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Determination of continuous shifts in the term structure of interest rates against which a bond portfolio is immunized*

by

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Abstract: In this paper we identify those shifts (continuous functions) of the term structure of interest rates, against which a given bond portfolio (BP) is immunized. The set of such shifts (IMMU) happens to be an $(m-1)$-dimensional linear subspace in an m-dimensional linear space of all admissible shifts. In the proof we use triangular (Lagrange) functions, by means of which we build a base for IMMU. How this IMMU space varies in response to changes in the cash flow generated by bond portfolio, BP, is also discussed in the last section of the paper.

Keywords: immunization, bond portfolio, Lagrange functions

1. Introduction

Consider an investor who, possessing C dollars today, must achieve an investment goal of L dollars $(L > C)$, q years from now by means of a purchase of appropriately selected bond portfolio BP. If not successful he/she will incur a severe penalty, while achieving more than L dollars will result in no rewards. Such investors are called bond immunizers. By the term structure of interest rates one understands a schedule of spot interest rates, which is estimated from the yields of all coupon-bearing bonds, available on the financial market (M) under consideration.

The standard immunization problem relies on the construction of such a bond portfolio BP with the present value of C dollars that the single liability to pay L dollars q years from now by means of BP will be secured. Having built such a portfolio, the investor protects (hedges) own investment in bonds against a loss in its value at time q. The loss is caused by unfavorable changes in the term structure of interest rates $s(t)$ due to various random market forces.

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The new term structure may clearly be written down as $s^*(t) = s(t) + a(t)$, with $a(t)$ standing for a shift/movement/shock of the term structure $s(t)$, which can be flat, rising, declining, humped, or twisted. The classical results refer to flat shifts $a(t)$ only and they go back as far as to the pioneering work of Macaulay (1938), Redington (1952), and Fisher (1971). In the subsequent period many duration models were investigated in the literature. A short description of these models is given by Soto (2001). Another approach to immunization strategy has been developed by Leibowitz and Weinberger (1982-83). Specifically, contingent immunization consists of forming a bond portfolio with a duration longer or shorter than the investor's planning period, depending on interest rate expectations, in order to take advantage of the manager's ability to forecast interest rate movements as long as their forecasts are successful, but switches to a pure immunization strategy should the stop loss limit be encountered. The contingent immunization was then implemented in many situations for different term structures of interest rates (e.g., Diaz et al., 2009).

In the present paper, we cover all types of continuous term structures $s(t)$, although the continuity assumption is nowhere explicitly used, with the only assumption referring to shifts $a(t)$ of $s(t)$, by supposing that they are continuous functions on a certain interval $[t_0, T]$. No specific stochastic or deterministic model is employed here, as is the case in many other publications, see, e.g., Bansal and Zhou (2002). Consequently, we are not exposed to any model misspecification risk. In this sense our approach is similar to the one presented by Zheng (2006-2007).

In this paper we do not solve the standard immunization problem, but simply want to identify those shifts of the term structure of interest rates that our bond portfolio BP is already protected against loss of its value at time q.

Such a problem has rarely been stated, despite the fact that it has some similarity to the standard immunization problem. It was already dealt with by Rzadkowski and Zaremba (2010) for two classes of shifts, namely polynomials and continuous functions, with two examples, illustrating the employed methodologies. More specifically, we stated this problem in two different mathematical settings. The basis for these two classes of shift spaces (IMMU) consisted of monomials. It turned out, however, that computed coefficients of the elements of the bases were very different as to the orders of their magnitudes. This caused some inconvenience in the calculation.

For dealing with continuous shifts we employed a Hilbert space (a notion borrowed from the mathematical discipline of functional analysis) approach, which made it possible for us to identify each of the two functions, provided they had the same values at specified instances of time $t_0, t_1, t_2, \ldots, t_m$.

