

THE INTERSECTION GRAPH OF ANNIHILATOR SUBMODULES OF A MODULE

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Abstract. Let R be a commutative ring and M be a Noetherian R -module. The intersection graph of annihilator submodules of M , denoted by $GA(M)$ is an undirected simple graph whose vertices are the classes of elements of $Z_R(M) \setminus \text{Ann}_R(M)$, for $a, b \in R$ two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0$. In this paper, we study diameter and girth of $GA(M)$ and characterize all modules that the intersection graph of annihilator submodules are connected. We prove that $GA(M)$ is complete if and only if $Z_R(M)$ is an ideal of R . Also, we show that if M is a finitely generated R -module with $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$ and $|m - \text{Ass}_R(M)| = 1$ and $GA(M)$ is a star graph, then $r(\text{Ann}_R(M))$ is not a prime ideal of R and $|V(GA(M))| = |\text{Min Ass}_R(M)| + 1$.

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1. INTRODUCTION

Let R be a commutative ring and M be an R -module. The intersection graph of ideals of a ring introduced and studied in [7] and then in [1] the intersection graph of submodules of a module was defined. The intersection graph of submodules of a module, denoted by $G(M)$, is a graph whose vertices are in one to one correspondence with all non-trivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules of M have non-zero intersection. Also, the complement of the intersection graph of submodules of a module studied in [2], for more works on the intersection graph of modules, see [10, 15].

The zero-divisor graph of R , denoted by $\Gamma(R)$, is a graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices a and b are adjacent if and only if $ab = 0$, see [4, 6]. The compressed zero-divisor graph of R , $\Gamma_E(R)$, that is constructed from equivalence classes of zero-divisors, rather than individual zero-divisors themselves was introduced in [11] and studied in some literatures, for examples [3, 8, 13]. This graph

has some advantages over the zero-divisor graph. For examples, in many cases, the compressed zero-divisor graph of R is finite when the zero-divisor graph is infinite and another important aspect of the compressed zero-divisor graph is the connection to the associated primes of R .

In this paper, with inspire by the above ideas, we introduce the intersection graph of annihilator submodules of M . Let $a, b \in R$, we say that $a \sim b$ if and only if $\text{Ann}_M(a) = \text{Ann}_M(b)$. As noted in [11], \sim is an equivalence relation. If $[a]$ denotes the class of a , then $[a] = \text{Ann}_R(M)$ whenever $a \in \text{Ann}_R(M)$ and $[a] = R \setminus Z_R(M)$ whenever $a \in R \setminus Z_R(M)$; the other equivalence classes form a partition of $Z_R(M)$. The intersection graph of annihilator submodules of M , denoted by $GA(M)$, is an undirected simple graph whose vertices are the classes of elements of $Z_R(M) \setminus \text{Ann}_R(M)$, for $a, b \in Z_R(M) \setminus \text{Ann}_R(M)$ two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0$. Let M be a Noetherian R -module. In section two, we study connectivity, diameter and the girth of $GA(M)$. We show that $GA(M)$ is a disconnected graph if and only if $m - \text{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 = 0$. In section three, we show that $GA(M)$ is a complete graph if and only if $Z_R(M)$ is a prime ideal of R and we show that if $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$ and $|m - \text{Ass}_R(M)| = 1$ and $GA(M)$ is a star graph with at least three vertices, then $r(\text{Ann}_R(M))$ is not a prime ideal of R and

$$|V(GA(M))| = |\text{Min Ass}_R(M)| + 1.$$

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For each pair of vertices $u, v \in V(G)$, if u is adjacent to v , then we write $u - v$. The degree of a vertex u , denoted by $\text{deg}(u)$, is the number of edges incident to u , and u is called end-vertex if $\text{deg}(u) = 1$. A graph with no edge is called null graph. The complement graph of G , denoted by \overline{G} , is a graph with the same vertices such that two vertices of G are adjacent if and only if they are not adjacent in \overline{G} . Recall that G is connected if there is a path between any two distinct vertices of G . For vertices x and y of G , let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of G is

$$\text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}.$$

The girth of G , denoted by $\text{gr}(G)$, is the length of the shortest cycle in G ($\text{gr}(G) = \infty$ if G contains no cycles). A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n . A complete bipartite graph is a graph G which may be partitioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex set is a singleton, then we call G a star graph. A clique of G is a complete subgraph of G and the number of vertices in the largest clique of G , denoted by $\omega(G)$, is called the clique number.

