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Professor Jakub Gutenbaum
on his 70th birthday

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Schauder's fixed-point theorem in nonlinear controllability problems

by

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Abstract: This paper refers to application of the Schauder's fixed point theorem together with linear controllability results in getting the sufficient controllability conditions for various kinds of controllability and for different types of nonlinear control systems. The following nonlinear control systems are considered: finite-dimensional systems, systems with delays in control or in the state variables, and infinite-dimensional systems. The paper presents the review of results existing in the literature which show how Schauder's fixed-point theorem can be practically used to solve several controllability problems for different types of nonlinear control systems.

Keywords: controllability, nonlinear control systems, continuous-time systems, fixed-point theorems, Schauder's fixed-point theorem.

1. Introduction

Controllability is one of the fundamental concepts in modern mathematical control theory. Many dynamical systems are such that control does not affect the complete state of the dynamical system but only a part of it. Therefore, it is important to determine whether or not the complete system control is possible. Roughly speaking, controllability means that it is possible to steer a dynamical system from an arbitrary initial to an arbitrary final state using the set of admissible controls. The literature presents many different definitions of controllability which depend on the type of a dynamical system. The extensive list of publications concerning various controllability problems, containing more than 500 items can be found in Klamka (1991). Moreover, a survey of recent results and the current state of the controllability theory for different types of dynamical systems can be found in Klamka (1993).

Controllability theory for linear control systems is well developed even in infinite-dimensional spaces, and the details can be found in several previous papers and monographs (see Klamka, 1991, or Klamka, 1993, for more details). However, advances in nonlinear control systems have been mostly limited to specific classes of nonlinearities. It seems that the development of nonlinear controllability theory is possible by the well known fixed-point techniques of nonlinear analysis.

There are generally two different methods of analysis, see Carmichael and Quinn (1988), Magnusson, Pritchard and Quinn (1985). In the first method, a control driving the linearized system from a given initial to the final state is constructed. This control is a function of the state at which the linearization was performed and of the boundary conditions. Substitution of this control function into the nonlinear state equation yields a nonlinear implicit equation for the state. To solve this equation, one of the known fixed-point theorems (Schauder, Leray-Schauder, Banach, Nussbaum, or Darbou) can be applied and thus sufficient controllability conditions can be found. The second method does not construct a particular control, but considers the set of all controls, which steer the linearized system from a given initial state to the required endpoint. In this case multivalued maps are obtained and different versions (strong or weak) of Bohnenblust and Karlin fixed-point theorem could be used to obtain a fixed point, see Carmichael and Quinn (1988), Magnusson, Pritchard and Quinn (1985).

In this paper Schauder's fixed point theorem together with linear controllability results are considered as the basis for sufficient controllability conditions for various kinds of controllability and different types of nonlinear control systems. A review of existing results is presented, showing how Schauder's fixed-point theorem can be practically used to solve several controllability problems for different types of nonlinear control systems.

Let us recall that the fundamental version of Schauder's fixed-point theorem can be formulated as follows:

"Every continuous operator which maps a closed convex subset of a Banach space into its compact subset has at least one fixed point".

Observe that Schauder's fixed-point theorem does not require any contraction assumption, as in the Banach fixed-point-theorem, but on the other hand it does not give us the uniqueness of the fixed point. Since in controllability analysis the uniqueness is not important, Schauder's fixed-point theorem is an appropriate tool.

The paper is organized as follows. In Section 2 different mathematical models of nonlinear finite-dimensional control systems are presented and short comments on local and global controllability problems are given. Relative local and global controllability for nonlinear control systems with different kinds of delays in control or in the state variables is shortly described in Section 3. Section 4 is devoted to a study of exact local controllability and exact global controllability for certain semilinear abstract infinite-dimensional control systems. Finally, Section 5 contains concluding remarks and comments.

2. Controllability of finite-dimensional nonlinear systems

In this section we shall present the mathematical models (differential state equations) of several types of nonlinear finite-dimensional control systems. It should be pointed out that for all these systems it is possible to obtain sufficient conditions for different kinds of controllability using Schauder's fixed-point theorem and the results taken from linear controllability theory.

The first controllability results for nonlinear problems were obtained for finite–dimensional control systems of the following form

$$x'(t) = A(t, x(t))x(t) + B(t, x(t))u(t) \qquad t \in [t_0, t_1] \tag{1}$$

where $x \in R^n$, $u \in R^p$ and the elements of matrices A and B are continuous functions of x (for fixed t) and piecewise continuous functions in t (for fixed x). Moreover, they are bounded on $[t_0, t_1]$.

