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An output sensitivity problem for a class of linear distributed systems with uncertain initial state

ABDELILAH LARRACHE, MUSTAPHA LHOUS, SOUKAINA BEN RHILA,
MOSTAFA RACHIK and ABDESSAMAD TRIDANE

In this paper, we consider an infinite dimensional linear systems. It is assumed that the initial state of system is not known throughout all the domain $\Omega \subset \mathbb{R}^n$, the initial state $x_0 \in L^2(\Omega)$ is supposed known on one part of the domain Ω and uncertain on the rest. That means $\Omega = \omega_1 \cup \omega_2 \cup \ldots \cup \omega_t$ with $\omega_i \cap \omega_j = \varnothing$, $\forall i, j \in \{1, \ldots, t\}$, $i \neq j$ where $\omega_i \neq \varnothing$ and $x_0(\theta) = \alpha_i$ for $\theta \in \omega_i$, $\forall i$, i.e., $x_0(\theta) = \sum_{i=1}^{t} \alpha_i \mathbb{1}_{\omega_i}(\theta)$ where the values $\alpha_1, \ldots, \alpha_r$ are supposed known and $\alpha_{r+1}, \ldots, \alpha_t$ unknown and $\mathbb{1}_{\omega_i}$ is the indicator function. The uncertain part $(\alpha_1, \ldots, \alpha_r)$ of the initial state x_0 is said to be $(\varepsilon_1, \ldots, \varepsilon_r)$-admissible if the sensitivity of corresponding output signal $(y_i)_{i \geqslant 0}$ relatively to uncertainties $(\alpha_k)_{1 \leqslant k \leqslant r}$ is less to the treshold ε_k, i.e., $\left\| \frac{\partial y_i}{\partial \alpha_k} \right\| \leqslant \varepsilon_k$, $\forall i \geqslant 0$, $\forall k \in \{1, \ldots, r\}$. The main goal of this paper is to determine the set of all possible gain operators that makes the system insensitive to all uncertainties. The characterization of this set is investigated and an algorithmic determination of each gain operators is presented. Some examples are given.

Key words: linear system, distributed system, uncertain initial state, gain operators, observability, stability, linear programming

A. Larrache, S. Ben Rhila and M. Rachik are with Laboratory of Analysis Modelling and Simulation, Department of Mathematics and Computer Science, Faculty of Sciences Ben M'sik, Hassan II University of Casablanca, B.P 7955 Sidi Othman Casablanca, Morocco.

M. Lhous (Corresponding author), E-mail: mlhous17@gmail.com, is with Laboratory of Modeling, Analysis, Control and Statistics, Department of Mathematics and Computer Science, Faculty of Sciences Ain Chock, Hassan II University of Casablanca, B.P 5366 Maarif Casablanca, Morocco.

A. Tridane is with Department of Mathematical Sciences, United Arab Emirates University, P.O. Box 15551, Al Ain, UAE.

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1. Introduction

During the measurement of a system state we are always confronted with the presence of certain unknown parameters and then we are not able to have a full access to the state variables. This uncertain parameters that come from the natural relationship which exists between a system and its environment, data errors and additives unknown internal and external noise. To better avoid damages being able to be caused by such uncertainties on the evolution of a system, many research has focused their work on the determination and characterization the set of this uncertainties, see [5, 6] and [16].

Output admissible sets have many important applications in the areas of stability analysis and design of closed-loop systems with state and control constraints. Although, the theory of output admissible sets has been appeared in a variety of contexts see [8, 9, 15]. The case of the disturbances which infect the initial state for linear system has considered in [2, 7] and [10]. The output admissible set in this case has determined based on the mathematical programming. However, in most of the studies available, the problem for infinite dimensional systems is not considered and hence their applicability is severely limited.

The aim of this work is to present a contribution to the study of the output admissible set for a class of infinite dimensional discrete systems. A control law is introduced in order to reduce the effects of these intolerable uncertainties and/or makes the system insensitive to the effects of all unknown parameters that infect the initial state.

Without loss of generality, we consider the linear system described by

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t \geqslant 0, \\ x(0) = x_0 = \alpha \mathbb{1}_{\omega_1} + \beta \mathbb{1}_{\omega_2}, \end{cases} \tag{1}$$

where $x(t) \in X = L^2(\Omega)$ is the state variable, Ω is an open bounded of $\mathbb{R}^n$, $\Omega = \omega_1 \cup \omega_2$ and $\omega_1 \cap \omega_2 = \emptyset$. A generates a continuous strongly semigroup $(S(t))_{t\geqslant 0}$ on the space X. The initial state x_0 is supposed to be known on ω_1 but not on ω_2.

The associated output of the system is discrete and is governed by

$$y(t_i) = Cx(t_i) + Dv_i, \quad \forall i \geqslant 0, \tag{2}$$

where $(t_i)_{i\geqslant 0}$ is a constant step subdivision of $[0, +\infty[$, i.e., $[0, +\infty[= \cup_{i=0}^{\infty}[t_i, t_{i+1}[$, $t_0 = 0$, $t_i = i\delta$ and δ is the step of sampling. The control is assumed closed loop, i.e., $v_i = Kx_i$ where $x_i = x(t_i)$ and $K \in \mathcal{L}(X, \mathbb{R}^p)$ with $C \in \mathcal{L}(X, \mathbb{R}^k)$ and $D \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^k)$.

We will propose a technique to determine among these controls law which makes the system insensitive the effects of these unknown parameters β.