Throughout, R denotes a commutative ring with non-zero identity, $Z(R)$ its set of zero-divisors and for ideal I of R ,

$$r(I) = \{r \in R : \text{there exists } n \in \mathbb{N} \text{ with } r^n \in I\}$$

denotes the radical of I . As usual, \mathbb{Z} and \mathbb{Z}_n will denote the ring of integers and the ring of integers modulo n , respectively. Let M be an R -module and

$$\text{Ass}_R(M) = \{\mathfrak{p} \in \text{Spec}(R) : \mathfrak{p} = \text{Ann}_R(m) \text{ for some } 0 \neq m \in M\}.$$

A proper submodule P of M is said to be prime submodule whenever for $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in \text{Ann}_R(M/P)$. Let $\text{Spec}_R(M)$ denote the set of prime submodules of M and

$$m - \text{Ass}_R(M) = \{P \in \text{Spec}_R(M) : P = \text{Ann}_M(a) \text{ for some } 0 \neq a \in R\},$$

where for $a \in R$, $\text{Ann}_M(a) = \{m \in M : am = 0\}$. For notations and terminologies not given in this article, the reader is referred to [12].

2. CONNECTIVITY, DIAMETER AND GIRTH OF $GA(M)$

Recall that R is a commutative ring and M is an R -module with property that its zero submodule is not a prime submodule. In this section, the annihilator submodules of M and the intersection graph of annihilator submodules of M are studied.

Theorem 2.1. *Let M be an R -module and $a, b \in R$. Then the following statements are true:*

- (i) [5, Theorem 2] *If $a \notin r(\text{Ann}_R(M))$ and $\text{Ann}_M(a)$ is a prime submodule of M , then it is a minimal prime submodule of M .*
- (ii) [5, Theorem 6(ii)] *If $a, b \notin r(\text{Ann}_R(M))$ and $\text{Ann}_M(a), \text{Ann}_M(b)$ are distinct prime submodules of M , then $abM = 0$.*
- (iii) [5, Theorem 5(ii)] *If $a \in r(\text{Ann}_R(M))$, then $\text{Ann}_M(a)$ is an essential submodule of M .*

Lemma 2.2. *Let M be a Noetherian R -module. Then*

$$r(\text{Ann}_R(M))M \subseteq \bigcap_{P \in m - \text{Ass}_R(M)} P.$$

Proof. By hypotheses and [9, Proposition 3.2], $m - \text{Ass}_R(M) \neq \emptyset$. Suppose that $a \in r(\text{Ann}_R(M))$ and $P \in m - \text{Ass}_R(M)$ we have to show that $aM \subseteq P$. By assumption there is a positive integer t such that $a^t M = 0$. Thus $a^t M \subseteq P$. By hypotheses P is a prime submodule of M , thus $\text{Ann}_R(M/P)$ is a prime ideal of R so by $a^t \in \text{Ann}_R(M/P)$ it follows that $a \in \text{Ann}_R(M/P)$. Hence, $aM \subseteq P$, we are done. \square

Theorem 2.3. *Let M be a Noetherian R -module with $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$ and $a, b \in R \setminus \text{Ann}_R(M)$. If $P_1 = \text{Ann}_M(a)$ and $P_2 = \text{Ann}_M(b)$ are prime submodules of M such that $P_1 \cap P_2 = 0$, then $|m - \text{Ass}_R(M)| = 2$.*

Proof. Let $P = \text{Ann}_M(c)$ is a prime submodule of M . Then $c \notin \text{Ann}_R(M)$. By hypotheses, $P_1 \cap P_2 = 0$. Thus $P_1 \cap P_2 \subseteq P$. If $P_1 \not\subseteq P$ and $P_2 \not\subseteq P$, then there exist $m_1 \in P_1 \setminus P$ and $m_2 \in P_2 \setminus P$ such that $am_1 = bm_2 = 0 \in P$ which implies

that $a, b \in \text{Ann}_R(M/\text{Ann}_M(c)) = \text{Ann}_R(cM)$. So that $cM \subseteq \text{Ann}_M(a) \cap \text{Ann}_M(b) = P_1 \cap P_2$. Hence, $cM = 0$ and so $c \in \text{Ann}_R(M)$, contrary to the assumption. Therefore, either $P_1 \subseteq P$ or $P_2 \subseteq P$ and so either $P_1 = P$ or $P_2 = P$ by Theorem 2.1(i). Thus $|m - \text{Ass}_R(M)| = 2$. \square

Corollary 2.4. *Let M be an R -module with $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$ and $a, b \in R \setminus \text{Ann}_R(M)$. If $P_1 = \text{Ann}_M(a)$ and $P_2 = \text{Ann}_M(b)$ are prime submodules of M such that $P_1 \cap P_2 = 0$, then $aM + bM \cong aM \oplus bM$.*

Proof. Let $a, b \in R \setminus \text{Ann}_R(M)$, $P_1 = \text{Ann}_M(a)$ and $P_2 = \text{Ann}_M(b)$ are prime submodules of M such that $P_1 \cap P_2 = 0$. Then by Theorem 2.1(ii) it follows that $abM = 0$. Hence $aM \cap bM \subseteq \text{Ann}_M(b) \cap \text{Ann}_M(a) = 0$. Thus $aM + bM \cong aM \oplus bM$. \square