In this paper we offer an alternative approach to this problem, by employing the triangular functions (tent functions). Sometimes, these triangular functions are called the Lagrange functions, because they occur in the formula for the Lagrange polynomials. Triangular functions play also an important role in functional analysis as examples of the so called Schauder bases (see Schauder, 1928; Semadeni, 1982). These functions seem to be well suited to represent all kinds of continuous shifts. Thomas Ho (1992) also uses triangular functions for modeling the shifts in the term structure. As previously, in the paper by Rządkowski and Zaremba (2010), we solve our problem with the help of Theorem 1 from Rządkowski and Zaremba (2000). A similar approach has been presented by Barber (1999) (see also Barber, 2013).

2. Problem formulation

Let BP be a bond portfolio, constructed from debt instruments available on a given financial market M. Let, moreover, t_0 stand for the very moment when an investor bought BP, while $t_1, t_2, \ldots, t_m = T$ comprise all instances from the interval $[t_0, T]$, representing the life span (expressed in years) for portfolio BP, when either BP generates payments at t_i, $1 \leq i \leq m$ (in the form of coupons or par values), or the owner of BP is required to pay his/her liabilities (q is one of the instances $t_1, t_2, \ldots, t_m$).

In addition, let $s(t)$ denote the term structure of interest rates on this market, which, for the clarity of presentation, is assumed to be a continuous function defined on $[t_0, T]$. We want to identify all such continuous shifts $a(t)$ of $s(t)$ that portfolio BP is already immunized against.

Since the bond portfolio BP generates payments at instances $t_1, t_2, \ldots, t_m = T$, its present and futures values (for $t \geq t_1$) under the new term structure $s^*(t) = s(t) + a(t)$ will not depend on $a(t_0)$.

Now, we recall from functional analysis the so-called triangular functions (tent functions, Lagrange functions) on the interval $[t_0, T]$. Each of these triangular functions $S_0(t), S_1(t), \ldots, S_m(t)$ is equal to zero everywhere except for a certain subinterval of $[t_0, T]$. Strictly speaking, the first $S_0(t)$ and the last $S_m(t)$ are given by (see Fig. 1)

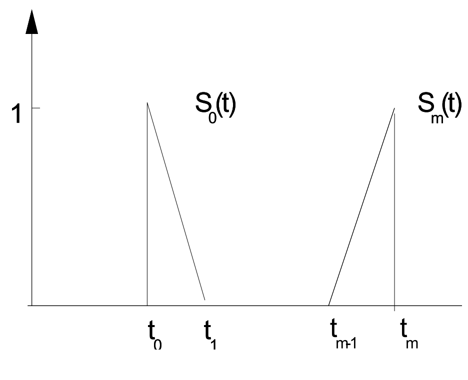

Figure 1. Triangular functions $S_0(t)$ and $S_m(t)$

$$S_0(t) = \frac{t - t_1}{t_0 - t_1}, \quad t \in [t_0, t_1] \quad \text{and } S_0(t) = 0 \text{ for } t \in [t_1, t_m], \tag{1}$$

$$S_m(t) = \frac{t - t_{m-1}}{t_m - t_{m-1}}, \ t \in [t_{m-1}, t_m] \quad \text{and } S_m(t) = 0 \text{ for } t \in [t_0, t_{m-1}]. \tag{2}$$

The remaining $(m-2)$ triangular functions $S_k(t)$, $k \neq 0, m$ (see Fig. 2), whose graphs have the shape of a triangle, are defined as follows:

$$\begin{aligned} S_k(t) &= \frac{t - t_{k-1}}{t_k - t_{k-1}}, \ t \in [t_{k-1}, t_k], \\ S_k(t) &= \frac{t - t_{k+1}}{t_k - t_{k+1}}, \ t \in [t_k, t_{k+1}]. \end{aligned} \tag{3}$$