A uncertainty is said to be admissible if

$$\left\| \frac{\partial y_i}{\partial \beta} \right\| \leqslant \varepsilon, \quad \text{for } \forall i \geqslant 0. \tag{3}$$

The constraints may be summarized by a single set inclusion

$$\mathcal{Y}_\varepsilon = \left\{ y_i \in \mathbb{R}^k / \left\| \frac{\partial y_i}{\partial \beta} \right\| \leqslant \varepsilon, \quad \text{for } \forall i \geqslant 0 \right\}. \tag{4}$$

If these constraints are violated for any $i \geqslant 0$, serious damage may happen. We say that makes the system insensitive the effects of these uncertainties, if the output of the system never exceed the specified constraints (3). With (1) and (3), it is desired to determine the set $\mathcal{K}$ of all gain operators K that have an admissible uncertainties, to be explicit:

$$\mathcal{K} = \left\{ K \in \mathcal{L}(X, \mathbb{R}^p) / \left\| \frac{\partial y_i}{\partial \beta} \right\| \leqslant \varepsilon, \quad \text{for } \forall i \geqslant 0 \right\}.$$

In this paper, we are interested in studied the output sensitivity for a class of infinite dimensional linear systems with uncertain initial state. We will show that, under some hypothesis, the output system will be insensitive to the effects of unknown parameters in initial state of system under a corresponding control law. We are interested with the investigation of the set of all gain operators those makes the system insensitive of the effects of uncertain initial state. Under some assumptions, we determine that this set cab be described by finite number of inequalities and an algorithmic procedure is established for computing this set.

This paper is organised as follows: In section 2 the characterization of the gain operator set is presented. A algorithmic determination for the characterization of the sets $\mathcal{S}_\varepsilon(K)$ for each gain operators will presented in section 3. In section 4 we give some assumption to determine the set $\mathcal{S}_\varepsilon(K)$ by a finite number of inequalities. In section 5 we give another approach to characterize the set of output sensitivity and the concluding remarks are given insection 6.

2. Characterization of the gain operators Set

The linear systems considered in this paper have the following form

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t \geqslant 0, \\ x(0) = x_0 = \alpha \mathbb{1}_{\omega_1} + \beta \mathbb{1}_{\omega_2} \end{cases} \tag{5}$$

the corresponding output is

$$y_i = Cx_{t_i} + Dv_i,$$

where $(t_i)_{i\geqslant 0}$ is a constant step subdivision of $[0,+\infty[$, i.e., $[0,+\infty[=\cup_{i=0}^{\infty}[t_i,t_{i+1}[$, $t_0=0$, $t_i=i\delta$ and δ is the step of sampling. $x(t)\in X=L^2(\Omega)$ is the state variable, Ω is an open bounded of $\mathbb{R}^n$, $\Omega=\omega_1\cup\omega_2$ and $\omega_1\cap\omega_2=\emptyset$. A generates a continuous strongly semigroup $(S(t))_{t\geqslant 0}$ on the space X. The initial state x_0 is supposed to be known on ω_1 defined by α but not on ω_2 where β is the uncertain parameter of the initial state. The control feedback is $v_i=Kx_i$ where $x_i=x(t_i)$ and $K\in\mathcal{L}(X,\mathbb{R}^p)$ with $C\in\mathcal{L}(X,\mathbb{R}^k)$ and $D\in\mathcal{L}(\mathbb{R}^p,\mathbb{R}^k)$.

Definition 1 *Output function is insensitive to the effects of the uncertainties, if the corresponding output satisfies the following condition*

$$\left\|\frac{\partial y_i}{\partial\beta}\right\|\leqslant\varepsilon,\quad \textit{for}\ \ \forall i\geqslant 0. \tag{6}$$

The control law

$$v_i=Kx_i$$

is introduced in order to reduce the effects of these intolerable uncertainties and/or makes the system insensitive to the effects of all unknown parameters that infect the initial state.

As $x(t)=S(t)x_0$ then

$$y_i=\alpha(C+DK)S(t_i)\mathbb{1}_{\omega_1}+\beta(C+DK)S(t_i)\mathbb{1}_{\omega_2}$$

then

$$\frac{\partial y_i}{\partial\beta}=(C+DK)S(t_i)\mathbb{1}_{\omega_2}=(C+DK)[S(\delta)]^i\mathbb{1}_{\omega_2}.$$

Our problem is to determine under some assumptions, the gain K such that

$$\left\|(C+DK)[S(\delta)]^i\mathbb{1}_{\omega_2}\right\|\leqslant\varepsilon,\quad \text{for}\ \ \forall i\geqslant 0.$$

Some of the control objectives are to stabilize the system and to maintain its output trajectory within the domain of constraints. The constraints may be summarized by a single set inclusion

$$y_i\in\mathcal{Y}_\varepsilon=\left\{y_i\in\mathbb{R}^k/\left\|\frac{\partial y_i}{\partial\beta}\right\|\leqslant\varepsilon,\quad \text{for}\ \ \forall i\geqslant 0\right\}.$$

Our goal is to characterize the set $\mathcal{K}$ of control law such that the output of system never exceed the specified constraints (3)

$$\mathcal{K}=\left\{K\in\mathcal{L}(X,\mathbb{R}^p)\,/\left\|\frac{\partial y_i}{\partial\beta}\right\|\leqslant\varepsilon,\quad \text{for}\ \ \forall i\geqslant 0\right\}.$$

Let $\varepsilon > 0$ and $K \in \mathcal{L}(X, \mathbb{R}^p)$, we note

$$\mathcal{S}_\varepsilon(K) = \left\{x \in X \;/\; \left\|(C + DK)[S(\delta)]^i x\right\| \leqslant \varepsilon, \;\; \forall i \geqslant 0\right\}$$

then

$$\begin{aligned} \mathcal{K} &= \left\{K \in \mathcal{L}(X, \mathbb{R}^p) \;/\; \left\|(C + DK)[S(\delta)]^i \mathbb{1}_{\omega_2}\right\| \leqslant \varepsilon, \;\; \forall i \geqslant 0\right\} \\ &= \left\{K \in \mathcal{L}(X, \mathbb{R}^p) \;/\; \mathbb{1}_{\omega_2} \in \mathcal{S}_\varepsilon(K)\right\}. \end{aligned}$$

We note that the set of all gain operator $\mathcal{S}_\varepsilon(K)$ is defined by an infinite number of inequalities. We will establish sufficient conditions which allow us to describe it by a finite number of inequalities. In order to characterize the set $\mathcal{S}_\varepsilon(K)$, we introduce the following notations

$$\mathcal{S}_\varepsilon(K) = \left\{x \in X \;/\; \left\|\widetilde{C}\widetilde{A}^i x\right\| \leqslant \varepsilon, \;\; \forall i \geqslant 0\right\}$$

where $\widetilde{C} = C + DK$ and $\widetilde{A} = S(\delta)$.

