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Relations for moments of generalised order statistics based on Weibull–G family of distributions

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Abstract

The Weibull distribution is one of the important distributions used in reliability theory and life-testing experiments. The generalised versions of the Weibull distribution give a more flexible model for these studies. The Weibull–G family of distributions is one of the extended versions extensively studied. In this paper, we have studied moments properties of generalised order statistics for the said distribution in terms of a single moment, product moments and characterisation. Several examples and special cases are presented. The results can be applied to all distributions belonging to this family.

Keywords: *generalised order statistics, order statistics, record values, expectation identities, characterisation*

1. Introduction

Order statistics (OSs) and related general models of ordered random variables (ORVs) are important in statistical theory and its applications. Kamps [17] introduced the concept of generalised order statistics (GOS) and showed that all well-known models of ORVs such as record values (RVs), OSs, Pfeifer's records, progressive type II censored order statistics (PT2COS), sequential order statistics (SOS), etc. are the submodels of GOS in the distributional and theoretical sense. There is no doubt that GOS and different models of ORVs will continue to arouse the interest of many researchers working in the fields of theoretical statistics and applications.

Recurrence relations for moments of GOS and characterisation through it for various distributions have been investigated by several authors in the literature. For a detailed review of the literature, see [2, 3, 6–12, 14, 18–28] and references therein. Furthermore, Alawady et al. [5] studied the concomitants of GOS from the iterated Farlie–Gumbel–Morgenstern-type bivariate distribution, while Abd

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Elgawad et al. [1] and Alawady et al. [4] studied the concomitants of GOS from the bivariate Cambanis family of distributions. Jamal and Chesneau [16] studied the moment properties of order statistics, reverse order statistics, and upper record values of the power Ailamujia distribution.

1.1. Definition of GOS

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \text{ for } 1 \leq i \leq n-1$$

Random variables (RVs) $X_{1,n,\tilde{m},k}, X_{2,n,\tilde{m},k}, \ldots, X_{n,n,\tilde{m},k}$ are said to be GOS from an absolutely continuous population with a cumulative distribution function (CDF) $F()$, and probability density function (PDF) $f()$ if their joint PDF is of the form

$$k \prod_{j=1}^{n-1} \gamma_j \Big(\prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i) \Big) [1 - F(x_n)]^{k-1} f(x_n) \tag{1}$$

in the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

The particular cases of the model (1) are given below:

- If $m_1 = m_2 = \cdots = m_{n-1} = 0$, and $k = 1$, then $\gamma_r = n - r + 1, 1 \leq r \leq n-1$. In this case, model (1) reduces to the joint density of the OSs.
- By choosing $n = m$, $m_i = R_i$ for $i = 1, 2, \ldots, m-1$ and $k = R_m + 1, \gamma_r = m - r + 1 + \sum_{i=r}^{m} R_i$, $1 \leq r \leq m$, whereR_i is a set of prefixed integers that shows random removal R_i in the ith failure of the surviving items of an experiment. Model (1) reduces to the joint density based on PT2COS.
- If $m_1 = m_2 = \ldots m_{n-1} = -1$ and $k = 1$, then $\gamma_r = 1, 1 \leq r \leq n-1$. In this case, model (1) reduces to the joint density of upper RVs.
- If $m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1$ and $k = \alpha_n, \alpha \in R^+, i = 1, 2, \ldots, n-1$, then $\gamma_r = (n - r + 1)\alpha_r, 1 \leq r \leq n-1$. Model (1) reduces to the joint density of the SOS.

Here, we may consider two cases.

Case I. $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n-1, i \neq j$

In view of (1), the PDF of rth GOS $X_{r,n,\tilde{m},k}$ is given as in [18]

$$f_{r,n,\tilde{m},k}(x) \; = \; C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i - 1} \tag{2}$$

where

$$C_{r-1} \; = \; \prod_{i=1}^{r} \gamma_i, \;\; \gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0, \;\; a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \qquad 1 \leq i \leq r \leq n$$

The joint PDF of $X_{r,n,\tilde{m},k}$ and $X_{s,n,\tilde{m},k}$, $1 \leq r < s \leq n$, is given as in [18]

$$f_{r,s,n,\tilde{m},k}(x,y) = C_{s-1} \sum_{j=r+1}^{s} a_j^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j} \left[\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \quad x < y \tag{3}$$

where

$$a_j^{(r)}(s) = \prod_{\substack{t=r+1 \\ t \neq j}}^{s} \frac{1}{(\gamma_t - \gamma_j)}, \quad r+1 \le j \le s \le n$$

Case II. $m_i = m$, $i = 1, 2, \ldots, n-1$
The PDF of rth GOS $X_{r,n,m,k}$ is given as in [17]

$$f_{r,n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{4}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + (n-i)(m+1)$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ \log\left(\dfrac{1}{1-x}\right), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \quad x \in [0,1)$$

The joint PDF of $X_{r,n,m,k}$ and $X_{s,n,m,k}$, $1 \le r < s \le n$, is given as in [25]

$$\begin{aligned} f_{r,s,n,m,k}(x,y) &= \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}(F(x)) \\ &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(x) f(y), \quad -\infty \le x < y \le \infty \end{aligned} \tag{5}$$

1.2. Weibull–G family of distributions

Bourguignon et al. [13] proposed a wider Weibull–G family of distributions being one of the most important distributions used in reliability theory. The CDF of the Weibull–G family of distribution is

$$F(x) = 1 - \exp\left[-\alpha \left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}\right], \quad \alpha, \beta > 0 \tag{6}$$

and the corresponding PDF is given by

$$f(x) = \alpha\beta g(x) \frac{(G(x))^{\beta-1}}{(\bar{G}(x))^{\beta+1}} \exp\left[-\alpha \left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}\right] \tag{7}$$

where $G(x)$ refers to the base distribution and $\bar{G}(x) = 1 - G(x)$.