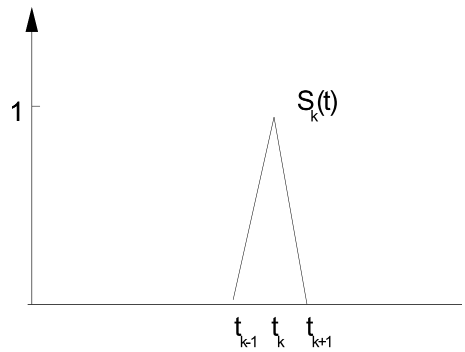

Figure 2. Triangular function $S_k(t)$

These functions have a very useful property, namely each function $f(t)$, continuous and piecewise linear (i.e., linear in any subinterval $[t_k, t_{k+1}]$) defined on $[t_0, T]$ with $f(t_i) = f_i$, $0 \leq i \leq m$ is the following linear combination of $(S_k(t))$:

$$f(t) = f_0 S_0(t) + f_1 S_1(t) + \cdots + f_m S_m(t). \tag{4}$$

In this paper we will be having in mind the following:

Fact 1 *Each continuous function $f(t)$ defined on $[t_0, T]$ with $f(t_i) = f_i$ can be approximated by the linear combination (4).*

3. Solution of the problem

We start by invoking Theorem 1 from Rządkowski and Zaremba (2000), which can be formulated as the following:

FACT 2 *If q denotes a future date, when a single liability of L dollars has to be discharged by the accumulated value of the inflows generated by bond portfolio BP, then the payment of L dollars at time q will be guaranteed (immunization will be secured) provided the following necessary and sufficient condition (we assume that $t_0 = 0$), having nothing to do with the kind of dynamics (deterministic or stochastic), as well as the shape of the continuous term structure $s(t)$ of interest rates, but referring solely to its continuous shifts $a(t)$, holds:*

$$a(q)q = \sum_{i=1}^{m} w_i a(t_i) t_i, \quad \textit{with} \quad w_i = \frac{c_k e^{-s(t_k)t_k}}{\sum\limits_{i=1}^{m} c_i e^{-s(t_i)t_i}} \tag{5}$$

meaning the weight of the coupon c_k, $1 \leq k \leq m$.

Looking at (5), one can see that what really matters in Equation (5) are the values of $a(t)$ at instances $t_1, t_2, \ldots, t_m$ only, that is, $a(t_i)$. This observation inspires us to define a new notion below.

DEFINITION 1 *We shall say that a function $f(t)$ represents a function $g(t)$, or that $f(t)$ is a representation of $g(t)$, if and only if $f(t_i) = g(t_i)$ for all instances $t_1, t_2, \ldots, t_m$ when a given bond portfolio promises to pay cash (coupons or par values).*

How to find a handy representation of a function $a(t)$ defined on $[t_0, T]$ by means of the triangular functions? Upon setting $a(t_k) = a_k$, $1 \leq k \leq m$, we can do it by creating a function $b(t)$ via the formula

$$b(t) = \sum_{k=1}^{m} a_k S_k(t), \tag{6}$$

where, as always, $S_k(t)$ stands for the kth triangular function.

LEMMA 1 *For all specified above instances $t_1, t_2, \ldots, t_m = T$ one has $S_i(t_i) = 1$ and $S_i(t_k) = 0$, $i \neq k$. Moreover, $b(t)$ is a piecewise linear representation of $a(t)$.*

PROOF The equalities $S_i(t_i) = 1$ and $S_i(t_k) = 0$, $i \neq k$ follow directly from (1)-(3). To demonstrate that $b(t_k) = a_k$, we can employ the relationship (6), to conclude that $b(t_k) = \sum\limits_{i=1}^{m} a_i S_i(t_k) = a_k S_k(t_k) = a_k$. Finally, $b(t)$ is piecewise linear because all $S_i(t)$ are piecewise linear. The proof is complete. □

From now on, we will be often replacing functions (shifts) $a(t)$ with $b(t)$ satisfying (6). Our nearest goal is to identify all shifts $a(t)$ of the term structure of interest rates $s(t)$, or equivalently, all linear combinations $b(t)$ of triangular functions $S_i(t)$, for which (5) holds.