Let consider the Banach space $Y = \left\{(x_i)_{i \geqslant 0}, \; x_i \in \mathbb{R}^k \;/\; \sup_{i \geqslant 0} \|x_i\| < \infty\right\}$ and we introduced the operator defined by

$$\begin{aligned} H : \; X &\longrightarrow Y \\ x &\longrightarrow (\widetilde{C}\widetilde{A}^i x)_{i \geqslant 0}. \end{aligned} \tag{7}$$

Remark 1 *The operator H is the observability operator of discrete-time linear system*

$$(S)\begin{cases} x_{i+1} = \widetilde{A} x_i, \; i \geqslant 0 \\ x_0 \in X \end{cases} \quad \textit{and} \quad (O)\left\{y_i = \widetilde{C} x_i, \; i \geqslant 0\right.$$

with $x_i \in X$ is the state of system (S), $X = L^2(\Omega)$, $y_i \in \mathbb{R}^k$ is the corresponding output. The system (S)-(O) is observable if the operator H is injective.

Proposition 1 *If $(\widetilde{A}, \widetilde{C})$ is observable and $\|\widetilde{A}\| < 1$ then $\mathcal{S}_\varepsilon(K)$ is bounded, i.e., there exist $\gamma > 0$ such that $\mathcal{S}_\varepsilon(K) \subset B(0, \gamma)$.*

Proof. If $\|\widetilde{A}\| < 1$ then the operaor H is bounded and if H is injective then $H^{-1} : ImH \longrightarrow X$ is bounded because the graph of H^{-1} is closed. Let consider $x \in \mathcal{S}_\varepsilon(K)$ and $z = (z_i)_i = Hx = (\widetilde{C}\widetilde{A}^i x)_i \in Y$ then $z_i \in B(0, \varepsilon), \;\; \forall i \geqslant 0$. Then with the norm of the Banach space Y, $z \in B(0, \varepsilon)$ and $x \in H^{-1}B(0, \varepsilon)$ which implies $\mathcal{S}_\varepsilon(K) \subset H^{-1}B(0, \varepsilon)$ then there exist $\gamma > 0$ such that $\mathcal{S}_\varepsilon(K) \subset B(0, \gamma)$. □

In order to characterize $\mathcal{S}_\varepsilon(K)$ we introduce for each integer i the set $\mathcal{S}^\gamma_{\varepsilon,i}(K)$ defined by

$$\mathcal{S}^\gamma_{\varepsilon,i}(K) = \left\{x \in X \cap B(0, \gamma) \;/\; \left\|\widetilde{C}\widetilde{A}^j x\right\| \leqslant \varepsilon, \;\; \forall j = 0, 1, \ldots, i\right\},$$

where γ is a real positive such that $\mathcal{S}_\varepsilon(K) \subset B(0, \gamma)$.

Proposition 2 *$\mathcal{S}^{\gamma}_{\varepsilon,i}(K)$ is a closed, convex and symmetric set.*

Proof. The results are easily checked from the definition of $\mathcal{S}^{\gamma}_{\varepsilon,i}(K)$. □

Remark 2 *For every integer $i \geqslant 0$ we have*

$$\mathcal{S}_{\varepsilon}(K) \subset \mathcal{S}^{\gamma}_{\varepsilon,i+1}(K) \subset \mathcal{S}^{\gamma}_{\varepsilon,i}(K).$$

Definition 2 *$\mathcal{S}_{\varepsilon}(K)$ is finitely determined if there exists an integer i such that $\mathcal{S}_{\varepsilon}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i}(K)$.*

The finite determination of $\mathcal{S}_{\varepsilon}(K)$ is characterized by the following theorem.

Theorem 1 *$\mathcal{S}_{\varepsilon}(K)$ is finitely determined if and only if there exists an integer i such that $\mathcal{S}^{\gamma}_{\varepsilon,i}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i+1}(K)$.*

Proof. If we suppose that exist an integer $i \geqslant 0$ such that

$$\mathcal{S}^{\gamma}_{\varepsilon,i}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i+1}(K)$$

then

$$x \in \mathcal{S}^{\gamma}_{\varepsilon,i}(K) \Rightarrow x \in \mathcal{S}^{\gamma}_{\varepsilon,i+1}(K) \Rightarrow \widetilde{A}x \in \mathcal{S}^{\gamma}_{\varepsilon,i}(K)$$

and by iteration we have

$$x \in \mathcal{S}^{\gamma}_{\varepsilon,i}(K) \Rightarrow \widetilde{A}^{j}x \in \mathcal{S}^{\gamma}_{\varepsilon,i}(K), \forall j \geqslant 0,$$

then

$$x \in \mathcal{S}_{\varepsilon}(K)$$

that implies

$$\mathcal{S}^{\gamma}_{\varepsilon,i}(K) \subset \mathcal{S}_{\varepsilon}(K).$$

And we know that $\mathcal{S}_{\varepsilon}(K) \subset \mathcal{S}^{\gamma}_{\varepsilon,i}(K)$ for every $i \geqslant 0$, hence $\mathcal{S}_{\varepsilon}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i}(K)$.