In view of (6) and (7), the relation between the survival function (SF) and PDF of this family of distributions can be seen as

$$\bar{F}(x) = \left[\frac{1}{\alpha\beta\lambda(x)} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}\right] \left(\bar{G}(x)\right)^{\beta+l+1} f(x) \tag{8}$$

where $\bar{F}(x) = 1 - F(x)$ is the SF and $\lambda(x) = \dfrac{g(x)}{G(x)}$ is the inverse failure rate (IFR).

The paper is organised as follows. Section 2 demonstrates a single moment of GOS for the Weibull–G family of distributions, as given in (6). In addition, some examples and special cases are discussed. The properties of product moments are studied in Section 3, while the characterisation results are presented in Section 4. In Section 5, a brief conclusion is given.

2. Single moment

Before coming to the main result, we reproduce the lemma given by Athar and Islam [7].

Lemma 1. For Case I with PDF given in (2) and $2 \le r \le n, n \ge 1, k_1 = 1, 2 \ldots$

$$E\left[X^{k_1}_{r,n,\tilde{m},k}\right] - E\left[X^{k_1}_{r-1,n,\tilde{m},k}\right] = k_1 C_{r-2} \int_{-\infty}^{\infty} x^{k_1-1} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i} dx \tag{9}$$

Proof. For $\gamma_i \ne \gamma_j, i, j = 1, 2, \ldots, n-1, i \ne j$, Athar and Islam [7] have shown that

$$E\left[\xi(X_{r,n,\tilde{m},k})\right] - E\left[\xi(X_{r-1,n,\tilde{m},k})\right] = C_{r-2} \int_{-\infty}^{\infty} \xi'(x) \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i} dx \tag{10}$$

where $\xi(x)$ is a Borel measurable function of $x \in (-\infty, \infty)$.

Let $\xi(x) = x^{k_1}$, then Lemma 1 can be established in view of (10). □

Theorem 1. Assume that Case I is satisfied. For the Weibull–G family of distributions as given in (6) and $n \in N, \tilde{m} \in \mathbb{R}, k > 0, 1 \le r \le n, k_1 = 1, 2, \ldots$

$$E\left[X^{k_1}_{r,n,\tilde{m},k}\right] - E\left[X^{k_1}_{r-1,n,\tilde{m},k}\right] = \frac{k_1}{\alpha\beta\gamma_r} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma(\beta)\Gamma(l+1)} E\left[B^{\beta+l+1}(X_{r,n,\tilde{m},k})\right] \tag{11}$$

where $B^{\beta+l+1} = \dfrac{x^{k_1-1}}{\lambda(x)} \left(\bar{G}(x)\right)^{\beta+l+1}$.

Proof. Based on (8) and (9), we have the following

$$E\left[X^{k_1}_{r,n,\tilde{m},k}\right] - E\left[X^{k_1}_{r-1,n,\tilde{m},k}\right] = \frac{k_1 C_{r-1}}{\gamma_r} \int_{-\infty}^{\infty} x^{k_1-1} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i-1}$$

$$\times \left[\frac{1}{\alpha\beta\lambda(x)}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}\left(\bar{G}(x)\right)^{\beta+l+1}\right] f(x)dx$$

$$= \frac{k_1 C_{r-1}}{\alpha\beta\gamma_r}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}\int_{-\infty}^{\infty}\left[B^{\beta+l+1}(X_{r,n,\tilde{m},k})\right]\sum_{i=1}^{r} a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1} f(x)dx$$

Therefore, we reached (11). Hence, the proof of Theorem 1 is completed. □

Corollary 1. For Case II and the condition as stated in Theorem 1

$$E\left[X_{r,n,m,k}^{k_1}\right] - E\left[X_{r-1,n,m,k}^{k_1}\right] = \frac{k_1}{\alpha\beta\gamma_r}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E\left[B^{\beta+l+1}(X_{r,n,m,k})\right] \tag{12}$$

Proof. Since for $\gamma_i \neq \gamma_j, i \neq j = 1, 2, \ldots, n-1$ but $m_i = m$

$$a_i(r) = \frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!}$$

Therefore, PDF given in (2) reduces to (4). Thus, relation (12) can be obtained by replacing $\tilde{m}$ with m in (11). □

Remark 1. If $m_i = 0, i = 1, 2, \ldots, n-1$ and $k = 1$, then the relation for a single moment of OS for Weibull–G family of distribution is given as

$$E\left[X_{r:n}^{k_1}\right] - E\left[X_{r-1:n}^{k_1}\right] = \frac{k_1}{\alpha\beta(n-r+1)}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E\left[B^{\beta+l+1}(X_{r:n})\right]$$

where $E(X_{r:n}^{k_1})$ is the k_1th moment of rth OS.