Our initial assumption that q, the instant when the liability to pay L dollars has to be discharged, is one of the points $t_1, t_2, \ldots, t_m$, say t_n, will simplify a little bit our reasoning. In fact, by virtue of Lemma 1, one will have $a(q) = a(t_n) = b(t_n) = a_n$, $a(t_i) = b(t_i) = a_i$, $1 \leq i \leq m$, and consequently (5) can be rewritten in the form of a single (easy to solve) linear equation

$$qa_n = \sum_{i=1}^{m} w_i t_i a_i, \tag{7}$$

Here, a_i, $1 \leq i \leq m$, stand for the unknown variables to be identified, q and t_i are exogenous variables, while the weights w_i have more complex nature because they depend both on endogenous factors, such as coupons, and the term structure, which is an exogenous complex factor.

If we knew that for all expected (likely to occur) shifts $a(t)$ one would have $a(q) \neq 0$, which is the case with parallel movements of the term structure when $a(t) \equiv \lambda$, then we might define the notion of duration via the formula

$$D = \sum_{i=1}^{m} w_i \frac{a_i}{a(q)} t_i, \tag{8}$$

and formulate an easy to prove (based on Fact 2) sufficient and necessary condition for immunization in the form given by the following:

Corollary 1 *Let q denote a future date when a single liability to pay L dollars has to be discharged by means of a bond portfolio BP. In addition, let $t_1, t_2, \ldots, t_m$ be instants when the bonds belonging to BP promise to pay coupons. Then, the portfolio BP is immunized against all such shocks/shifts $a(t)$ of $s(t)$ for which $D = q$.*

When we cannot ensure that $a(q) \neq 0$, we prefer to treat (5) as our sufficient and necessary condition for immunization. Let us note that Equation (5) remains the same if we replace $a(t)$ with $\alpha \cdot a(t)$, which means that the immunization against shift $a(t)$ implies immunization against $\alpha \cdot a(t)$, and vice versa. In case of parallel movements ($a(t) = \lambda$), this new notion reduces to the classical concept of duration.

How can we better characterize those shifts $a(t)$ which will make no harm to the value of our bond portfolio at time q when the liability to pay L dollars has to be discharged ? Denote the set of such shifts by IMMU. To prove that IMMU is a linear space, let us recall this notion.

Definition 2 *A set V of elements (called also vectors) is a linear space if the sum of each two elements $a \in V$ and $b \in V$ also belongs to V $(a + b \in V)$, and for any real number r, and any vector $a \in V$ the product of r and a belongs to V as well $(r \cdot a \in V)$.*

Fact 3 *The set IMMU of all shifts against which a given bond portfolio BP is immunized against loss at time q is a linear space.*

To prove this, it suffices to check whether the sum of two arbitrary shifts, which satisfy (5), verifies (5), too. But this observation is trivial, similarly as trivial is implication that (5) holds for $r \cdot a(t)$ if and only if it holds for $a(t)$.

Definition 3 *A set of vectors $v_1, v_2, \ldots, v_k$ from a linear space V (for example IMMU) is said to be linearly independent if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \neq 0 \in V$ whenever real numbers $\alpha_i \in V$ are not all equal to $0 \in R$.*

Definition 4 *A set of vectors $v_1, v_2, \ldots, v_k$ from a linear space V is said to be a base for V if those vectors are linearly independent and, additionally, each vector $v \in V$ is a linear combination of them, that is, $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$.*

It is well known from linear algebra that each finite base has the same number of vectors. Besides, each set of $n < k$ linearly independent vectors in V can be extended to a base, by simply adding properly chosen $(k - n)$ vectors. We shall say that a linear space V is said to be k-dimensional if a certain base of V consists of k vectors.