Conversely, if $\mathcal{S}_{\varepsilon}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i}(K)$ for some $i \geqslant 0$, then obviously $\mathcal{S}^{\gamma}_{\varepsilon,i}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i+1}(K)$. Which complete the proof. □

Remark 3 *Suppose that $\mathcal{S}_{\varepsilon}(K)$ is finitely determined and let i^{*} be the smallest i such that $\mathcal{S}^{\gamma}_{\varepsilon,i}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i+1}(K)$, then $\mathcal{S}_{\varepsilon}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i^{*}}(K)$ for all $i \geqslant i^{*}$.*

3. Algorithmic determination

As a natural consequence of the previous proposition, we shall give the following conceptual algorithm for determining the index i^* such that $\mathcal{S}_\varepsilon(K) = \mathcal{S}^\gamma_{\varepsilon,i^*}(K)$ and consequently the characterization of the set $\mathcal{S}_\varepsilon(K)$.

Algorithm I

step 1: Set $i = 0$
step 2: If $\mathcal{S}^\gamma_{\varepsilon,i+1}(K) = \mathcal{S}^\gamma_{\varepsilon,i}(K)$ then set $i^* = i$ and stop,
else continue.
step 3: Replace i by $i+1$ and return to step 2.

Clearly, the algorithm I will produce i^* and $\mathcal{S}_\varepsilon(K)$ if and only if $\mathcal{S}_\varepsilon(K)$ is finitely determined. There appears to be not finite algorithmic procedure for showing that $\mathcal{S}_\varepsilon(K)$ is not finitely determined.

Algorithm I is not practical because it does not describe how the test $\mathcal{S}^\gamma_{\varepsilon,i}(K) = \mathcal{S}^\gamma_{\varepsilon,i+1}(K)$ is implemented. In order to overcome this difficulty, let $\mathbb{R}^k$ be endowed with the following norm

$$\|x\| = \max_{1\leqslant i\leqslant n} |x_i|, \quad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^k.$$

Let with $h_l : \mathbb{R}^k \longrightarrow \mathbb{R}$ is described for all $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ by

$$\begin{cases} h_{2r-1}(x) = x_r - \varepsilon, & \text{for } r \in \{1, 2, \ldots, k\}, \\ h_{2r}(x) = -x_r - \varepsilon, & \text{for } r \in \{1, 2, \ldots, k\}. \end{cases}$$

In this case, for every integer i, $\mathcal{S}^\gamma_{\varepsilon,i}(K)$is given by

$$\mathcal{S}^\gamma_{\varepsilon,i}(K) = \{x \in X \cap B(0,\gamma);\ \ h_j(\widetilde{C}\widetilde{A}^s x) \leqslant 0, \quad j = 1, \ldots, 2k;\ \ s = 0, \ldots, i\},$$

on the other hand

$$\begin{aligned} \mathcal{S}^\gamma_{\varepsilon,i+1}(K) &= \left\{x \in \mathcal{S}^\gamma_{\varepsilon,i}(K);\ \ \|\widetilde{C}\widetilde{A}^{i+1}(x)\| \leqslant \varepsilon\right\} \\ &= \left\{x \in \mathcal{S}^\gamma_{\varepsilon,i}(K);\ \ h_j(\widetilde{C}\widetilde{A}^{i+1}(x)) \leqslant 0, \ \text{for } j = 1, \ldots, 2k\right\}. \end{aligned}$$

Now, since $\mathcal{S}^\gamma_{\varepsilon,i+1}(K) \subset \mathcal{S}^\gamma_{\varepsilon,i}(K)$ for every integer i, then

$$\begin{aligned} \mathcal{S}^\gamma_{\varepsilon,i+1}(K) = \mathcal{S}^\gamma_{\varepsilon,i}(K) &\Longleftrightarrow \mathcal{S}^\gamma_{\varepsilon,i}(K) \subset \mathcal{S}^\gamma_{\varepsilon,i+1}(K) \\ &\Longleftrightarrow x \in \mathcal{S}^\gamma_{\varepsilon,i}(K); \quad h_j(\widetilde{C}\widetilde{A}^{k+1}(x)) \leqslant 0, \quad \text{for all } j = 1, \ldots, s \end{aligned}$$

$$\iff \sup_{x\in\mathcal{S}^{\gamma}_{\varepsilon,i}(K)} h_j(\widetilde{C}\widetilde{A}^{i+1}(x)) \leqslant 0, \quad \text{for all } \ j = 1,\ldots,s$$

$$\iff \sup_{\substack{\underbrace{h_j(\widetilde{C}\widetilde{A}^{l}(x_0))\leqslant 0}\\ j\in\{1,\ldots,2k\},\ l\in\{0,\ldots,i\}}} h_j(\widetilde{C}\widetilde{A}^{i+1}x) \leqslant 0, \quad \text{for all } \ j \in \{1,\ldots,2k\}.$$

Consequently the test $\mathcal{S}^{\gamma}_{\varepsilon,i}(K) = \mathcal{S}^{\gamma}_{\varepsilon,i+1}(K)$ leads to a set of mathematical programming problems, and algorithm I can be implemented as follows.

Algorithm II

step 1: Let $i = 0$;
step 2: For $j = 1,\ldots,s$, do :
Maximize $J_j(x) = h_j(\widetilde{C}\widetilde{A}^{i+1}(x))$
$$\begin{cases} h_r(\widetilde{C}\widetilde{A}^{l}x) \leqslant 0 \\ j = 1,2,\ldots,2k, l = 0,\ldots,i. \end{cases}$$
Let J_j^* be the maximum value of $J_j(x)$.
If $J_j^* \leqslant 0$, for $j = 1,\ldots,s$ then set $i^* := i$ and stop.
Else continue.
step 3: Replace i by $i + 1$ and return to step 2.