Remark 2. Let $m_i \to -1; i = 1, 2, \ldots, n-1$, then single moment of kth upper RVs is obtained as

$$E[X_{U^{(k)}(n)}^{k_1}] - E[X_{U^{(k)}(n-1)}^{k_1}] = \frac{k_1}{\alpha\beta k}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E[B^{\beta+l+1}(X_{U^{(k)}(n)})]$$

where $E[X_{U^{(k)}(n)}^{k_1}]$ is the k_1th moment of sequence of kth upper RVs.

2.1. Examples

2.1.1. Weibull–uniform distribution (WU)

Suppose the parent distribution is a uniform distribution in the interval $(0, \theta)$. Thus, its CDF and PDF are

$$G(x;\theta) = \frac{x}{\theta}, \quad 0 \le x \le \theta$$

and

$$g(x;\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta$$

Thus, the CDF and PDF of WU distribution, respectively, given by

$$F(x;\, \alpha,\, \beta,\, \theta) = 1 - \exp\left[-\alpha\left(\frac{x}{\theta - x}\right)^{\beta}\right], \quad 0 \leq x \leq \theta,\, \alpha,\, \beta > 0 \tag{13}$$

$$f(x;\, \alpha,\, \beta,\, \theta) = \frac{\alpha\beta\theta}{(\theta - x)^2}\left(\frac{x}{\theta - x}\right)^{\beta-1} \exp\left[-\alpha\left(\frac{x}{\theta - x}\right)^{\beta}\right], \quad 0 \leq \theta,\ \alpha, \beta > 0 \tag{14}$$

Now, it is easy to see that

$$B^{\beta+l+1}(x) \;=\; \frac{x^{k_1-1}}{\lambda(x)}\left(\bar{G}(x)\right)^{\beta+l+1} = x^{k_1}\left(1 - \frac{x}{\theta}\right)^{\beta+l+1} = \sum_{u=0}^{\beta+l+1}(-1)^u\binom{\beta+l+1}{u}\frac{1}{\theta^u}x^{k_1+u}$$

Now, using (11), we get

$$E\left[X^{k_1}_{r,n,\tilde{m},k}\right] - E\left[X^{k_1}_{r-1,n,\tilde{m},k}\right] = \frac{k_1}{\alpha\beta\theta^u\gamma_r}\sum_{l=0}^{\infty}\sum_{u=0}^{\beta+l+1}(-1)^u\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}\binom{\beta+l+1}{u}E\left[X^{(k_1+u)}_{r,n,\tilde{m},k}\right]$$

2.1.2. Weibull–Weibull distribution (WW)

Consider the base distribution to be the Weibull distribution. The CDF and PDF of the Weibull distribution are given by

$$G(x; \lambda, \theta) = 1 - \mathrm{e}^{-\lambda x^{\theta}}, \quad x > 0; \lambda, \theta > 0$$

and

$$g(x; \lambda, \theta) = \theta\lambda x^{\theta-1}\mathrm{e}^{-\lambda x^{\theta}}, \quad x > 0; \lambda, \theta > 0$$

Now, the CDF and PDF of the WW distribution can be written as

$$F(x; \alpha, \beta, \lambda, \theta) = 1 - \exp\left[-\alpha\left(e^{\lambda x^{\theta}} - 1\right)^{\beta}\right], \quad x > 0; \alpha, \beta > 0 \tag{15}$$

and

$$f(x; \alpha, \beta, \lambda, \theta) = \alpha\beta\lambda\theta x^{\theta-1}\left(1 - \mathrm{e}^{-\lambda x^{\theta}}\right)^{\beta-1}\exp\left[\lambda\beta x^{\theta} - \alpha\left(\mathrm{e}^{\lambda x^{\theta}} - 1\right)^{\beta}\right] \tag{16}$$

Further, we have

$$\begin{aligned}
B^{\beta+l+1}(x) &= \frac{x^{k_1-1}}{\lambda(x)}\left(\bar{G}(x)\right)^{\beta+l+1} \\
&= x^{k_1-1}\frac{(1 - \mathrm{e}^{-\lambda x^{\theta}})}{\lambda\theta x^{\theta-1}\mathrm{e}^{-\lambda x^{\theta}}}\mathrm{e}^{-\lambda x^{\theta}(\beta+l+1)} = \frac{x^{k_1-\theta}}{\lambda\theta}\left[\mathrm{e}^{-\lambda x^{\theta}(\beta+l)} - \mathrm{e}^{-\lambda x^{\theta}(\beta+l+1)}\right] \\
&= \frac{1}{\lambda\theta}\left[\sum_{a=1}^{\infty}(-1)^a\frac{[\lambda(\beta+l)]^a}{a!}x^{k_1+\theta(a-1)} - \sum_{b=1}^{\infty}(-1)^b\frac{[\lambda(\beta+l+1)]^b}{b!}x^{k_1+\theta(b-1)}\right]
\end{aligned}$$

