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LOCAL EXISTENCE FOR A VISCOELASTIC KIRCHHOFF TYPE EQUATION WITH THE DISPERSIVE TERM, INTERNAL DAMPING, AND LOGARITHMIC NONLINEARITY

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Abstract. This paper concerns a viscoelastic Kirchhoff-type equation with the dispersive term, internal damping, and logarithmic nonlinearity. We prove the local existence of a weak solution via a modified lemma of contraction of the Banach fixed-point theorem. Although the uniqueness of a weak solution is still an open problem, we proved uniqueness locally for specifically suitable exponents. Furthermore, we established a result for local existence without guaranteeing uniqueness, stating a contraction lemma.

Keywords: viscoelastic equation, dispersive term, logarithmic nonlinearity, local existence.

Mathematics Subject Classification: 35A01, 35L20, 35L70.

1. INTRODUCTION

In elasticity, the existing theory accounts for materials that can store mechanical energy with no energy dissipation. On the other hand, a Newtonian viscous fluid in a non-hydrostatic stress state can dissipate energy without keeping it. Materials outside the scope of these two theories would be those for which some work done to deform can be recovered. This material has a capacity for storage and dissipation of mechanical energy. An example of this kind of material is viscoelastic.

Viscoelastic materials are those for which the behavior combines liquid-like and solid-like characteristics. Viscoelasticity is essential in biomechanics, the power industry or heavy construction, synthetic polymers, wood, human tissue, cartilage, metals at high temperatures, and concrete, among others. Polymers, for instance, are viscoelastic materials since they exhibit an intermediate position between viscous liquids and elastic solids.

Physically, Boltzmann's theory inspired the relationship between stress and strain history. The formulation of Boltzmann's superposition principle leads to a memory term involving a relaxation function of exponential type. But, it was observed that the relaxation functions of some viscoelastic materials are not necessarily exponential. See [21].

In this work, we are concerned with the following viscoelastic problem in $\Omega \times (0, \infty)$,

$$\begin{cases} |u_t|^{\rho}_{\mathbb{R}} u_{tt} + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int\limits_0^t g(t-s)\Delta u(s)ds + u_t = u|u|^{p-2}_{\mathbb{R}} \ln |u|^k_{\mathbb{R}}, \\ u = 0 \text{ on } \partial\Omega \times [0, \infty), \\ u(x, 0) = u_0(x) \text{ in } \Omega, \\ u_t(x, 0) = u_1(x) \text{ in } \Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $p > 2$ and $\rho > 0$ are constants and $g : \mathbb{R}^+ \to \mathbb{R}^+$, $M : [0, \infty) \to \mathbb{R}$ are C^1 functions, left to be defined later.

Dafermos first considered the viscoelastic problem with the power source term in 1970 [9]. Since then, several works have been considered, including combining damping and source terms. The source given by logarithmic nonlinearity appears in several branches of physics, such as inflationary cosmology, nuclear physics, optics, and geophysics, see [22].

With all this specific underlying meaning in physics, the global-in-time well-posedness of a solution to the problem of an evolution equation with such logarithmic-type nonlinearity captures lots of attention. See [7, 22] for the references to each branch listed above.

The dispersive term Δu_{tt} arises in the study of extensional vibrations of thin rods, see Love [20], via the model

$$u_{tt} - \Delta u - \Delta u_{tt} = f.$$

The function $M(\lambda)$ related to the Kirchhoff term in (1.1) has its motivation in the mathematical description of vibration of an elastic stretched string, modeled by the equation

$$u_{tt} - M\left(\int\limits_\Omega |\nabla u|^2 dx \right) \Delta u = 0,$$

which for $M(\lambda) \geq m_0 > 0$ was studied in [8, 17, 18, 23, 25]. Some recent works on hyperbolic wave equation with Kirchhoff-type term can be found in [19, 24, 26].

As a benchmark model, one may also take as in Love [20] the equation

$$\frac{d}{dt}(\rho(u_t)) - \Delta u_{tt} - \gamma \Delta u_t = \Delta u$$

with $\gamma \geq 0$ and nonlinearity density $\rho(s)$ a monotone increasing function, which models problems in the mechanics of solids, which account for variable material density (depending on the velocity u_t) and potential mechanical damping. It is of a particular interest when it is assumed of the form $\rho(s) = \frac{1}{\rho+1}|s|^\rho s$ (see also [3]).

From the physical point of view, the logarithmic nonlinearity if of much use in physics, since it naturally appears in inflation cosmology and supersymmetric field theories, quantum mechanics, and nuclear physics (see refs [1, 10]). The mathematical approach for equation with logarithmic nonlinearity involving global and local existence and blow of solution can be checked in [5, 12, 14, 15].

Although the uniqueness of a weak solution of equation (1.1) is still an open problem, in this work, we have proved uniqueness locally for a specific set of ρ, namely $\rho \geq 1$, through the method of Banach fixed point. Now, for ρ small, we have also established a result for local existence, without guarantee of uniqueness, stating a contraction lemma.