It is a nice experience to prove that the triangular functions are independent vectors (functions) in the sense of Definition 3 in the set of all continuous functions defined on $[t_1, T]$. Define V^S as the set of all linear combinations $\sum_{i=1}^{m} a_i S_i(t)$ of m triangular functions defined on $[t_1, T]$. By the definition of V^S, they form a base for V^S, because they are independent and span the entire space V^S.

Corollary 2 *The linear space IMMU of all shifts, against which our bond portfolio BP is immunized, is an $(m - 1)$-dimensional linear subspace in V^S.*

The proof follows from the observation that the set of solutions of the single linear equation (7) is $(m - 1)$-dimensional.

Example 1 *Let our bond portfolio BP be reduced to a single bond B paying 10 coupons $c_i = 10,\ 1 \leq i \leq 10$, at instances $t_1 = 0.5, t_2 = 1, t_3 = 1.5, t_3 = 2, \ldots, t_9 = 4.5, t_{10} = 5$; plus the par value of $c_{10} = 100$, at maturity $(t_{10} = 5)$. Assume that the term structure of interest rates* $s(t) = 0.065 - 0.0005t$, and the liability of L dollars (the present value of L equals the value of B) has to be discharged $q = 3.5$ years from now. Our goal is to determine the linear space IMMU by specifying its $(m - 1)$ base vectors (shifts) by means of Lagrange functions; see (10a)-(10e).*

Table 1. Numerical results of Example 1

i	t_i	$s(t_i)$	c_i	$PV(c_i)$	weights	$w(t_i)\cdot t_i$
1	0.5	0.0648	10	9.6814	0.0615	0.03073
2	1	0.0645	10	9.3754	0.0595	0.05951
3	1.5	0.0643	10	9.0812	0.0576	0.08647
4	2	0.0640	10	8.7985	0.0559	0.11170
5	2.5	0.0638	10	8.5268	0.0541	0.13532
6	3	0.0635	10	8.2655	0.0525	0.15740
7	3.5	0.0633	10	8.0142	0.0509	**0.17806**
8	4	0.0630	10	7.7724	0.0493	0.19735
9	4.5	0.0628	10	7.5399	0.0479	0.21538
10	5	0.0625	110	80.4777	0.5109	2.55431

Solution. We will be looking for $m-1$ base shifts $a_i(t)$, $1 \le i \le 9$, each of them in the form of $\sum a_j S_j(t)$, against which bond B is immunized at time $q = 3.5$. We will identify the first of these shifts, $a_1(t)$, and then the second, third, fourth, and so on, each one in two steps. To determine $a_1(t)$, in the first step we solve the equation $3.5\cdot a_7 = \sum_{i=1}^{10} w_i t_i a_i$ (compare (7)) with a_i standing for $a_1(t_i)$ and next, making use of identity (4), we will find the resulting base shift (10a). To solve $3.5\cdot a_7 = \sum_{i=1}^{10} w_i t_i a_i$, let us start with the numerical computations given in Table 1 to arrive at the following equation:

$$
\begin{aligned}
&3.5a_7 = \\
&\quad 0.03073a_1 + 0.05951a_2 + 0.08647a_3 + 0.11170a_4 + 0.13532a_5 + 0.15740a_6 \\
&\quad +0.17806a_7 + 0.19735a_8 + 0.21538a_9 + 2.55431a_{10},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&0.03073a_1 + 0.05951a_2 + 0.08647a_3 + 0.11170a_4 + 0.13532a_5 + 0.15740a_6 \\
&- 3.3219a_7 + 0.19735a_8 + 0.21538a_9 + 2.55431a_{10} = 0. \qquad (9)
\end{aligned}
$$

Our idea is to assume that just two of the coefficients a_i are different from 0, and then solve (9). This method of solving (9) could be implemented in several ways, but we select such a way of solution of (9), which will appear as advantageous in our later investigations.