Now, in the view of (11), we get

$$E\big[X_{r,n,\tilde{m},k}^{k_1}\big] - E\big[X_{r-1,n,\tilde{m},k}^{k_1}\big] = \frac{k_1}{\alpha\beta\theta\lambda\gamma_r}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma(\beta)\Gamma(l+1)} \times \left\{\sum_{a=1}^{\infty}(-1)^a\frac{[\lambda(\beta+l)]^a}{a!}E\left[X_{r,n,\tilde{m},k}^{k_1+\theta(a-1)}\right] - \sum_{b=1}^{\infty}(-1)^b\frac{[\lambda(\beta+l+1)]^b}{b!}E\left[X_{r,n,\tilde{m},k}^{k_1+\theta(b-1)}\right]\right\}$$

2.1.3. Weibull–Pareto distribution (WP)

Let the base distribution be the Pareto distribution with CDF and PDF given by

$$G(x;\theta,\rho) = 1-\rho^{\theta}x^{-\theta}, \quad \rho<x<\infty; \rho>0, \theta>1$$

and

$$g(x;\theta,\rho) = \theta\rho^{\theta}x^{-(\theta+1)}, \quad \rho<x<\infty; \rho>0, \theta>1$$

Thus, the CDF and PDF of the WP distribution are

$$F(x;\alpha,\beta,\theta,\rho) = 1-\exp\left[-\alpha\left(\rho^{-\theta}x^{\theta}-1\right)^{\beta}\right], \quad \rho<x<\infty, \alpha,\beta,\rho>0; \theta>1 \tag{17}$$

and

$$f(x;\alpha,\beta,\theta,\rho) = \alpha\beta\theta\rho^{-\theta}x^{\theta-1}\left(\rho^{-\theta}x^{\theta}-1\right)^{\beta-1}\exp\left[-\alpha\left(\rho^{-\theta}x^{\theta}-1\right)^{\beta}\right] \tag{18}$$

Also, $B^{\beta+l+1}(x)$ is computed as follows

$$B^{\beta+l+1}(x) = \frac{x^{k_1-1}}{\lambda(x)}\left(\bar{G}(x)\right)^{\beta+l+1} = x^{k_1-1}\frac{(1-\rho^{\theta}x^{-\theta})}{\theta\rho^{\theta}x^{-(\theta+1)}}\left(\rho^{\theta}x^{-\theta}\right)^{\beta+l+1} = \theta^{-1}\left[\rho^{\theta(\beta+l)}x^{k_1-\theta(\beta+l)} - \rho^{\theta(\beta+l+1)}x^{k_1-\theta(\beta+l+1)}\right]$$

Now, using (11), we have

$$E\big[X_{r,n,\tilde{m},k}^{k_1}\big] - E\big[X_{r-1,n,\tilde{m},k}^{k_1}\big] = \frac{k_1}{\alpha\beta\theta\gamma_r}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma(\beta)\Gamma(l+1)} \times \left\{\rho^{\theta(\beta+l)}E\left[X_{r,n,\tilde{m},k}^{k_1-\theta(\beta+l)}\right] - \rho^{\theta(\beta+l+1)}E\left[X_{r,n,\tilde{m},k}^{k_1-\theta(\beta+l+1)}\right]\right\}$$

2.1.4. Weibull extreme value distribution (WE)

Consider the base distribution to be the extreme value distribution with CDF and PDF given by

$$G(x) = 1-\mathrm{e}^{-\mathrm{e}^{x}}, \quad -\infty<x<\infty$$

and

$$g(x) = \mathrm{e}^{(x-\mathrm{e}^{x})}, \quad -\infty<x<\infty$$

Thus, the CDF and PDF of WE distribution are given, respectively, by

$$F(x;\alpha,\beta)=1-\exp\left[-\alpha\left(\mathrm{e}^{\mathrm{e}^{x}}-1\right)^{\beta}\right],\quad -\infty<x<\infty;\alpha,\beta,>0 \tag{19}$$

and

$$f(x;\alpha,\beta)=\alpha\beta\mathrm{e}^{(\beta\mathrm{e}^{x}+x)}\left(1-\mathrm{e}^{-\mathrm{e}^{x}}\right)^{\beta-1}\exp\left[-\alpha\left(\mathrm{e}^{\mathrm{e}^{x}}-1\right)^{\beta}\right] \tag{20}$$

Furthermore, we require the following computation

$$\begin{aligned}
B^{\beta+l+1}(x) &= \frac{x^{k_1-1}}{\lambda(x)}\left(\bar{G}(x)\right)^{\beta+l+1}=x^{k_1-1}\frac{\left(1-\mathrm{e}^{-\mathrm{e}^{x}}\right)}{\mathrm{e}^{(x-\mathrm{e}^{x})}}\left(\mathrm{e}^{-\mathrm{e}^{x}}\right)^{\beta+l+1}\\
&= x^{k_1-1}\mathrm{e}^{-x}\left[\mathrm{e}^{-\mathrm{e}^{x}(\beta+l)}-\mathrm{e}^{-\mathrm{e}^{x}(\beta+l+1)}\right]\\
&= x^{k_1-1}\mathrm{e}^{-x}\left[\sum_{b=0}^{\infty}(-1)^{b}\frac{(\mathrm{e}^{x}(\beta+l))^{b}}{b!}-\sum_{c=0}^{\infty}(-1)^{c}\frac{(\mathrm{e}^{x}(\beta+l+1))^{c}}{c!}\right]\\
&= \sum_{b=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{b}\frac{(\beta+l)^{b}}{b!}\frac{(b-1)^{m}}{m!}x^{k_1+m-1}\\
&- \sum_{c=0}^{\infty}\sum_{h=0}^{\infty}(-1)^{c}\frac{(\beta+l+1)^{c}}{c!}\frac{(c-1)^{h}}{h!}x^{k_1+h-1}
\end{aligned}$$