Corollary 2.5. *Let M be an R -module with $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$ and $a \in R \setminus \text{Ann}_R(M)$. If $\text{Ann}_M(a)$ is a prime submodule of M , then $Z_R(aM)$ is a prime ideal of R .*

Proof. First, we show that $Z_R(aM) = \text{Ann}_R(aM)$. Suppose that $c \in R$. We show that either $\text{Ann}_{aM}(c) = 0$ or $\text{Ann}_{aM}(c) = aM$. Assume that $\text{Ann}_{aM}(c) \neq 0$ and $0 \neq am \in \text{Ann}_{aM}(c)$. Thus $cam = 0$. Hence, $cm \in \text{Ann}_M(a)$ and $\text{Ann}_M(a)$ is a prime submodule of M , it follows that $caM = 0$. Thus $\text{Ann}_{aM}(c) = aM$. Hence, $Z_R(aM) = \text{Ann}_R(aM) = \text{Ann}_R(M/\text{Ann}_M(a))$ is a prime ideal of R , as claimed. \square

Let R be a commutative ring and M be an R -module. Assume $Z_R^*(M)$ denotes the set of non-zero zero-divisors of M . For $a, b \in R$, we say that $a \sim b$ if and only if $\text{Ann}_M(a) = \text{Ann}_M(b)$. As noted in [11], \sim is an equivalence relation. If $[a]$ denotes the class of a , then $[a] = \text{Ann}_R(M)$ whenever $a \in \text{Ann}_R(M)$ and $[a] = R \setminus Z_R(M)$ for all $a \in R \setminus Z_R(M)$; the other equivalence classes form a partition of $Z_R(M)$. The intersection graph of annihilator submodules of M , denoted by $GA(M)$, is an undirected simple graph whose vertices are the classes of elements in $Z_R^*(M) \setminus \text{Ann}_R(M)$, and two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0$.

Lemma 2.6. *Let M be an R -module. If $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$, then $GA(M)$ is a connected graph.*

Proof. Let $a \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$. Then by Theorem 2.1(iii), $\text{Ann}_M(a)$ is an essential submodule of M . So that $[a]$ is a universal vertex in $GA(M)$, which show that $GA(M)$ is connected. \square

Theorem 2.7. *Let M be a Noetherian R -module. Then $GA(M)$ is a disconnected graph if and only if $m - \text{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 = 0$.*

Proof. (\Leftarrow) Suppose that P_1 and P_2 are two prime annihilator submodules of M such that $P_1 \cap P_2 = 0$. Thus by Lemma 2.2, $r(\text{Ann}_R(M))M \subseteq P_1 \cap P_2 = 0$ and so $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$. Assume that $a, b \in R \setminus \text{Ann}_R(M)$ and $P_1 = \text{Ann}_M(a)$ and $P_2 = \text{Ann}_M(b)$. It is our claim that, there is no path between two vertices $[a]$ and $[b]$. Set $X = \{\text{Ann}_M(c) : c \notin \text{Ann}_R(M)\}$. If P_1 is not a maximal element of X , then there exists a maximal element P in X different from P_2 such that $P_1 \subseteq P$. By [9, Proposition 3.2], P is a prime submodule of M so that $|m - \text{Ass}_R(M)| \geq 3$,

contrary to hypotheses. Therefore, P_1 and P_2 are maximal elements of X . Thus for each $[c] \in V(GA(M))$ we have either $\text{Ann}_M(c) \subseteq P_1$ or $\text{Ann}_M(c) \subseteq P_2$. On the contrary, suppose that $[a] - [c] - [d] - [b]$ is a path between $[a]$ and $[b]$, where $\text{Ann}_M(c) \subseteq P_1$ and $\text{Ann}_M(d) \subseteq P_2$. Then $0 \neq \text{Ann}_M(c) \cap \text{Ann}_M(d) \subseteq P_1 \cap P_2$ that is a contradiction. In any other cases we have the same contradiction. Hence, there is no path between $[a]$ and $[b]$, as claimed. Therefore, $GA(M)$ is a disconnected graph.

(\Rightarrow) By hypotheses and Lemma 2.6 we can assume that $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$. If $|m - \text{Ass}_R(M)| = 1$, then $GA(M)$ has a universal vertex such as $[a]$, where $a \in R \setminus \text{Ann}_R(M)$ and $\text{Ann}_M(a) = P \in m - \text{Ass}_R(M)$. So that $|m - \text{Ass}_R(M)| \geq 2$. Assume that $P' \cap P'' \neq 0$, for each prime annihilator submodules P' and P'' of M and we look for a contradiction. Let $[c]$ and $[d]$ be two arbitrary vertices of $GA(M)$. Then there exist two prime annihilator submodules $P_1 = \text{Ann}_M(c)$ and $P_2 = \text{Ann}_M(d)$ of M such that $\text{Ann}_M(c) \subseteq P_1$ and $\text{Ann}_M(d) \subseteq P_2$. Consequently, $[c] - [a] - [b] - [d]$ is a path between two vertices $[c]$ and $[d]$. Hence, $GA(M)$ is a connected graph contrary to assumption. So that there exist two annihilator prime submodules P' and P'' of M such that $P' \cap P'' = 0$. Now, the assertion follows from Theorem 2.3. \square

Corollary 2.8. *Let M be a Noetherian R -module. Then the following statements are equivalent:*

- (i) $GA(M)$ is a connected graph,
- (ii) Either $m - \text{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 \neq 0$ or $|m - \text{Ass}_R(M)| \neq 2$,
- (iii) Any two distinct elements of $m - \text{Ass}_R(M)$ are adjacent in $GA(M)$.