*We use here the same term structure as in the examples from the paper by Rządkowski and Zaremba (2000)

Namely, we first set $a_1 = a_1(t_1) = a_1(0.5) = 1$ and then calculate $a_2 = a_1(t_2) = a_1(1) = -0.51632$ because (see Table 1)

$$a_2 = \frac{-w(t_1)t_1}{w(t_2)t_2} = \frac{-0.03073}{0.051632} = -0.51632 \tag{10}$$

with the remaining $8 = 10 - 2$ unknown variables being equal to 0. This is the end of step 1. At step 2 we make use of the property of triangular functions stated in identity (4) which enables us to express the first base shift in the form

$$a_1(t) = 1 \cdot S_1(t) - 0.51632 \cdot S_2(t). \tag{10a}$$

In the same manner we will be determine the remaining base shifts

$$a_2(t) = 1 \cdot S_1(t) - 0.35536 \cdot S_3(t), \quad a_3(t) = 1 \cdot S_1(t) - 0.27509 \cdot S_4(t), \tag{10b}$$

$$a_4(t) = 1 \cdot S_1(t) - 0.22708 \cdot S_5(t), \quad a_5(t) = 1 \cdot S_1(t) - 0.19522 \cdot S_6(t), \tag{10c}$$

$$a_6(t) = 1 \cdot S_1(t) - 0.17258 \cdot S_7(t), \quad a_7(t) = 1 \cdot S_1(t) - 0.15570 \cdot S_8(t), \tag{10d}$$

$$a_8(t) = 1 \cdot S_1(t) - 0.14267 \cdot S_9(t), \quad a_9(t) = 1 \cdot S_1(t) - 0.01203 \cdot S_{10}(t). \tag{10e}$$

To better illustrate the mechanism of determining the above base functions, let us demonstrate in detail how we have, for example, derived the formula for the fourth base vector (shift) $a_4(t) = 1 \cdot S_1(t) - 0.22708 \cdot S_5(t)$; in an analogous fashion one could prove the validity of the remaining seven formulas.

Searching for the fourth solution of equation (10), $a_4(t)$, in step 1 we solve as always the equation $3.5 \cdot a_7 = \sum_{i=1}^{10} w_i t_i a_i$, with a_i standing for $a_4(t_i),\ 1 \le i \le 10$, and then make use of identity (4), by letting $a_1 = a_4(t_1) = a_4(0.5) = 1$ and compute

$$a_5 = a_4(t_5) = a_4(2.5) = \frac{-w(t_1)t_1}{w(t_5)t_5} = \frac{-0.03073}{0.13532} = -0.22708 \tag{11}$$

from Table 1, with the remaining eight coefficients $a_i = a_4(t_i) = 0$. In the second step, based on identity (4), we conclude that

$$a_4(t) = 1 \cdot S_1(t) - 0.22708 \cdot S_5(t).$$

Summing up, we shall say that the set IMMU of all shifts, against which bond B is immunized, consists of such continuous functions $a(t)$, which are represented in the sense of Definition 1 by all the possible linear combinations of the above nine base shifts (10a)-(10e), defined on the interval $[0,\ 5]$.

Table 2. Numerical results of Example 2

i	t_i	$s(t_i)$	$\overline{c}_i$	$PV(\overline{c}_i)$	$\overline{w}(t_i)$	$\overline{w}(t_i)\cdot t_i$
1	0.5	0.0648	10	9.6814	0.0606	0.03028
2	1	0.0645	10	9.3754	0.0586	0.05865
3	1.5	0.0643	10	9.0812	0.0568	0.08521
4	2	0.0640	10	8.7985	0.0550	0.11007
5	2.5	0.0638	10	8.5268	0.0533	0.13334
6	3	0.0635	10	8.2655	0.0517	0.15511
7	3.5	0.0633	10	8.0142	0.0501	**0.17546**
8	4	0.0630	13	10.1042	0.0632	0.25282
9	4.5	0.0628	10	7.5399	0.0472	0.21224
10	5	0.0625	110	80.4777	0.5034	2.51706

4. Continuity properties of the linear space IMMU

Since all entries in Table 1 depend in a continuous way on instances $t_1, t_2, \ldots, t_m = T$, and the cash flow $c_1, c_2, \ldots, c_m$, as well as the term structure $s(t)$ of interest rates, the resulting equation (7) depends continuously on these parameters, too. As a result of this, the solutions of (7) and the resulting base vectors (shifts) $a_1(t), a_2(t), \ldots, a_m(t)$, constructed by us earlier, will also vary in a continuous fashion with these parameters.