The system (1) is said to be controllable on $[t_0, t_1]$ if for any initial state $x(t_0) \in R^n$ and any given vector $x_1 \in R^n$ there exists a control $u(t), t \in [t_0, t_1]$ which steers the system from $x(t_0)$ to x_1 at time t_1.

In order to formulate the controllability problem in the form suitable for application of Schauder's fixed point theorem, it is assumed that the linear control system

$$x'(t) = A(t, z(t))x(t) + B(t, z(t))u(t) \tag{2}$$

is controllable, where z is a specified function belonging to the space $C(t_0, t_1; R^n)$. The solution of the linear system (2) in terms of the state transition matrix $F(t, t_0; z)$ is given by

$$x(t) = F(t, t_0; z)x(t_0) + \int_{t_0}^{t} F(t, s, z)B(s, z)u(s)ds. \tag{3}$$

Let us define

$$H(t_0, s; z) = F(t_0, s; z)B(s, z)$$

and

$$G(t_0, t; z) = \int_{t_0}^{t} H(t_0, s; z)H^T(t_0, s; z)ds.$$

Let us assume that the linear system (2) is controllable on $[t_0, t_1]$ for each function $z \in C(t_0, t_1; R^n)$, i.e. for each z there exists the inverse matrix $G(t_0, t_1; z)^{-1}$. Then it is easy to show that the following control function

$$u(t) = H^T(t_0, t; z)G(t_0, t_1; z)^{-1}(F(t_1, t_0; z)^{-1}x_1 - x(t_0)) \tag{4}$$

drives the system (2) from $x(t_0)$ to x_1 in finite time t_1. Substituting control $u(t)$ given by (4) into the right-hand side of equation (3) yields a nonlinear operator $P : C(t_0, t_1; R^n) \rightarrow C(t_0, t_1; R^n)$ of the following form

$$(Pz)(t) = F(t, t_0; z)x(t_0)+$$
$$\int_{t_0}^{t} F(t, s, z)B(s, z)H^T(t_0, s; z)G^{-1}(t_0, t_1; z)(F(t_1, t_0; z)^1 x_1 - x(t_0))ds. \quad (5)$$

Using Schauder's fixed-point theorem it can be shown that the nonlinear operator P has a fixed point z^* which is a solution to the nonlinear equation (5), where the control u^* is given by (4) evaluated at z^*. Moreover, $Pz^*(t_0) = x(t_0)$ and $Pz^*(t_1) = x_1$. Therefore, the control u^* steers the nonlinear system (1) from $x(t_0)$ to x_1 at t_1, i.e., the system (1) is controllable on $[t_0, t_1]$.

The above considerations show that sufficient conditions for nonlinear controllability can be obtained using Schauder's fixed-point theorem together with the well known linear controllability results, see Klamka (1991), Klamka (1993).

Let us observe that a similar method can be used to derive sufficient nonlinear controllability conditions for rather broad class of nonlinear control systems. After 1970, a great number of nonlinear controllability results based on Schauder's fixed-point theorem were obtained in the literature. Now, we shall briefly present the main types of nonlinear control systems considered in different papers. Schauder's fixed-point theorem together with linear controllability conditions are used to obtain sufficient conditions for different types of controllability.

The papers of Davison and Kunze (1970), Do (1990), Mirza and Womach (1997a) consider a perturbed linear control system of the form

$$x'(t) = A(t)x(t) + B(t)u(t) + f(t, x(t), u(t)). \quad (6)$$

The nonlinearity f satisfies a Lipschitz-type condition in x and u.

The papers Klamka (1975a, b) extend the controllability results to nonlinear control systems of the following type

$$x'(t) = A(t, x(t))x(t) + B(t, x(t))u(t) + f(t, x(t)), \quad (7)$$

where elements of matrices A and B satisfy the same conditions as in (1), whereas the nonlinear perturbation term f is uniformly bounded and satisfies a Lipschitz-type condition in x.

Wei (1976) provides controllability results for nonlinear system of the form

$$x'(t) = A(t, x(t), u(t))x(t) + B(t)u(t) + f(t, x(t), u(t)), \quad (8)$$

where elements of matrix A satisfy the same conditions as in (1), while the nonlinear perturbation term f is uniformly bounded and satisfies a Lipschitz-type condition in x and u. Moreover, matrix B is assumed to have a continuous first derivative.