Remark 4 *The optimization problem cited in step 2 is a mathematical programming problem and can be solved by standard methods.*

4. Sufficient conditions for finite determination of $\mathcal{S}_{\varepsilon}(K)$

In order to show that the finite determination property is not so restrictive, we give the following result.

Theorem 2 *If $\widetilde{A}$ is asymptotically stable ($\lambda < 1$ for every λ eigenvalue of $\widetilde{A}$), then there exists an integer i_0 such that the output function y_i is not sensitive to uncertainties β for every $i > i_0$.*

Proof. Let $\varepsilon > 0$, the asymptotic stability of $\widetilde{A}$ implies that there exists a certain i_0 such that

$$\left\|\widetilde{C}\widetilde{A}^{i}\right\| < \frac{\varepsilon}{M}, \forall\, i \geqslant i_0$$

where $M > 0$ is the bounded of $\mathbb{1}_{\omega_2}$, i.e., $\|\mathbb{1}_{\omega_2}\|_X < M$ (Ω is bounded in $\mathbb{R}^n$) then

$$\left\|\widetilde{C}\widetilde{A}^{i}\mathbb{1}_{\omega_2}\right\| \leqslant \varepsilon, \quad \forall i \geqslant i_0\,.$$

Then

$$\left\| \frac{\partial y_i}{\partial \beta} \right\| \leqslant \varepsilon, \quad \text{for } \forall i \geqslant i_0,$$

and the output function y_i is not sensitive to the uncertainties β for every $i > i_0$.□

Theorem 3 *Suppose the following assumptions to hold:*

1. *the pair* $\left(\widetilde{A}, \widetilde{C}\right)$ *is observable, i.e., the operator* H *defined by (7) is injective.*

2. $\widetilde{A}$ *is asymptotically stable.*

Then the output sensitivity set $\mathcal{S}_\varepsilon(K)$ *is finitely determined.*

Proof. By the observability of $\left(\widetilde{A}, \widetilde{C}\right)$ and by Proposition 2 there exists a $\gamma > 0$ such that $\mathcal{S}_\varepsilon(K) \subset B(0, \gamma)$. $\widetilde{A} = A + BK$ is asymptotically stable, then there exists an integer i_0 such that

$$\|\widetilde{C}\widetilde{A}^i\| \leqslant \frac{\varepsilon}{\gamma}, \quad \forall i \geqslant i_0 .$$

Let $x \in \mathcal{S}^{\gamma}_{\varepsilon,i_0}(K)$ then $\|x\| \leqslant \gamma$ and $\|\widetilde{C}\widetilde{A}^i x\| \leqslant \varepsilon, \quad \forall i \leqslant i_0$ and by the asymptotically stable of $\widetilde{A} = A + BK$ we have

$$\|\widetilde{C}\widetilde{A}^i\| \leqslant \varepsilon, \quad \forall i \geqslant 0$$

then $\mathcal{S}^{\gamma}_{\varepsilon,i_0}(K) \subset \mathcal{S}_\varepsilon(K)$ and $\mathcal{S}_\varepsilon(K)$ is finitely determined. □

Example 1. Let consider $X = L^2(0, 1)$ and the evolution equation defined by

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geqslant 0, \\ x(0) = x_0 = \alpha \mathbb{1}_{\omega_1} + \beta \mathbb{1}_{\omega_2}, \end{cases} \tag{8}$$

where $x(t) \in L^2(0, 1)$, A is the Laplacian operator with

$$D(A) = \{y \in X \mid \Delta z \in X, \ \text{ and } \ z(0) = z(1) = 0\} .$$

The operator A generate a strongly continuous semigroup $S(t)_{t \geqslant 0}$ defined by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \langle z, \phi_n \rangle \phi_n ,$$

where $\langle ., . \rangle$ is the usual inner product on $L^2(0, 1)$ and $\phi_n(s) = \sqrt{2}\sin(n\pi s)$ is a basis of $L^2(0, 1)$.

The system (8) is augmented with the output function

$$y_i = Cx_i + Dv_i, \quad i \geqslant 0$$

where

$$\begin{cases} C: \ L^2(0,1) \longrightarrow \mathbb{R} \\ \qquad\quad x \quad \longrightarrow \langle x, g\rangle, \ \text{ where } \ g(s) = s^2, \ \ \forall s \in]0,1[\\ D: \mathbb{R} \longrightarrow \mathbb{R}, \quad Dv = v. \end{cases}$$

The observability operator is

$$(Hx)_i = \widetilde{C}\widetilde{A}^i x = \sum_{n=1}^{\infty} e^{-n^2\pi^2\delta} \langle x, \phi_n\rangle(\langle \phi_n, g\rangle + K\phi_n), \ \ \forall i \geqslant 0$$

then if $K\phi_n \neq -\langle \phi_n, g\rangle = -\dfrac{\sqrt{2}}{n\pi}(-1)^n + \dfrac{2\sqrt{2}}{(n\pi)^3}[1-(-1)^n], \ \ \forall n \geqslant 1$, we deduce that the operator H is injective. Also we have $\|S(\delta)z\|^2 = \sum\limits_{n=1}^{\infty} e^{-2n^2\pi^2\delta}\langle z, \phi_n\rangle^2$, then we verify that $\|\widetilde{A}\| = \|S(\delta)\| < 1$.