The independence number of a graph G is the maximum size of an independent vertex set and is denoted by $\alpha(G)$.

Corollary 2.9. *Let M be a Noetherian R -module. If $GA(M)$ is a connected graph, then*

$$\alpha(GA(M)) \leq |V(GA(M))| - |m - \text{Ass}_R(M)| + 1.$$

Proof. By Corollary 2.8, any two distinct elements of $m - \text{Ass}_R(M)$ are connected by an edge in $GA(M)$, so that $|m - \text{Ass}_R(M)| \leq \omega(GA(M))$. On the other hand, by [14, proposition 5.1.7]

$$\omega(GA(M)) \leq |V(GA(M))| - \alpha(GA(M)) + 1.$$

Thus the result follows. \square

Remark 2.10. Let $G(M)$ be a disconnected graph. Then by [1, Theorem 2.1], $M = P_1 \oplus P_2$ and $m - \text{Ass}_R(M) = \{P_1, P_2\}$, where P_1, P_2 are prime submodules of M and $P_1 \cap P_2 = 0$, so that $GA(M)$ is a disconnected graph by Theorem 2.7, whenever $|V(GA(M))| \geq 2$.

The converse is not true in general. Consider $M = \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$ as a \mathbb{Z} -module. It is easy to see that $GA(M)$ is a null graph with two vertices, but $G(M)$ is a connected graph with more than two vertices.

Theorem 2.11. *Let M be a Noetherian R -module and $GA(M)$ be a connected graph. Then $\text{diam}(GA(M)) \leq 2$.*

Proof. Let $[a]$ and $[b]$ be two arbitrary vertices of $GA(M)$. If $\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0$, then $d([a], [b]) = 1$. Let $\text{Ann}_M(a) \cap \text{Ann}_M(b) = 0$. If $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$, then by Theorem 2.1(iii), there is a universal vertex in $GA(M)$ denoted by $[x]$ such that $x \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$. Hence $GA(M)$ has the path $[a] - [x] - [b]$ as a subgraph and so $d([a], [b]) = 2$.

Suppose that $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$. If $ab \notin \text{Ann}_R(M)$, then $[ab] \in V(GA(M))$ and $\text{Ann}_M(a) \subset \text{Ann}_M(ab)$. In this case, if $\text{Ann}_M(ab) = \text{Ann}_M(a)$, then we must have $\text{Ann}_M(b) \subseteq \text{Ann}_M(a)$ which contradicts the fact that $\text{Ann}_M(a) \cap \text{Ann}_M(b) = 0$. So $\text{Ann}_M(a) \subset \text{Ann}_M(ab)$ and a similar argument shows that $\text{Ann}_M(b) \subset \text{Ann}_M(ab)$. Hence, $GA(M)$ has the path $[a] - [ab] - [b]$ as a subgraph and so $d([a], [b]) = 2$.

In the sequel, if $ab \in \text{Ann}_R(M)$, then there exist two vertices $[x], [y]$ of $GA(M)$ such that $\text{Ann}_M(a) \subseteq \text{Ann}_M(x)$ and $\text{Ann}_M(b) \subseteq \text{Ann}_M(y)$ where $\text{Ann}_M(x), \text{Ann}_M(y) \in m - \text{Ass}_R(M)$ and $x, y \notin r(\text{Ann}_R(M))$. Thus by Corollary 2.8, $GA(M)$ has the path $[a] - [x] - [y] - [b]$ as a subgraph. Suppose that $0 \neq m \in \text{Ann}_M(x) \cap \text{Ann}_M(y)$. If either $m \in \text{Ann}_M(a)$ or $m \in \text{Ann}_M(b)$, then $GA(M)$ has the path $[a] - [y] - [b]$ or $[a] - [x] - [b]$ as a subgraph, respectively. Now, let $m \notin \text{Ann}_M(a) \cup \text{Ann}_M(b)$. By $abM = 0$ it follows that $bM \subset \text{Ann}_M(a)$. Hence $bm \in \text{Ann}_M(a) \cap \text{Ann}_M(y)$ which implies that $GA(M)$ has the path $[a] - [y] - [b]$ as a subgraph. Hence, $d([a], [b]) = 2$ and therefore $\text{diam}(GA(M)) \leq 2$. \square