Thus, from (11), we can write

$$\begin{aligned}
E\left[X_{r,n,\tilde{m},k}^{k_1}\right]-E\left[X_{r-1,n,\tilde{m},k}^{k_1}\right] &= \frac{k_1}{\alpha\beta\gamma_r}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma(\beta)\Gamma(l+1)}\\
&\times\left\{\sum_{b=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{b}\frac{(\beta+l)^{b}}{b!}\frac{(b-1)^{m}}{m!}E\left[X_{r,n,\tilde{m},k}^{k_1+m-1}\right]\right.\\
&\left.-\sum_{c=0}^{\infty}\sum_{h=0}^{\infty}(-1)^{c}\frac{(\beta+l+1)^{c}}{c!}\frac{(c-1)^{h}}{h!}E\left[X_{r,n,\tilde{m},k}^{k_1+h-1}\right]\right\}
\end{aligned}$$

3. Product moments

Lemma 2. For Case I with PDF as given in (3) and $1\le r<s\le n, n\ge 1$, $k>0$, $k_1,k_2=1,2,\ \ldots$

$$\begin{aligned}
&E\left[X_{r,n,\tilde{m},k}^{k_1}X_{s,n,\tilde{m},k}^{k_2}\right]-E\left[X_{r,n,\tilde{m},k}^{k_1}X_{s-1,n,\tilde{m},k}^{k_2}\right]\\
&=k_2C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^{k_1}y^{k_2-1}\left[\sum_{c=r+1}^{s}a_c^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_c}\right]\\
&\times\left[\sum_{h=1}^{r}a_h(r)[\bar{F}(x)]^{\gamma_h}\right]\frac{f(x)}{\bar{F}(x)}dydx
\end{aligned} \tag{21}$$

Proof. Athar and Islam [7] have shown that

$$\begin{aligned} & E\big[\xi\{X_{r,n,\tilde{m},k}, X_{s,n,\tilde{m},k}\}\big] - E\big[\xi\{X_{r,n,\tilde{m},k}, X_{s-1,n,\tilde{m},k}\}\big] \\ &= C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty} \frac{\partial}{\partial y}\xi(x,y)\left[\sum_{c=r+1}^{s} a_c^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_c}\right] \\ &\times \left[\sum_{h=1}^{r} a_h(r)[\bar{F}(x)]^{\gamma_h}\right]\frac{f(x)}{\bar{F}(x)}dydx \end{aligned} \tag{22}$$

The lemma can be established by letting $\xi(x,y) = x^{k_1}y^{k_2}$ in (22). □

Theorem 2. Let Case I be satisfied. For the Weibull–G family of distributions as given in (6) and $n \in N, \tilde{m} \in \mathbb{R}, K > 0, 1 < s \le k_1, k_2 = 1, 2 \ldots$

$$\begin{aligned} & E\big[X^{k_1}_{r,n,\tilde{m},k}X^{k_2}_{s,n,\tilde{m},k}\big] - E\big[X^{k_1}_{r,n,\tilde{m},k}X^{k_2}_{s-1,n,\tilde{m},k}\big] \\ &= \frac{k_2}{\alpha\beta\gamma_s}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E\big[A^{\beta+l+1}\{X_{r,n,\tilde{m},k}, X_{s,n,\tilde{m},k}\}\big] \end{aligned} \tag{23}$$

where

$$A^{\beta+l+1}(x,y) = x^{k_1}y^{k_2-1}\frac{1}{\lambda(y)}\left(\bar{G}(y)\right)^{\beta+l+1} \text{ and } \lambda(y) = \frac{g(y)}{G(y)}$$

Proof. Using (8) and (21), we get the following

$$\begin{aligned} & E\big[X^{k_1}_{r,n,\tilde{m},k}.X^{k_2}_{s,n,\tilde{m},k}\big] - E\big[X^{k_1}_{r,n,\tilde{m},k}.X^{k_2}_{s-1,n,\tilde{m},k}\big] \\ &= \frac{k_2C_{s-1}}{\alpha\beta\gamma_s}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^{k_1}y^{k_2-1}\left[\sum_{c=r+1}^{s} a_c^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_c}\right]\left[\sum_{h=1}^{r} a_h(r)[\bar{F}(x)]^{\gamma_h}\right] \\ &\times \frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}\frac{1}{\lambda(y)}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}\left(\bar{G}(y)\right)^{\beta+l+1}dydx \\ &= \frac{k_2C_{s-1}}{\alpha\beta\gamma_s}\int_{-\infty}^{\infty}\int_{x}^{\infty} A^{\beta+l+1}(x,y)\left[\sum_{c=r+1}^{s} a_c^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_c}\right] \\ &\times \left[\sum_{h=1}^{r} a_h(r)[\bar{F}(x)]^{\gamma_h}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx \end{aligned}$$