Dannon and Kartsatos (1987) and Yamamoto (1977) consider nonlinear control system of the form

$$x'(t) = A(t, x(t), u(t))x(t) + B(t, x(t), u(t))u(t) + f(t, x(t), u(t)), \tag{9}$$

where elements of matrices A and B are continuous functions of x and u for fixed t and piecewise continuous functions of t for fixed x and u, and are bounded for $t \in [t_0, t_1]$ and $x \in R^n$. The nonlinear perturbation term f is bounded and satisfies a Lipschitz-type condition in both x and u.

Controllability and the so called total controllability (controllability connected with stabilizability) of the nonlinear control system

$$x'(t) = f(t, x(t)) + B(t)u(t), \tag{10}$$

and its pertrbation

$$x'(t) = f(t, x(t)) + B(t)u(t) + g(t, x(t), u(t)), \tag{11}$$

are considered in Balachandran (1988) and Eke (1990). It is generally assumed that the nonlinear functions f and g are continuous with respect to all their arguments and additionally satisfy Lipschitz-type conditions: f in x, and g both in x and u. Moreover, nonlinear functions f and g satisfy certain additional growth conditions.

In Deo (1997) controllability of first order time-varying semilinear Volterra integrodifferential control system is considered. The state equation of the control system has the following form

$$x'(t) = A(t)x(t) + \int_{t_0}^{t} K(t, s)x(s)ds + B(t)u(t) \qquad t \in [t_0, t_1], \tag{12}$$

where the kernel $K(t, s)$ is an $n \times n$ continuous matrix.

Applications of the nonlinear controllability theory in mathematical population dynamics can be found in Joshi and George (1989, 1992), Klamka (1991b, 1994). Schauder's fixed-point theorem can be also used in stochastic controllability problems. Sample controllability, in a given finite time interval, for a quite general nonlinear stochastic control system is investigated in Daner and Balachandran (1997). Moreover, several sufficient conditions for local or global controllability in a given time interval and for various types of nonlinear control systems can be found in the recently published papers Balachandran (1985), Balachandran and Balasubramaniam (1994), Balachandran and Dauer (1987, 1989, 1990, 1993, 1996), Balachandran and Somasundram (1983, 1985), Dauer (1972, 1976). In all these papers Schauder's fixed-point theorem or Leray-Schauder's fixed-point theorem are used to formulate and prove various controllability results.

3. Controllability for nonlinear systems with delays

Controllability theory for nonlinear control systems with delays is much more complicated and difficult than for systems without delays. First of all, let us observe that there are many different types of delays in control or state variables: multiple lumped constant or time-varying delays and distributed delays. Moreover, for control systems with delays there exist two different definitions of the state: instantaneous state and absolute state. Without going into details it should be pointed out that instantaneous state of a delayed control system is finite-dimensional and it is represented by an n-dimensional vector in R^n state space. On the other hand the absolute state of a delayed control system is represented by a suitably defined piece of trajectory of control system and hence it is an element of the appropriate infinite-dimensional function state space. Therefore, for delayed control systems it is necessary to introduce two fundamental concepts of controllability: relative controllability or Euclidean controllability connected with the finite-dimensional state space and absolute controllability or function controllability related to infinite-dimensional state space (see Klamka, 1991, or Klamka, 1993, for more details). Most of the results given in this section concern the relative controllability of nonlinear control systems with different types of delays in the control or state variables.

The paper by Balachandran and Balasubramaniam (1994) presents sufficient conditions for the global relative controllability of a semilinear control system with lumped and distributed delays in the state variables, described by the following differential equation

$$\begin{aligned} x'(t) &= A_0(t)x(t) + A_1(t)x(t-1) + \\ &\int_{-1}^{0} K(t,s)x(t+s)ds + B(t)u(t) + f(t,x(t),x(t-1),u(t)), \end{aligned} \tag{13}$$

where $A_0(t)$, $A_1(t)$, $B(t)$, $K(t,s)$, are time-varying matrices of appropriate dimensions and f is a nonlinear perturbation.