Theorem 2.12. *Let M be a Noetherian R -module and $GA(M)$ be a disconnected graph. Then $\overline{GA(M)}$ is a connected graph and $\text{diam}(\overline{GA(M)}) \leq 2$.*

Proof. By Theorem 2.7, there exist two prime annihilator submodules $P_1 = \text{Ann}_M(a)$ and $P_2 = \text{Ann}_M(b)$ of M such that $P_1 \cap P_2 = 0$. Thus $[a]$ and $[b]$ are adjacent in $\overline{GA(M)}$. Also, P_1 and P_2 are the only maximal elements of $X = \{\text{Ann}_M(d) : d \notin \text{Ann}_R(M)\}$ so for every $[c] \in V(GA(M)) \setminus \{[a], [b]\}$, we have either $\text{Ann}_M(c) \subset \text{Ann}_M(a)$ or $\text{Ann}_M(c) \subset \text{Ann}_M(b)$. Assume that $[c]$ and $[d]$ are two arbitrary vertices of $\overline{GA(M)}$. If either $\text{Ann}_M(c), \text{Ann}_M(d) \subset P_1$ or $\text{Ann}_M(c), \text{Ann}_M(d) \subset P_2$, then $\overline{GA(M)}$ has the path $[c] - [b] - [d]$ or $[c] - [a] - [d]$ as a subgraph, respectively. Also, if $\text{Ann}_M(c) \subset P_1$ and $\text{Ann}_M(d) \subset P_2$, then $\overline{GA(M)}$ has a cycle $[c] - [b] - [a] - [d] - [c]$ as a subgraph. Hence, $\overline{GA(M)}$ is a connected graph and $\text{diam}(\overline{GA(M)}) \leq 2$. \square

Theorem 2.13. *Let M be a Noetherian R -module and let $GA(M)$ be a connected graph. Then the following statements are true:*

- (i) *If either $|m - \text{Ass}_R(M)| = 1$ or $|m - \text{Ass}_R(M)| = 2$ and $V(GA(M)) \leq 3$, then $\text{gr}(GA(M)) \in \{3, \infty\}$.*
- (ii) *If either $|m - \text{Ass}_R(M)| \geq 3$ or $|m - \text{Ass}_R(M)| = 2$ and $V(GA(M)) \geq 4$, then $\text{gr}(GA(M)) = 3$.*

Proof. If $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$, then it follows from Theorem 2.1(iii) that $GA(M)$ has a universal vertex such as $[a]$, where $a \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$. Thus $\text{gr}(GA(M)) \in \{3, \infty\}$. So in the following we can assume that $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$. If $m - \text{Ass}_R(M) = \{P\}$, then $P = \text{Ann}_M(a)$ for some $a \in R \setminus \text{Ann}_R(M)$. So $[a]$ is a universal vertex of $GA(M)$ and the result follows. For $|m - \text{Ass}_R(M)| \geq 3$ the result follows from Corollary 2.8.

Now, suppose that $m - \text{Ass}_R(M) = \{P_1, P_2\}$, where $P_1 = \text{Ann}_M(a)$, $P_2 = \text{Ann}_M(b)$ and $a, b \notin \text{Ann}_R(M)$. If $|V(GA(M))| < 4$, then by Corollary 2.8 the proof is obvious. Thus we can assume that $|V(GA(M))| \geq 4$. Let $|V(GA(M))| > 4$. Then there exist two vertices $[x], [y]$ which are adjacent to $[a]$. If $0 = \text{Ann}_M(x) \cap \text{Ann}_M(y)$, then $\text{Ann}_M(x) \cap \text{Ann}_M(y) \subseteq P_2$ so by an argument similar to that of Theorem 2.3 one can show that either $\text{Ann}_M(x) \subseteq P_2$ or $\text{Ann}_M(y) \subseteq P_2$. Thus $GA(M)$ has the cycle $[x] - [b] - [a] - [x]$ or $[y] - [b] - [a] - [y]$ as a subgraph, respectively and $\text{gr}(GA(M)) = 3$. If $0 \neq \text{Ann}_M(x) \cap \text{Ann}_M(y)$, then $[a] - [x] - [y] - [a]$ is a cycle in $GA(M)$. Thus the result follows. Finally, assume that $|V(GA(M))| = 4$ and $[x], [y] \in V(GA(M)) \setminus \{[a], [b]\}$. If both of $[x]$ and $[y]$ are adjacent to either $[a]$ or $[b]$, then the proof is similar to the previous procedure. Otherwise, without loss of generality, let $[x] - [a] - [b] - [y]$. Then $\text{Ann}_M(x) \subseteq \text{Ann}_M(a)$ and $\text{Ann}_M(y) \subseteq \text{Ann}_M(b)$. Since $\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0$, then there exist $0 \neq m \in M$ such that $m \in \text{Ann}_M(a) \cap \text{Ann}_M(b)$. If $m \in \text{Ann}_M(x)$ or $m \in \text{Ann}_M(y)$, then $GA(M)$ has the cycle $[x] - [a] - [b] - [x]$ or $[y] - [b] - [a] - [y]$ as a subgraph, respectively. Now, let $m \notin \text{Ann}_M(x) \cup \text{Ann}_M(y)$. If $xyM = 0$, then we have $yM \subseteq \text{Ann}_M(x)$ and so $ym \in \text{Ann}_M(x)$. It follows that $ym \in \text{Ann}_M(x) \cap \text{Ann}_M(b)$ which implies that $GA(M)$ has the cycle $[x] - [b] - [a] - [x]$ as a subgraph. If $xyM \neq 0$, then $xy \neq 0$ and we have the following two cases, since $|V(GA(M))| = 4$,