This yields (23). Therefore, the proof of Theorem 2 is complete. □

Corollary 2. For Case II and the condition as stated in Theorem 2

$$E\big[X^{k_1}_{r,n,m,k}X^{k_2}_{s,n,m,k}\big]-E\big[X^{k_1}_{r,n,m,k}X^{k_2}_{s-1,n,m,k}\big]$$
$$=\frac{k_2}{\alpha\beta\gamma_s}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E\big[A^{\beta+l+1}\{X_{r,n,m,k},X_{s,n,m,k}\}\big] \tag{24}$$

Proof. Since for $\gamma_h\neq\gamma_c; c\neq h=1,2,\ \ldots,\ n-1$ but $m_h=m$

$$a_h^{(r)}(s)=\frac{1}{(m+1)^{s-r-1}}(-1)^{s-h}\frac{1}{(h-r-1)!(s-h)!}$$

Therefore, joint PDF of $X_{r,n,\tilde{m},k}$ and $X_{s,n,\tilde{m},k}$ given in (3) reduces to (5). Thus, relation (24) can be established by replacing $\tilde{m}$ with m in (23). □

Remark 3. If $m_h=0; h=1,2,\ \ldots,\ n-1$ and $k=1$, then the relation for product moment of OSs for Weibull–G family of distribution is given by

$$E\big[X^{k_1}_{r:n}X^{k_2}_{s:n}\big]-E\big[X^{k_1}_{r:n}X^{k_2}_{s-1:n}\big]=\frac{k_2}{\alpha\beta\gamma_s}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E\big[A^{\beta+l+1}\{X_{r:n},X_{s:n}\}\big]$$

Remark 4. Let $m_h\to -1; h=1,2,\ \ldots,\ n-1$, then the product moment of the kth upper RVs is given as

$$E[X^{k_1}_{U^{(k)}(n)}X^{k_2}_{U^{(k)}(m)}]-E[X^{k_1}_{U^{(k)}(n)}X^{k_2}_{U^{(k)}(m-1)}]$$

$$=\frac{k_2}{\alpha\beta k}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E[A^{\beta+l+1}(X_{U^{(k)}(n)},X_{U^{(k)}(m)})]$$

3.1. Examples

3.1.1. Weibull–uniform distribution (WU)

For the given CDF in (13), we obtain

$$A^{\beta+l+1}(x,y)=\frac{x^{k_1}y^{k_2-1}}{\lambda(y)}\left(\bar{G}(y)\right)^{\beta+l+1}$$
$$=x^{k_1}y^{k_2}\left(1-\frac{y}{\theta}\right)^{\beta+l+1}=\sum_{u=0}^{\beta+l+1}(-1)^u\binom{\beta+l+1}{u}\frac{1}{\theta^u}x^{k_1}y^{k_2+u}$$

Thus, in view of (23), it is easy to see that

$$E\big[X^{k_1}_{r,n,\tilde{m},k}X^{k_2}_{s,n,\tilde{m},k}\big]-E\big[X^{k_1}_{r,n,\tilde{m},k}X^{k_2}_{s-1,n,\tilde{m},k}\big]$$
$$=\frac{k_2}{\alpha\beta\theta^u\gamma_s}\sum_{l=0}^{\infty}\sum_{u=0}^{\beta+l+1}(-1)^u\frac{\Gamma(\beta+l)}{\Gamma(\beta)\Gamma(l+1)}\binom{\beta+l+1}{u}E\big[X^{k_1}_{r,n,\tilde{m},k}X^{k_2+u}_{s,n,\tilde{m},k}\big]$$

3.1.2. Weibull–Weibull distribution (WW)

For the CDF given in (15), we get

$$A^{\beta+l+1}(x,y) = \frac{x^{k_1} y^{k_2-1}}{\lambda(y)} \left(\bar{G}(y)\right)^{\beta+l+1} = x^{k_1} y^{k_2-1} \frac{\left(1-\mathrm{e}^{-\lambda y^{\theta}}\right)}{\lambda\theta y^{\theta-1}\mathrm{e}^{-\lambda y^{\theta}}} \left(\mathrm{e}^{-\lambda y^{\theta}}\right)^{\beta+l+1}$$

$$= \frac{x^{k_1} y^{k_2-\theta}}{\lambda\theta} \left[\mathrm{e}^{-\lambda y^{\theta}(\beta+l)} - \mathrm{e}^{-\lambda y^{\theta}(\beta+l+1)}\right]$$

$$= \frac{1}{\lambda\theta} \left\{ \sum_{a=1}^{\infty} (-1)^a \frac{[\lambda(\beta+l)]^a}{a!} x^{k_1} y^{k_2+\theta(a-1)} \right.$$

$$\left. - \sum_{b=1}^{\infty} (-1)^b \frac{[\lambda(\beta+l+1)]^b}{b!} x^{k_1} y^{k_2+\theta(b-1)} \right\}$$