The set $\mathcal{K}$ of control law is given by

$$\begin{aligned} \mathcal{K} &= \left\{ K \in \mathcal{L}(X, \mathbb{R}^p) \ / \ \left\| \widetilde{A} S(\delta)^i \mathbb{1}_{\omega_2} \right\| \leqslant \varepsilon, \ \ \forall i \geqslant 0 \right\} \\ &= \left\{ K \in \mathcal{L}(X, \mathbb{R}^p) \ / \ \left| \sum_{m=0}^{\infty} \sqrt{2} e^{-i(2m+1)^2\pi^2\delta} \left(\frac{-1 + (-1^m)}{(2m+1)\pi} \right) \left(\frac{\sqrt{2}}{(2m+1)\pi} + \right.\right.\right. \\ &\qquad \left. + \frac{4\sqrt{2}}{(2m+1)^3\pi^3} + K\phi_{2m+1} \right) + \\ &\qquad \left.\left. + \sum_{m=1}^{\infty} \sqrt{2} e^{-i4m^2\pi^2\delta} \left(\frac{1 + (-1^m)}{(2m)\pi} \right) \left(-\frac{\sqrt{2}}{(2m)\pi} + K\phi_{2m} \right) \right| \leqslant \varepsilon, \ \ \forall i \geqslant 0 \right\}. \end{aligned}$$

For

$$\begin{aligned} K: \ X &\longrightarrow \mathbb{R} \\ x &\longrightarrow Kx = -\int_0^1 x(s)\,\mathrm{d}s = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\pi}((-1)^n - 1)\langle x, \phi_n\rangle \end{aligned}$$

the system is observable and the set $\mathcal{S}_\varepsilon(K)$ is given by

$$\mathcal{S}_\varepsilon(K) = \left\{ x \in X \ / \left| \sum_{n=1}^{\infty} \left(\frac{4}{(n\pi)^3}[1-(-1)^n] - \frac{2}{n\pi} \right) \int_0^1 x(s)\sin(n\pi s)\,\mathrm{d}s \right| \leqslant \varepsilon, \ \forall i \geqslant 0 \right\}.$$

5. The output sensitivity problem for parabolic systems

Let consider the system (1) with the corresponding output (2) and we will to determine the set of all gain operator K, such that $v_i = Kx_i$ where $\|K\| \leqslant k$ with k is a fixed positive real, those makes the system insensitive of the effects of disturbances, i.e., those verify (3). The operator A generate a continuous strongly semigroup $(S(t)_{t\geqslant 0})$ on the space $X = L^2(\Omega)$.

$$S(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle x, \phi_n\rangle\phi_n\,,$$

where $(\phi)_n$ is a basis of $L^2(\Omega)$ and λ_n are the eigenvalues of A which verify

$$\lambda_n < 0, \quad \lim_{n\to+\infty}(\lambda_{n+1}-\lambda_n) = -\infty, \quad \lambda_n \text{ and } (\lambda_{n+1}-\lambda_n) \text{ are decreasing}$$

we have

$$\begin{aligned}
\left\|\frac{\partial y_i}{\partial\beta}\right\| \leqslant \varepsilon,\ \forall i\geqslant 0 \Longleftrightarrow & \left\|(C+DK)[S(\delta)^i]\mathbb{1}_{\omega_2}\right\| \leqslant \varepsilon,\ \ \forall i\geqslant 0\\
\Longleftrightarrow & \left\|\sum_{n=1}^{\infty}\left\langle \mathbb{1}_{\omega_2},\phi_n\right\rangle(C+DK)[S(\delta)^i]\phi_n\right\| \leqslant \varepsilon,\ \forall i\geqslant 0\\
\Longleftrightarrow & \left\|\sum_{n=1}^{N}\left\langle \mathbb{1}_{\omega_2},\phi_n\right\rangle(C+DK)[S(\delta)^i]\phi_n + \right.\\
& \left.\sum_{n=N+1}^{\infty}\left\langle \mathbb{1}_{\omega_2},\phi_n\right\rangle(C+DK)[S(\delta)^i]\phi_n\right\| \leqslant \varepsilon,\\
& \forall i\geqslant 0,\ \ \forall N\geqslant 1.
\end{aligned} \tag{9}$$

Let consider the integer

$$N = E\left(\frac{1}{2\delta(\lambda_2-\lambda_1)}\ln\left(\frac{\varepsilon}{2}\frac{(1-e^{2(\lambda_2-\lambda_1)\delta})}{e^{2\lambda_1\delta}}\frac{1}{(\|C\|+k\|D\|)\|\mathbb{1}_{\omega_2}\|}\right)\right)+1$$

where E is the whole party, then

$$N \geqslant \frac{1}{2\delta(\lambda_2-\lambda_1)}\ln\left(\frac{\varepsilon}{2}\frac{(1-e^{2(\lambda_2-\lambda_1)\delta})}{e^{2\lambda_1\delta}}\frac{1}{(\|C\|+k\|D\|)\|\mathbb{1}_{\omega_2}\|}\right)$$

witch implies that

$$(\|C\|+k\|D\|)\|\mathbb{1}_{\omega_2}\|\frac{e^{2\lambda_1\delta}}{(1-e^{2(\lambda_2-\lambda_1)\delta})}\left(e^{2\delta(\lambda_2-\lambda_1)}\right)^N \leqslant \frac{\varepsilon}{2} \tag{10}$$

and

$$\begin{aligned}\left\|\sum_{n=N+1}^{\infty}\langle \mathbb{1}_{\omega_2},\phi_n\rangle(C+DK)[S(\delta)^i]\phi_n\right\| &\leqslant \left\|\sum_{n=N+1}^{\infty}\langle \mathbb{1}_{\omega_2},\phi_n\rangle(C+DK)e^{i\lambda_n\delta}\phi_n\right\|\\ &\leqslant (\|C\|+k\|D\|)\sum_{n=N+1}^{\infty}|\langle \mathbb{1}_{\omega_2},\phi_n\rangle|\,e^{i\lambda_n\delta}\\ &\leqslant (\|C\|+k\|D\|)\|\mathbb{1}_{\omega_2}\|\left(\sum_{n=N+1}^{\infty}e^{2i\lambda_n\delta}\right)\\ &\leqslant (\|C\|+k\|D\|)\|\mathbb{1}_{\omega_2}\|\sum_{n=N+1}^{\infty}e^{2\lambda_n\delta}\end{aligned}$$

If we put $u_n = e^{2\lambda_n\delta}$ then $\dfrac{u_{n+1}}{u_n} = e^{2(\lambda_{n+1}-\lambda_n)\delta} \leqslant e^{2(\lambda_2-\lambda_1)\delta} < 1$.