Case 1. If $\text{Ann}_M(xy) = \text{Ann}_M(a)$, then $\text{Ann}_M(y) \subseteq \text{Ann}_M(a)$. Thus $[y] - [a] - [b] - [y]$ is a cycle in $GA(M)$. Similarly, if $\text{Ann}_M(xy) = \text{Ann}_M(b)$, then $[x] - [b] - [a] - [x]$ is a cycle in $GA(M)$.

Case 2. If $\text{Ann}_M(xy) = \text{Ann}_M(x)$, then $\text{Ann}_M(y) \subseteq \text{Ann}_M(x)$. Thus $[y] - [a] - [b] - [y]$ is a cycle in $GA(M)$. Similarly, for $\text{Ann}_M(xy) = \text{Ann}_M(y)$, we have $[x] - [b] - [a] - [x]$ is a cycle in $GA(M)$.

Therefore, $\text{gr}(GA(M)) = 3$ and the proof is completed. □

Corollary 2.14. *Let M be a Noetherian R -module and $|m - \text{Ass}_R(M)| = 2$. If $GA(M)$ is a connected triangle-free graph with at least three vertices, then $GA(M) \cong K_{1,2}$.*

Proof. Let $\text{Ann}_M(a), \text{Ann}_M(b) \in m - \text{Ass}_R(M)$. Then by hypotheses $a, b \notin r(\text{Ann}_R(M))$ and by the proof of Theorem 2.13, we have $|V(GA(M))| = 3$. Assume that $[c] \in V(GA(M)) \setminus \{[a], [b]\}$. Then it follows that either $\text{Ann}_M(c) \subseteq \text{Ann}_M(a)$ or $\text{Ann}_M(c) \subseteq \text{Ann}_M(b)$ since $GA(M)$ is triangle-free. Thus $GA(M)$ is either $[b] - [a] - [c]$ or $[a] - [b] - [c]$, we are done. □

3. MODULES WHOSE $GA(M)$ IS COMPLETE OR STAR

In this section, R is a commutative ring and M is an R -module. The subset

$$T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$$

of M is called the torsion subset of M .

Theorem 3.1. *Let M be a Noetherian R -module. Then $GA(M)$ is a complete graph if and only if $Z_R(M)$ is a prime ideal of R .*

Proof. First, let $GA(M)$ is a complete graph and $a, b \in Z_R(M)$. If $\text{Ann}_M(a) = \text{Ann}_M(b)$, then $a + b \in Z_R(M)$ and the proof is completed. Otherwise, there is an edge between $[a]$ and $[b]$, since $GA(M)$ is a complete graph, thus $\text{Ann}_M(a) \cap \text{Ann}_M(b) \neq 0$. Hence, there exists $0 \neq m \in M$ such that $am = bm = 0$. Therefore, $(a + b)m = 0$ and so $a + b \in Z_R(M)$ which complete the proof.

Conversely, suppose that $[a], [b]$ are two disjoint arbitrary vertices of $GA(M)$ and suppose that $Z_R(M)$ is an ideal of R . Let $I = \text{Ann}_R(M)$. It is easy to see that R/I is a Noetherian ring and M is Noetherian R/I -module. By [12, Corollary 9.36], $Z_{R/I}(M) = \cup_{\mathfrak{p} \in \text{Ass}_{R/I}(M)} \mathfrak{p}$. It follows from The Prime Avoidance Theorem, [12, Theorem 3.61], that $Z_{R/I}(M) = \text{Ann}_{R/I}(m)$, for some $0 \neq m \in M$. We have $a + I, b + I \in Z_{R/I}^*(M)$ so that $am = bm = 0$. Thus $m \in \text{Ann}_M(a) \cap \text{Ann}_M(b)$. Hence, $GA(M)$ is a complete graph. \square