Therefore, using (23), we obtain

$$E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s,n,\tilde{m},k}^{k_2}\right] - E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s-1,n,\tilde{m},k}^{k_2}\right]$$

$$= \frac{k_2}{\alpha\beta\theta\lambda\gamma_s} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} \left\{ \sum_{a=1}^{\infty} (-1)^a \frac{[\lambda(\beta+l)]^a}{a!} E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s,n,\tilde{m},k}^{k_2+\theta(a-1)}\right] \right.$$

$$\left. - \sum_{b=1}^{\infty} (-1)^b \frac{[\lambda(\beta+l+1)]^b}{b!} E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s,n,\tilde{m},k}^{k_2+\theta(b-1)}\right] \right\}$$

3.1.3. Weibull–Pareto distribution (WP)

From the CDF in (17), we have

$$A^{\beta+l+1}(x,y) = \frac{x^{k_1} y^{k_2-1}}{\lambda(y)} \left(\bar{G}(y)\right)^{\beta+l+1} = x^{k_1} y^{k_2-1} \frac{\left(1-\rho^{\theta} y^{-\theta}\right)}{\theta\rho^{\theta} y^{-(\theta+1)}} \left(\rho^{\theta} y^{-\theta}\right)^{\beta+l+1}$$

$$= \theta^{-1} \left[\rho^{\theta(\beta+l)} x^{k_1} y^{k_2-\theta(\beta+l)} - \rho^{\theta(\beta+l+1)} x^{k_1} y^{k_2-\theta(\beta+l+1)}\right]$$

Now, using (23), it is easy to see that

$$E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s,n,\tilde{m},k}^{k_2}\right] - E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s-1,n,\tilde{m},k}^{k_2}\right]$$

$$= \frac{k_2}{\alpha\beta\theta\gamma_s} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} \left\{ \rho^{\theta(\beta+l)} E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s,n,\tilde{m},k}^{k_2-\theta(\beta+l)}\right] \right.$$

$$\left. - \rho^{\theta(\beta+l+1)} E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s,n,\tilde{m},k}^{k_2-\theta(\beta+l+1)}\right] \right\}$$

3.1.4. Weibull–extreme value distribution (WE)

For the given CDF in (19), we obtain

$$A^{\beta+l+1}(x,y) = \frac{x^{k_1}y^{k_2-1}}{\lambda(y)}\left(\bar{G}(y)\right)^{\beta+l+1} = x^{k_1}y^{k_2-1}\frac{(1-\mathrm{e}^{-\mathrm{e}^y})}{\mathrm{e}^{(y-\mathrm{e}^y)}}\left(\mathrm{e}^{-\mathrm{e}^y}\right)^{\beta+l+1}$$

$$= x^{k_1}y^{k_2-1}\mathrm{e}^{-y}\left[\mathrm{e}^{-\mathrm{e}^y(\beta+l)} - \mathrm{e}^{-\mathrm{e}^y(\beta+l+1)}\right]$$

$$= x^{k_1}y^{k_2-1}\mathrm{e}^{-y}\left[\sum_{b=0}^{\infty}(-1)^b\frac{(\mathrm{e}^y(\beta+l))^b}{b!} - \sum_{c=0}^{\infty}(-1)^c\frac{(\mathrm{e}^y(\beta+l+1))^c}{c!}\right]$$

$$= \sum_{b=0}^{\infty}\sum_{m=0}^{\infty}(-1)^b\frac{(\beta+l)^b}{b!}\frac{(b-1)^m}{m!}x^{k_1}y^{k_2+m-1}$$

$$-\sum_{c=0}^{\infty}\sum_{h=0}^{\infty}(-1)^c\frac{(\beta+l+1)^c}{c!}\frac{(c-1)^h}{h!}x^{k_1}y^{k_2+h-1}$$

Thus, in view of (23), we have

$$E\left[X^{k_1}_{r,n,\tilde{m},k}X^{k_2}_{s,n,\tilde{m},k}\right] - E\left[X^{k_1}_{r,n,\tilde{m},k}X^{k_2}_{s-1,n,\tilde{m},k}\right]$$

$$= \frac{k_2}{\alpha\beta\gamma_s}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}\left\{\sum_{b=0}^{\infty}\sum_{m=0}^{\infty}(-1)^b\frac{(\beta+l)^b}{b!}\frac{(b-1)^m}{m!}E\left[X^{k_1}_{r,n,\tilde{m},k}X^{k_2+m-1}_{s,n,\tilde{m},k}\right]\right.$$

$$\left.-\sum_{c=0}^{\infty}\sum_{h=0}^{\infty}(-1)^c\frac{(\beta+l+1)^c}{c!}\frac{(c-1)^h}{h!}E\left[X^{k_1}_{r,n,\tilde{m},k}.X^{k_2+h-1}_{s,n,\tilde{m},k}\right]\right\}$$

4. Characterisation

In this section, the characterisation of the Weibull–G family of distributions, which is defined in (6), is discussed through recurrence relations between the moments of GOS.