And with $\lambda = e^{2(\lambda_2-\lambda_1)\delta}$ we have $|u_n| \leqslant \lambda|u_{n-1}|$ and $|u_n| \leqslant \lambda^{n-1}|u_1|$, then $\sum\limits_{n=N+1}^{\infty} \leqslant \dfrac{\lambda^N|u_1|}{1-\lambda}$.

Thus

$$(\|C\|+k\|D\|)\,\|\mathbb{1}_{\omega_2}\|\sum_{n=N+1}^{\infty}e^{2\lambda_n\delta} \leqslant (\|C\|+k\|D\|)\,\|\mathbb{1}_{\omega_2}\|\frac{e^{2(\lambda_2-\lambda_1)\delta N}e^{2\lambda_1\delta}}{1-e^{2(\lambda_2-\lambda_1)\delta}}$$

and by (10) we deduce that

$$\left\|\sum_{n=N+1}^{\infty}\langle \mathbb{1}_{\omega_2},\phi_n\rangle(C+DK)[S(\delta)^i]\phi_n\right\| \leqslant \frac{\varepsilon}{2}.$$

Then to have (6) it enough that

$$\begin{aligned}\mathbb{1}_{\omega_2}\in\mathcal{T}^N(\mathcal{K},\varepsilon) &= \left\{x\in L^2(\Omega)\;|\left\|\sum_{n=N+1}^{\infty}\langle \mathbb{1}_{\omega_2},\phi_n\rangle(C+DK)[S(\delta)^i]\phi_n\right\|\leqslant\frac{\varepsilon}{2},\ \forall i\geqslant 0\right\}\\ &= \left\{x\in L^2(\Omega)\;|\left\|\bar{C}\bar{A}\begin{pmatrix}\langle x,\phi_1\rangle\\ \vdots\\ \langle x,\phi_N\rangle\end{pmatrix}\right\|\leqslant\frac{\varepsilon}{2},\quad \forall i\geqslant 0\right\}\end{aligned}$$

where the matrices $\bar{C}$ and $\bar{A}$ are given by

$$\begin{aligned}\bar{C}:\quad &\mathbb{R}^N \longrightarrow \mathbb{R}^p\\ &\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}\longrightarrow (C+DK)\sum_{i=1}^{N}x_i\phi_i\end{aligned}$$

then

$$\bar{C}_{ij} = \langle (C + DK)\phi_j, e_i\rangle, \;\; i \in \{1, \dots, p\} \;\text{ and }\; j \in \{1, \dots, N\}$$

and

$$\bar{A} = \begin{pmatrix} e^{\lambda_1 \delta} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & e^{\lambda_N \delta} \end{pmatrix}.$$

If we pose $\xi = \begin{pmatrix} \langle \mathbb{1}_{\omega_2}, \phi_1 \rangle \\ \vdots \\ \langle \mathbb{1}_{\omega_2}, \phi_N \rangle \end{pmatrix}$ and $\mathcal{S}^N(K,\varepsilon) = \left\{ x \in \mathbb{R}^N \mid \|\bar{C}\bar{A}^i x\| \leqslant \frac{\varepsilon}{2}, \;\; \forall i \geqslant 0 \right\}$,

then

$$\mathbb{1}_{\omega_2} \in \mathcal{T}^N(\mathcal{K}, \varepsilon) \iff \xi \in \mathcal{S}^N(K, \varepsilon).$$

We note that the set $\mathcal{S}^N(K,\varepsilon)$ is defined by an infinite number of inequalities. We will establish sufficient conditions which allow us to describe it by a finite number of inequalities. In order to characterize the set $\mathcal{S}^N(K,\varepsilon)$, we introduce for each integer k the set $\mathcal{S}^N_k(K,\varepsilon)$ defined by

$$\mathcal{S}^N_k(K,\varepsilon) = \left\{ x \in \mathbb{R}^N \mid \|\bar{C}\bar{A}^i x\| \leqslant \frac{\varepsilon}{2}, \;\; \forall i \in \{0, \dots, k\} \right\}.$$

Definition 3 *The set $\mathcal{S}^N(K,\varepsilon)$ is said to be finitely determined, if there exists an integer k such that $\mathcal{S}^N(K,\varepsilon) = \mathcal{S}^N_k(K,\varepsilon)$.*

The finite determination of T is characterized by the following theorem

Theorem 4 *$\mathcal{S}^N(K,\varepsilon)$ is finitely determined if and only if there exists an integer k such that $\mathcal{S}^N_k(K,\varepsilon) = \mathcal{S}^N_{k+1}(K,\varepsilon)$.*

Proof. The proof of the theorem is similar to that of Theorem 1. □

For the characterization of $\mathcal{S}^N(K,\varepsilon)$ we give the following algorithm.

Algorithm III

step 1 : Let $i = 0$;
step 2 : For $j = 1, \dots, s$, do :
  Maximize $J_j(x) = h_j(\bar{C}\bar{A}^{i+1}(x))$
  $\begin{cases} h_r(\bar{C}\bar{A}^l x) \leqslant 0 \\ j = 1, 2, \dots, 2k, l = 0, \dots, i. \end{cases}$
  Let J_j^* be the maximum value of $J_j(x)$.
  If $J_j^* \leqslant 0$, for $j = 1, \dots, s$ then set $i^* := i$ and stop.
  Else continue.
step 3 : Replace i by $i + 1$ and return to step 2.