Theorem 3.2. *Let M be a Noetherian R -module such that $r(\text{Ann}_R(M)) \neq \text{Ann}_R(M)$ and $|m - \text{Ass}_R(M)| = 1$. If $GA(M)$ is a star graph with at least three vertices, then $r(\text{Ann}_R(M))$ is not a prime ideal of R and*

$$|V(GA(M))| = |\text{Min Ass}_R(M)| + 1.$$

Proof. Suppose that $GA(M)$ is a star graph and $a \in R \setminus \text{Ann}_R(M)$ and $P = \text{Ann}_M(a) \in m - \text{Ass}_R(M)$. By [9, Proposition 3.2], each maximal element of $X = \{\text{Ann}_M(d) : d \notin \text{Ann}_R(M)\}$ is a prime submodule of M which implies that $\text{Ann}_M(b) \subseteq P$, for any $b \in Z_R(M) \setminus \text{Ann}_R(M)$. Thus $[a]$ is the only universal vertex of $GA(M)$. Let $[b], [c]$ be two end-vertices of $GA(M)$. Then by Theorem 2.2(iii), $b, c \notin r(\text{Ann}_R(M))$. We show that $bc \in r(\text{Ann}_R(M))$ and so $r(\text{Ann}_R(M))$ is not a prime ideal of R . If $bc \in \text{Ann}_R(M)$ we are done. Now, assume that $bc \in Z_R(M) \setminus \text{Ann}_R(M)$. If $\text{Ann}_M(b) = \text{Ann}_M(bc)$, then two vertices $[b]$ and $[c]$ are adjacent in $GA(M)$ which contradicts the fact that $GA(M)$ is a star graph. Thus $\text{Ann}_M(b) \neq \text{Ann}_M(bc)$, a similar argument shows that $\text{Ann}_M(c) \neq \text{Ann}_M(bc)$. Hence, $[b] - [bc] - [c]$ is a path in $GA(M)$ which implies that $\text{Ann}_M(a) = \text{Ann}_M(bc)$. Therefore, the previous paragraph shows $\text{Ann}_M(d) \subseteq \text{Ann}_M(bc)$, for any $d \in Z_R(M) \setminus \text{Ann}_R(M)$. Now, it is easy to see that for all $\mathfrak{p} \in \text{Ass}_R(M)$ we have $\text{Ann}_R(M) \subset \mathfrak{p}$ and $bc \in \mathfrak{p}$. Thus $bc \in r(\text{Ann}_R(M))$.

Let $[b]$ be an end-vertex of $GA(M)$. We show that $[b]$ lies in a minimal element of $\text{Ass}_R(M)$. By hypotheses $b \in Z_R(M) \setminus r(\text{Ann}_R(M))$ and an argument similar to that of Theorem 3.1 shows that $Z_R(M) = \cup_{\mathfrak{p} \in \text{Ass}_R(M)} \mathfrak{p}$. So that $b \in \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}_R(M)$. Suppose that $\mathfrak{p} = \text{Ann}_R(m)$, where $m \in M$. If \mathfrak{p} is a minimal element of $\text{Ass}_R(M)$ we are done. Otherwise, there exists a minimal element \mathfrak{q} of $\text{Ass}_R(M)$ such that $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} = \text{Ann}_R(m')$, for some $m' \in M$. Again, if $b \in \mathfrak{q}$ we are done. Assume that $b \notin \mathfrak{q}$. By the previous paragraph $r(\text{Ann}_R(M)) \subset \mathfrak{q}$. Assume that $c \in \mathfrak{q} \setminus r(\text{Ann}_R(M))$. Thus $c \neq b$. Now, $\text{Ann}_M(b) \neq \text{Ann}_M(c)$ since $cm' = 0$ but $bm' \neq 0$ and $m \in \text{Ann}_M(b) \cap \text{Ann}_M(c)$. This is a contradiction with the fact that $GA(M)$ is a star graph. Thus every end-vertices of $GA(M)$ lies in a minimal element of $\text{Ass}_R(M)$. If $b, c \in \mathfrak{p} = \text{Ann}_R(m)$, for two end-vertices $[b], [c]$ of $GA(M)$, then $m \in \text{Ann}_M(b) \cap \text{Ann}_M(c)$ which is a contradiction. Hence, any end-vertex is located in a unique minimal associated prime ideal of R . Therefore,

$$|V(GA(M))| \leq |\text{Min Ass}_R(M)| + 1.$$

By hypotheses and the previous paragraph $|\text{Min Ass}_R(M)| \geq 2$. Let $\mathfrak{p} \in \text{Min Ass}_R(M)$. Then $\mathfrak{p} \not\subseteq r(\text{Ann}_R(M))$. Assume that $b \in \mathfrak{p} \setminus r(\text{Ann}_R(M))$ thus $[b]$ is a end-vertices of $GA(M)$ and so

$$|V(GA(M))| \geq |\text{Min Ass}_R(M)| + 1. \quad \square$$

A graph G is planar if it has a drawing without crossings. In a graph G subdivision of an edge with endpoints u and v is a operation of replacing the edge $u - v$ with the path $u - w - v$, through a new vertex w . The subdivision of a graph G is a graph obtained from G by subdividing some of the edges of G .