Theorem 3. Fix a positive integer k, and let k_1 be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with PDF given in (7) is that

$$E\left[X^{k_1}_{r,n,\tilde{m},k}\right] - E\left[X^{k_1}_{r-1,n,\tilde{m},k}\right] = \frac{k_1}{\alpha\beta\gamma_r}\sum_{l=0}^{\infty}\frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}E\left[B^{\beta+l+1}(X_{r,n,\tilde{m},k})\right] \quad (25)$$

where $B^{\beta+l+1}(x) = \dfrac{x^{k_1-1}}{\lambda(x)}\left(\bar{G}(x)\right)^{\beta+l+1}$.

Proof. The necessary part follows from (11). On the other hand, assume the relation in (25) holds. Now, using (2) and (9) in (25), we get

$$k_1 C_{r-2} \int_{-\infty}^{\infty} x^{k_1-1} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i} dx = \frac{k_1 C_{r-1}}{\alpha\beta\gamma_r} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)}$$

$$\int_{-\infty}^{\infty} \frac{x^{k_1-1}}{\lambda(x)} \left(\bar{G}(x)\right)^{\beta+l+1} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i-1} f(x) dx$$

This implies

$$\frac{k_1 C_{r-1}}{\alpha\beta\gamma_r} \int_{-\infty}^{\infty} x^{k_1-1} \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i-1}$$
$$\times \left\{ \alpha\beta\bar{F}(x) - \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} \frac{\left(\bar{G}(x)\right)^{\beta+l+1}}{\lambda(x)} f(x) \right\} dx = 0 \tag{26}$$

Applying the extension of Müntz–Szász theorem (see, for example, [15]) to (26), we obtain

$$\frac{\bar{F}(x)}{f(x)} = \frac{1}{\alpha\beta\lambda(x)} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} \left(\bar{G}(x)\right)^{\beta+l+1}$$

Thus, $f(x)$ has a PDF as given in (7). Therefore, Theorem 3 holds. □

Theorem 4. Fix a positive integer k and let k_1, k_2 are non-negative integers. A necessary and sufficient condition for a random variable X to be distributed with PDF as stated in (7) is that

$$[E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s,n,\tilde{m},k}^{k_2}\right] - E\left[X_{r,n,\tilde{m},k}^{k_1} X_{s-1,n,\tilde{m},k}^{k_2}\right]$$
$$= \frac{k_2}{\alpha\beta\gamma_s} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} E\left[A^{\beta+l+1}\{X_{r,n,\tilde{m},k}, X_{s,n,\tilde{m},k}\}\right] \tag{27}$$

where $A^{\beta+l+1}(x,y) = x^{k_1} y^{k_2-1} \dfrac{\left(\bar{G}(y)\right)^{\beta+l+1}}{\lambda(y)}$.

Proof. The necessary part follows from (23). Now, suppose the relation in (27) is satisfied. Thus in view of (3) and (21), we have

$$k_2 C_{s-2} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^{k_1} y^{k_2-1} \left[\sum_{c=r+1}^{s} a_c^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_c} \right] \left[\sum_{h=1}^{r} a_h(r) [\bar{F}(x)]^{\gamma_h} \right] \frac{f(x)}{\bar{F}(x)} dy dx$$

$$= \frac{k_2 C_{s-1}}{\alpha\beta\gamma_s} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} \int_{-\infty}^{\infty}\int_{x}^{\infty} x^{k_1} y^{k_2-1} \frac{\left(\bar{G}(y)\right)^{\beta+l+1}}{\lambda(y)} \left[\sum_{c=r+1}^{s} a_c^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_c}\right]$$

$$\times \left[\sum_{h=1}^{r} a_h(r)[\bar{F}(x)]^{\gamma_h}\right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dydx$$

This implies

$$\frac{k_2 C_{s-1}}{\alpha\beta\gamma_s} \int_{-\infty}^{\infty}\int_{x}^{\infty} x^{k_1} y^{k_2-1} \left[\sum_{h=1}^{r} a_h(r)[\bar{F}(x)]^{\gamma_h}\right] \left[\sum_{c=r+1}^{s} a_c^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_c}\right] \frac{f(x)}{\bar{F}(x)}$$
$$\times \left\{\alpha\beta - \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} \frac{\left(\bar{G}(y)\right)^{\beta+l+1}}{\lambda(y)} \frac{f(y)}{\bar{F}(y)}\right\} dydx = 0 \tag{28}$$

Applying the extension of Müntz–Szász theorem (see, for example, [15]) to (28), we have

$$\frac{\bar{F}(y)}{f(y)} = \frac{1}{\alpha\beta\lambda(y)} \sum_{l=0}^{\infty} \frac{\Gamma(\beta+l)}{\Gamma\beta\Gamma(l+1)} \left(\bar{G}(y)\right)^{\beta+l+1}$$

Thus, $f(y)$ is a PDF as stated in (7) and Theorem 4 is satisfied. □

5. Conclusions

The Weibull–G family of distributions with two additional shape parameters is proposed by Bourguignon et al. [13]. It includes a broad family of continuous distributions and gives a better fit to generated distributions. The GOS is a unified approach for several ORVs, like OSs, RVs, SOS etc. The main purpose of this study is to find moments of GOS for several continuous distributions belonging to this class. Moreover, the characterisation of a probability distribution is essential, plays an important role in statistical studies, and has significant applications in natural and applied sciences. Thus, the characterisation of this general class of distribution is also studied using moment properties.

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