In this section, we want to describe more broadly the continuous dependence of these base shifts with respect to the cash flow $c_1, c_2, \ldots, c_m$.

To explain what we precisely have in mind, let us consider Example 2, when the cash flow from Example 1 is slightly modified by adding a single inflow at $t_8 = 4$. It will appear that this change will have an effect on the base vector $a_7(t)$ only, while the remaining base vectors (shifts) $a_1(t)-a_6(t)$ and $a_8(t)$, $a_9(t)$ will not change at all.

Example 2 *Let portfolio BP generate the same payments as bond B in Example 1 did, plus a single payment of \$3 at time $t_8 = 4$, with all other data being the same.*

Solution. Repeating the same line of reasoning as the one in Example 1 and using the figures calculated in Table 2, one arrives at

$$3.5\overline{a}_7 =$$
$$0.03028\overline{a}_1 + 0.05865\overline{a}_2 + 0.08521\overline{a}_3 + 0.11007\overline{a}_4 + 0.13334\overline{a}_5 + 0.15511\overline{a}_6$$
$$+0.17546\overline{a}_7 + 0.25282\overline{a}_8 + 0.21224\overline{a}_9 + 2.51706\overline{a}_{10},$$

which is equivalent to

$$\begin{aligned}&0.03028\overline{a}_1 + 0.05865\overline{a}_2 + 0.08521\overline{a}_3 + 0.11007\overline{a}_4 + 0.13334\overline{a}_5 + 0.15511\overline{a}_6\\&- 3.325\overline{a}_7 + 0.25282\overline{a}_8 + 0.21224\overline{a}_9 + 2.51706\overline{a}_{10} = 0.\end{aligned} \tag{12}$$

Proceeding in exactly the same way as in Example 1, we are led to base shifts $\overline{a}_i(t)$, almost the same as shifts $a_i(t)$ in Example 1. Strictly speaking, letting as usual $\overline{a}_1 = \overline{a}_1(t_1) = \overline{a}_1(0.5) = 1$ and $\overline{a}_i = \overline{a}_1(t_i) = 0,\ 3 \le i \le 10$, we solve (12) for $\overline{a}_2 = \overline{a}_1(t_2) = \overline{a}_1(1)$ obtaining

$$\overline{a}_2 = -0.51632 = \frac{-0.03028}{0.05865} = a_2. \tag{13}$$

Since (13) gives rise to the same result of calculation as (10) did, the first base function $\overline{a}_1(t)$ remains exactly the same as $a_1(t)$ was before the cash flow $c_1, c_2, \ldots, c_m$ has been modified, that is,

$$\overline{a}_1(t) = \overline{a}_1 \cdot S_1(t) + \overline{a}_2 \cdot S_2(t) = 1 \cdot S_1(t) - 0.51632 \cdot S_2(t). \tag{14a}$$

