian).

Lions, J.-L. and Magenes, E. (1972) *Non-Homogeneous Boundary Value Problems and Applications* **2**. *Grundlehren der mathematischen Wissenschaften,* **181**. Springer, Berlin.

Minoux, M. (1990) *Mathematical programming.* Moscow, Nauka (in Russian).

Vorontsov, M. A. and Shmalgauzen, V. I. (1985) *The principles of adaptive optics.* Moscow, Nauka (in Russian).

Yagub G., Ibrahimov, N.S. and Zengin, M. (2015) Solvability of the initial-boundary value problems for the nonlinear Schrödinger equation with a special gradient terms. *Abstracts of the XXV International Conference: Problems of Decision Making under Uncertainties (PDMU-2015),* Skhidnytsia, Ukraine, May 11-15, 53-54.

Yagub, G., İbrahimov, N.S. and Aksoy, N.Y. (2016) On the initial-boundary value problems for the nonlinear Schrödinger equation with special gradient terms. *Abstracts of the XXVII International Conference: Problems of Decision Making under Uncertainties (PDMU-2016),* Tbilisi-Batumi, Georgia, May 23-27, 170-171.

Yagub, G., Ibragimov, N., Musaeva, M. and Zenghin, M. (2017) Variational method of solving the inverse problem of determining quantum potential in the nonlinear non-stationary Schrödinger equation with complex coefficient in the nonlinear part. *Vestnik Lenkoranskogo Gosudarstvennogo Universiteta. Estestvennye Nauki, seriya 2.* Lenkoran', 7-39 (in Russian).

Yagub, G., İbrahimov, N.S. and Zengin, M. (2018) The solvability of the initial-boundary value problems for a nonlinear Schrödinger equation with a special gradient term. *Journal of Mathematical Physics, Analysis, Geometry,* 2, 214-232.

Yagubov, G. Ya. and Musaeva, M. A. (1997) On an identification problem for the nonlinear Schrödinger equation. *Differents. uravnenya*, **33** (12), 1691-1698 (in Russian).

Yagubov, G., Toyoğlu, F. and Subaşi, M. (2012) An optimal control problem for two-dimensional Schrödinger equation. *Applied Mathematics and Computation,* **218**, 11, 6177-6187.

Yagubov, G., Salmanov, V., Yagubov, V. and Zenghin, M. (2017) Solvability of the initial-boundary value problems for the nonlinear two-dimen–sional Schrödinger equation. *Nauchnyie Trudy Nakhichevanskogo Gosudarstvennogo Universiteta. Seriya fiziko-matematicheskikh i tekhnicheskikh nauk*, **4**(85), 7-21 (in Russian).

Yosida, K. (1967) *Functional analysis.* Moscow, Nauka (in Russian).

Zhuravlev, V. M. (2001) *Nonlinear waves in multi-component systems with dispersion and diffusion.* Ul'yanovsk, UlGU (in Russian).