Theorem 3.3. *Let R_1, R_2, R_3 be commutative rings and $R = R_1 \times R_2 \times R_3$ and let $M = M_1 \times M_2 \times M_3$, where M_i is an R_i -module, for $i = 1, 2, 3$. Then $GA(M)$ is a planar graph if and only if $Z_{R_i}(M_i) = \text{Ann}_{R_i}(M_i)$, for $i = 1, 2, 3$.*

Proof. First, we assume that $Z_{R_i}(M_i) = \text{Ann}_{R_i}(M_i)$, for all $1 \leq i \leq 3$. Then

$$V(GA(M)) = \{x_1 = [(0, 0, 1)], x_2 = [(0, 1, 0)], x_3 = [(1, 0, 0)], \\ x_4 = [(1, 1, 0)], x_5 = [(1, 0, 1)], x_6 = [(0, 1, 1)]\}.$$

Here, it is easy to check that the graph $GA(M)$ is isomorphic to the Figure 1 which contains no subdivision of K_5 and $K_{3,3}$. Thus by [14, Theorem 6.2.2], $GA(M)$ is a planar graph.

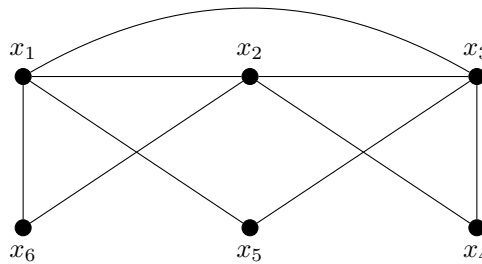


Fig. 1. A graph with no subdivision of K_5 and $K_{3,3}$

Conversely, suppose that the graph $GA(M)$ is planar. On the contrary and with no loss of generality, let $Z_{R_1}(M_1) \neq \text{Ann}_{R_1}(M_1)$. Then there exists $r \in R_1 \setminus \{0, 1\}$ such that $0 \neq \text{Ann}_{M_1}(r) \subset M$. It is easy to check that the vertex set

$$\mathcal{F} = \{x_1 = [(1, 1, 0)], x_2 = [(1, 0, 0)], x_3 = [(0, 0, 1)], x_4 = [(1, 0, 1)], \\ x_5 = [(r, 0, 1)], x_6 = [(r, 1, 0)], x_7 = [(r, 0, 0)], x_8 = [(0, 1, 0)]\}$$

induces a subgraph of $GA(M)$ which contains a subdivision of $K_{3,3}$, see Figure 2 for more details. By [14, Theorem 6.2.2], $GA(M)$ is not planar which is a contradiction. So $Z_{R_i}(M_i) = \text{Ann}_{R_i}(M_i)$, for all $1 \leq i \leq 3$. □

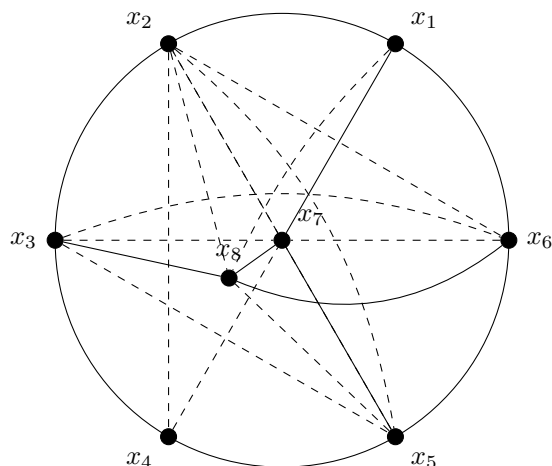


Fig. 2. An induced subgraph of $GA(M)$ whose bold lines form a subdivision of $K_{3,3}$

Theorem 3.4. Let R_1, \dots, R_n ($n \geq 4$), be commutative rings and $R = R_1 \times \dots \times R_n$ and let $M = M_1 \times \dots \times M_n$, where M_i is an R_i -module, for all $1 \leq i \leq n$. Then $GA(M)$ is not a planar graph.

Proof. It is sufficient to show that $GA(M)$ is not a planar graph for $n = 4$. Consider the vertex set

$$\mathcal{F} = \{x_1 = [(1, 1, 0, 0)], x_2 = [(0, 0, 1, 0)], \\ x_3 = [(1, 1, 1, 0)], x_4 = [(0, 1, 0, 0)], x_5 = [(1, 0, 0, 0)]\}.$$

It is easy to see that $\text{Ann}_M(x_i) \neq \text{Ann}_M(x_j)$ and $\text{Ann}_M(x_i) \cap \text{Ann}_M(x_j) \neq 0$, for all i, j with $i \neq j$ and $1 \leq i, j \leq 5$. Thus the set \mathcal{F} induces a complete subgraph of $GA(M)$. Hence, by [14, Theorem 6.2.2] the result follows. \square

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