Onwuatu (1993) considers relative null controllability for the semilinear infinite neutral control system with multiple time-varying point delays and distributed delays in the state variables

$$\begin{aligned} \frac{d}{dt}D(t,x_t) &= \sum_{k=0}^{k=N} A_k(t)x(h_k(t)) + B(t)u(t) + \\ &\int_{-\infty}^{0} A(s)x(t+s)ds + f(t,x_t,u(t)), \end{aligned} \tag{14}$$

where D is a linear differentiable matrix function, f is a nonlinear continuous matrix function, symbol x_t denotes a piece of trajectory, $h_k(t), k = 0,1,2,\ldots,N$ are lumped time-varying delays, $A_k(t), k = 0,1,2,\ldots,N$, are $n \times n$ dimensional time-varying matrices. The term $\int_{-\infty}^{0} A(s)x(t+S)ds$ represents infinite delay.

Similarly, relative null controllability problem for semilinear neutral control system of the form (14) but additionally with multiple time-varying point delays in the control

$$\begin{aligned}\frac{d}{dt}D(t,x_t) &= \sum_{k=0}^{k=N} A_k(t)x(h_k(t)) + \sum_{i=0}^{i=N} B_i(t)u(h_i(t)) + \\ &\int_{-\infty}^{0} A(s)x(t+s) + f(t,x_t,u(t)) \end{aligned} \qquad (15)$$

is discussed in Dauer, Balachandran and Anthoni (1998). The symbols $B_i, i = 0,1,2,\ldots,N$ in (15) denote time-varying matrices of appropriate dimensions.

Global relative controllability of the nonlinear system with distributed delays in control

$$x'(t) = f(t,x(t),u(t)) + \int_{-h}^{0} (d_sB(t,s))u(t+s), \qquad (16)$$

and its generalization

$$x'(t) = A(t,x(t),u(t))x(t) + f(t,x(t),u(t)) + \int_{-h}^{0} (d_sB(t,s))u(t+s), \qquad (17)$$

is considered in Onwuatu (1989). Here, symbol $\int_{-h}^{0}(d_sB(t,s))u(t+s)$ denotes the Stieltjes integral, Klamka (1991a). The nonlinear term f and the $n \times n$-dimensional matrix A satisfy certain additional assumptions.

Global and local relative controllability of the nonlinear system with multiple point delays in the control

$$x'(t) = A(t,x(t))x(t) + \sum_{i=0}^{i=N} B_i(t,x(t))u(h_i(t)), \qquad (18)$$

and of its perturbation

$$x'(t) = A(t,x(t))x(t) + \sum_{i=0}^{i=N} B_i(t,x(t))u(h_i(t)) + f(t,x(t)), \qquad (19)$$

is discussed in details in a series of papers Klamka (1975a, b, c, 1976).

Similar controllability problems but for nonlinear systems with additional distributed delay in control represented by the term $\int_{-h}^{0}(d_sB(t,s))u(t+s)$ are investigated in Klamka (1978 and 1980).

Global relative controllability of a more general nonlinear system with multiple time-varying point delays in the perturbation term

$$\begin{aligned}x'(t) &= g(t,x(t),u(t)) + \int_{-h}^{0} d_\eta B(t,\eta,x(t),u(t))u(t+\eta) + \\ &f(t,x(t),u(h_0(t)),u(h_1,(t)),\ldots,u(h_i(t)),\ldots,u(h_N(t)))\end{aligned} \qquad (20)$$

is studied in Klamka (1999b).

It should be mentioned that the methods used in controllability considerations for nonlinear control systems with different types of delays in control, or in the state variables, are quite similar to those presented in Section 2 for problems and certain difficulties in finding the form of the nonlinear operator P, especially for control systems with distributed delays represented by the Stieltjes integral.

4. Controllability for infinite-dimensional nonlinear systems

Most of the controllability results for nonlinear infinite-dimensional control systems concern the so called semilinear control systems which consist of a linear and a nonlinear part, Klamka (1995, 1996, 1998, 1999a). This type of nonlinear control systems arises if a local linear approximation around a nominal trajectory of a nonlinear control system is considered (Carmichael and Quinn, 1988, Mirza and Womack, 1972a). Moreover, it should be stressed that for infinite-dimensional systems two different concepts of controllability are analysed. Namely, the exact (or strong) controllability and the approximate (or weak) controllability, Klamka (1991a, 1993). However, most of the results given in this section concern exact local or global controllability.