Continuing in this way of reasoning, we identify the remaining eight base functions:

$$\overline{a}_2(t) = 1 \cdot S_1(t) - 0.35536 \cdot S_3(t), \quad \overline{a}_3(t) = 1 \cdot S_1(t) - 0.27509 \cdot S_4(t), \tag{14b}$$

$$\overline{a}_4(t) = 1 \cdot S_1(t) - 0.22708 \cdot S_5(t), \quad \overline{a}_5(t) = 1 \cdot S_1(t) - 0.19522 \cdot S_6(t), \tag{14c}$$

$$\overline{a}_6(t) = 1 \cdot S_1(t) - 0.17258 \cdot S_7(t), \quad \overline{a}_7(t) = 1 \cdot S_1(t) - 0.11977 \cdot S_8(t), \tag{14d}$$

$$\overline{a}_8(t) = 1 \cdot S_1(t) - 0.14267 \cdot S_9(t), \quad \overline{a}_9(t) = 1 \cdot S_1(t) - 0.01203 \cdot S_{10}(t), \tag{14e}$$

which, with the exception of $\overline{a}_7(t)$, appear to be exactly the same as their counterparts in Example 1. The question arises: why is $\overline{a}_7(t)$ different? The intuition tells us that this is probably so, because the formula for $\overline{a}_7(t)$ involves the shift $S_8(t)$, the only one among all the base shifts $(S_i(t))$, which captures the greater than previously (\$10) payment of \$13 at $t_8 = 4$.

Remark 1 *The natural question arises: why the result of calculation performed and shown in (14) must have been the same as the one performed and shown in (10)? Going further in this direction, one might ask why the base shifts shown in (10a)-(10e) are the same, except for one of them, as those in (14a)-(14e). Fortunately, the answer is not really very difficult.*

As far as the first question is concerned, all we have to demonstrate is to show the identity

$$a_2 = \frac{-w(t_1)t_1}{w(t_2)t_2} = \frac{-\overline{w}(t_1)t_1}{\overline{w}(t_2)t_2} = \overline{a}_2,$$

which is equivalent to the relationship $\frac{\overline{w}(t_2)}{w(t_2)} = \frac{\overline{w}(t_1)}{w(t_1)}$, and explain why this is so.

By the definition, $\overline{w}(t_1) = \frac{PV(\overline{c}_1)}{PV(BP)}$, $w(t_1) = \frac{PV(c_1)}{PV(B)}$, with $PV(c_1) = PV(\overline{c}_1)$, because $\overline{c}_1 = c_1 = 10$, so that $\frac{\overline{w}(t_1)}{w(t_1)} = \frac{PV(B)}{PV(BP)}$. Similarly, $PV(c_2) = PV(\overline{c}_2)$ and $\overline{w}(t_2) = \frac{PV(\overline{c}_2)}{PV(BP)}$, $w(t_2) = \frac{PV(c_2)}{PV(B)}$, so that $\frac{\overline{w}(t_2)}{w(t_2)} = \frac{PV(B)}{PV(BP)}$. The first question has thus been answered.

Remark 2 *When proving that the first base shift $\overline{a}_1(t)$ remains the same as $a_1(t)$ was before the cash flow $c_1, c_2, \ldots, c_m$ has been modified, we relied exclusively on the following two equalities:*

$$PV(\overline{c}_1) = PV(c_1) \quad \textit{and} \quad PV(\overline{c}_2) = PV(c_2).$$

Therefore, the above reasoning can be extended over to the remaining base vectors (shifts) $a_j(t)$, $j \neq 1$, as long as $PV(\overline{c}_1) = PV(c_1)$ and $PV(\overline{c}_{j+1}) = PV(c_{j+1})$.

In other words, as long as the payments generated by a bond portfolio BP are not modified at instances t_1 and t_{j+1} (equivalently $\overline{c}_1 = c_1$, $\overline{c}_{j+1} = c_{j+1}$), the base vector $a_j(t)$ for IMMU will remain the same for the new space IMMU corresponding to the new cash flow, that is, $\overline{a}_j(t) = a_j(t)$.

For example, since in Example 2 the payments generated by BP at instances t_1 and t_{8+1} are the same ($\overline{c}_1 = c_1 = 10, \overline{c}_9 = c_9 = 10$), the base vector $a_8(t)$ for IMMU will remain the same for the new space IMMU, that is, $\overline{a}_8(t) = a_8(t)$.

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