In order to show that the finite determination property is not so restrictive, we give the following result.

Theorem 5 *Suppose the following assumptions to hold:*

1. *The pair* $\left(\bar{A},\bar{C}\right)$ *is observable, i.e.,* $\left[\bar{C}^\top \left|\bar{A}^\top\bar{C}^\top\right|\ldots\left|\left(\bar{A}^\top\right)^{N-1}\bar{C}^\top\right.\right]$ *has rank* N.
2. $\bar{A}$ *is asymptotically stable (* $|\lambda| < 1$ *for every* λ *eigenvalue of* $\bar{A}$*).*

Then the set $\mathcal{S}^N(K,\varepsilon)$ *is finitely determined.*

Proof. By the observability of $\left(\bar{A},\bar{C}\right)$, the rank of the matrix H is N, where

$$H = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{N-1} \end{bmatrix}$$

which implies that H^TH is invertible, so there exists $c = \inf_{\lambda\in\sigma(H^TH)}\lambda > 0$ such that

$$c\,\|x\|^2 \leqslant \left\langle H^THx, x\right\rangle, \quad \forall x\in\mathbb{R}^N$$

which implies that

$$c\,\|x\|^2 \leqslant \left\|H^\top\right\|\,\|Hx\|\,\|x\|, \quad \forall x\in\mathbb{R}^N.$$

We have

$$Hx \in \overbrace{B\left(0,\frac{\varepsilon}{2}\right)\times B\left(0,\frac{\varepsilon}{2}\right)\times\ldots\times B\left(0,\frac{\varepsilon}{2}\right)}^{N\text{-time}}, \quad \forall x\in\mathcal{S}^N_{N-1}(K,\varepsilon)$$

where $B\left(0,\frac{\varepsilon}{2}\right) = \left\{\forall x\in\mathbb{R}^N/\,\|x\|\leqslant\frac{\varepsilon}{2}\right\}$, since

$\overbrace{B\left(0,\frac{\varepsilon}{2}\right)\times B\left(0,\frac{\varepsilon}{2}\right)\times\ldots\times B\left(0,\frac{\varepsilon}{2}\right)}^{N\text{-time}}$ is bounded, then $\exists\epsilon_0$ such that $\|Hx\|\leqslant\epsilon_0$ and

$$c\,\|x\|^2 \leqslant \varepsilon_0\left\|H^\top\right\|\,\|x\|, \quad \forall x\in\mathcal{S}^N_{N-1}(K,\varepsilon).$$

then there exists $\gamma > 0$ such that

$$\|x\| \leqslant \gamma = \frac{\varepsilon_0\,\|H^\top\|}{c}, \quad \forall x\in\mathcal{S}^N_{N-1}(K,\varepsilon). \tag{11}$$

Hence

$$\mathcal{S}^N_{N-1}(K,\varepsilon) \subset B(0,\gamma) = \left\{\forall x \in \mathbb{R}^N / \|x\| \leqslant \gamma\right\}.$$

The asymptotic stability of $\bar{A}$ implies that $\exists i_0 \geqslant N-1$ such that

$$\left\|\bar{C}\bar{A}^{i_0+1}\right\| \leqslant \frac{\varepsilon}{2\gamma},$$

then since $\mathcal{S}^N_{i_0}(K,\varepsilon) \subset \mathcal{S}^N_{N-1}(K,\varepsilon) \subset B(0,\gamma)$ and $\bar{C}\bar{A}^{i_0+1}B(0,\gamma) \subset B\left(0,\frac{\varepsilon}{2}\right)$ then for $x \in \mathcal{S}^N_{i_0}(K,\varepsilon)$ we have $\|\bar{C}\bar{A}^{i_0+1}x\| \leqslant \varepsilon$ and then $x \in \mathcal{S}^N_{i_0+1}(K,\varepsilon)$ which implies that $\mathcal{S}^N_{i_0}(K,\varepsilon) = \mathcal{S}^N_{i_0+1}(K,\varepsilon)$. Then $\mathcal{S}^N(K,\varepsilon)$ is finitely determined. □

Example 2. Let consider the Example 1, with the operators

$$A = \Delta, \quad Cx = \langle x, g\rangle \ \text{ with } \ g(s) = s^2, \ \text{ and } \ Dx = v.$$

If we take

$$Kx = -\int_0^1 x(s)\,\mathrm{d}s, \quad \varepsilon = 0.01 \ \text{ and } \ \delta = 0.005$$

then, we have $N = 2$, $\bar{A} = \begin{pmatrix} e^{-\pi^2\delta} & 0 \\ 0 & e^{-4\pi^2\delta} \end{pmatrix}$ and $\bar{C} = \left[\frac{\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi^3}, \ -\frac{\sqrt{2}}{2\pi}\right]$.

Using the Algorithm III, we find that $k^* = 0$ and

$$\mathcal{S}^N(K,\varepsilon) = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \left|\left(\frac{\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi^3}\right)x - \frac{\sqrt{2}}{2\pi}y\right| \leqslant 0.01\right\}.$$

6. Conclusion

In this paper the output sensitivity problem for infinite dimensional linear system with uncertain initial state is considered. A control law is introduced in order to reduce the effet of these intolerable incertainties and which makes the system insensitive to the effects of all unknown parameters. Necessary conditions are given to described the set of all gain operators by a finite number of inequalities and some examples are given. In a future work, we plan to extend the approach developed in section 5 for the parabolic case to another type of system, as the hyperbolic systems or generally the distributed systems whose state space is separable, i.e., proceeds a countable basis.

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