Let us consider a semilinear time-invariant infinite-dimensional control system of the form

$$x'(t) = Ax(t) + Bu(t) + F(x(t)) \qquad t \in [0, t_1], \tag{21}$$

where $x(t) \in X, u(t) \in U, X$ and U are Banach space, $A : X \supset D(A) \to X$ is a linear generally unbounded operator which generates strongly continuous semigroup of linear bounded operators $S(t) : X \to X$, for $t > 0, B : U \to X$ is a linear bounded operator and $F : X \to X$ is a nonlinear operator. For the initial condition $x(0) = 0$, a mild solution of abstract differential equation (21) satisfies the following nonlinear integral equation

$$x(t) = \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)F(x(s))ds. \tag{22}$$

Controllability results for a semilinear control system (21) strongly depend on the controllability properties of the linear part of the system given by

$$x'(t) = Ax(t) + Bu(t). \tag{23}$$

Now, let us introduce for the linear system (23) the controllability operator $C : L^p(0, t_1; U) \to X$ defined by

$$Cu = \int_0^{t-1} S(t_1 - s)Bu(s)ds.$$

The system (21) is said to be exactly controllable to a linear subspace $V \subset X$ on $[0, t_1]$ if V is contained in $RangeC$.

If the linear system (21) is exactly controllable to the subspace V, then we can assume without loss of generality that $RangeC = V$ and construct an invertible operator C_1 defined on $L^p(0, t_1; U)/KerC$,

Let us now consider the control

$$u(t) = C_1^{-1}\left(\nu - \int_0^{t_1} S(t_1 - s)F(x(s))ds\right)(t). \tag{24}$$

The same approach as in Section 2 can be applied. Control $u(t)$ given by (24) steers the semilinear control system (21) from zero initial state to $\nu \in V$ in time t_1. Substituting control $u(t)$ given by (24) into the right-hand side of the equality (22) yields the nonlinear operator P of the form

$$\begin{aligned}(Px)(t) \quad = \quad & \int_0^t S(t-s)BC_1^{-1}\left(\nu - \int_0^{t_1} S(t_1 - \tau)F(x(\tau))d\tau\right)ds + \\ & \int_0^t S(t-s)F(x(s))ds. \end{aligned} \tag{25}$$

It can be verified that if the operator P has a fixed point x^*, then the control u^*, given by (24) evaluated at x^*, steers the semilinear control system (21) from zero initial state to $\nu \in V$ in time t_1 i.e., the semilinear control system (21) is exactly controllable to V in time t_1, see Carmichael and Quinn (1988), Magnusson, Pritchard and Quinn (1985).

Therefore, the general idea is to look for a fixed point of the nonlinear map P defined by (25) in some ball contained in an appropriately defined Banach space. It should be pointed out that the application of Schauder's fixed point theorem requires both certain compactness and growth conditions imposed on the nonlinear operator F. Under these assumptions, Schauder's fixed-point theorem can be used to prove that the operator P has a fixed point, i.e., that the semilinear control system (21) is exactly controllable to V in time t_1, Carmichael and Quinn (1988), Magnusson, Pritchard and Quinn (1985).

Various kinds of sufficient conditions for exact or approximate controllability for different types of nonlinear infinite-dimensional control systems can be found in many recently published papers. Balachandran and Dauer (1998) consider local exact controllability of semilinear evolution control systems with time-varying nonlinear term of the form

$$x'(t) = Ax(t) + Bu(t) + F(t, x(t)) \qquad t \in [t_0, t_1], \tag{26}$$

where the nonlinear operator F satisfies certain growth condition both in t and x and the linear unbounded operator A generates an analytic semigroup of bounded linear operators $S(t) : X \to X$, for $t > 0$.

The second order semilinear Volterra integrodifferential control system is considered in Balachandran, Dauer and Balasubramaniam (1995), Balachandram, Park and Anthoni (1999), Park and Han (1997) where sufficient conditions for global exact controllability are formulated and proved. The control system is described by the following state equation

$$x''(t) = Ax(t) + \int_0^t g(t,s,x(s))ds + Bu(t) \qquad t \in [0,t_1], \tag{27}$$

where linear unbounded operator A generates a strongly continuous one parameter cosine family of bounded linear operators $S(t) : X \rightarrow X$, for $t > 0$, and g is a nonlinear unbounded mapping with certain additional assumptions concerning continuity.

5. Concluding remarks

The paper contains a short review of controllability results for nonlinear systems presented in the literature. The results referred to show that Schauder's fixed-point theorem can be effectively used in nonlinear control problems to obtain sufficient conditions for different kinds of controllability applied to different dynamical systems.

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