This work is split into three parts. Section 2 presents the notation and results underlying the methods used in this paper. Section 3 proves the global solution for an elliptic-associated problem. Section 4 treats the proof of the local existence of the solution via a modified lemma of contraction of the Banach fixed-point theorem.

2. PRELIMINARIES AND ASSUMPTIONS

Definition 2.1. Let B be a Banach space and $u : [0, T] \to B$ a measurable function. The vector function spaces $L^p(0, T; B)$, $1 \leq p \leq \infty$, are defined by

$$L^p(0,T;B) = \left\{ u : \left(\int_0^T \|u\|_B^p dt \right)^{1/p} < \infty,\ 1 \leq p < \infty \right\}$$

and

$$L^\infty(0,T;B) = \left\{ u : \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_B < \infty,\ p = \infty \right\}.$$

If B is reflexive, $1 \leq p < \infty$, and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, the dual of $L^p(0, T; B)$ is algebraic and topologically identified with $L^q(0, T; B')$ (see [13]).

Definition 2.2. If V and W are Banach spaces and $1 \leq p \leq \infty$, then we define

$$W_p(0,T;V,W) \coloneqq \{f \in L^p(0,T;V) \,:\, f' \in L^p(0,T;W)\}.$$

The spaces in Definition 2.2 are Banach spaces with the natural norms.

Proposition 2.3. *Let V and H be two separable Hilbert spaces, with V dense and injectively included in H. Then is valid the canonic injection*

$$W_2(0,T,V,V') \subset C([0,T],H).$$

A very useful result from Measure Theory will be required.

Lemma 2.4 ([28, p. 171]). *Given $1 \leq p \leq \infty$, every sequence in L^p that converges in L^p has a subsequence converging almost everywhere.*

Now we present a well-known compactness result. The compactness is needed to extract a sequence in the set of approximate solutions, which converges strongly.

Lemma 2.5 (Aubin–Lions Lemma, [16, Theorem 5.1, p. 58]). *Let B_0, B and B_1 be Banach spaces, B_i, $i = 0, 1$, reflexive spaces with $B_0 \hookrightarrow B$ compactly, $B \hookrightarrow B_1$ continously. Defining*

$$W = \{u : u \in L^{p_0}(0, T; B_0), u_t \in L^{p_1}(0, T; B_1)\},$$

where $T > 0$ and $1 < p_i < \infty$, $i = 0, 1$. Then, $W \subset L^{p_0}(0, T; B)$ equipped with the norm

$$\|w\| = \|u\|_{L^{p_0}(0,T;B_0)} + \|u_t\|_{L^{p_1}(0,T;B_1)}$$

is a Banach space and $W \hookrightarrow L^{p_0}(0, T; B)$ is compact.

For simplicity of notations hereafter we denote by $|\cdot|$ the Lebesgue space $L^2(\Omega)$-norm, by

$$\|\cdot\| := \int_\Omega |\nabla(\cdot)|^2_{\mathbb{R}^n}\, dx$$

the Sobolev space $H^1_0(\Omega)$-norm, and $\|\cdot\|_r := \|\cdot\|_{L^r(\Omega)}$. The absolute value in $\mathbb{R}$ will be denoted by $|\cdot|_{\mathbb{R}}$.

We start setting some hypotheses for the problem (1.1). Firstly, we shall assume that

$$\rho \in I_\rho \quad \text{and} \quad p \in I_p, \tag{2.1}$$

where

$$I_\rho := (0, \infty) \text{ if } n = 1, 2 \quad \text{or} \quad I_\rho := \left(0, \frac{2}{3}\right) \cup [1, 2] \text{ if } n = 3, \tag{2.2}$$

$$I_p := (2, \infty) \text{ if } n = 1, 2 \quad \text{or} \quad I_p := \left(\frac{11}{5}, 3\right] \text{ if } n = 3. \tag{2.3}$$

Secondly, we assume that:

(H_1) $M \in C^1([0, \infty), \mathbb{R})$ is such that

$$M(\lambda) \geq m_0 > 0, \quad \forall \lambda \in [0, \infty),$$

(H_2) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable and $C^1(\mathbb{R}^+)$ function such that

$$g(0) > 0, \quad g'(t) < 0, \quad 1 - \int_0^\infty g(s)ds = l > 0.$$

We will need the very useful relation

$$\int_0^t g(t-\tau)(\nabla u(\tau), \nabla u_t(t))d\tau = \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}(g \diamond \nabla u)'(t) + \frac{d}{dt}\left\{\frac{1}{2}\left(\int_0^t g(s)ds\right)|\nabla u(t)|^2\right\} - \frac{1}{2}g(t)|\nabla u(t)|^2 \tag{2.4}$$

that can be checked directly, where

$$(g \diamond y)(t) = \int_0^t g(t-s)|y(t) - y(s)|^2 ds. \tag{2.5}$$

Let us denote $\hat{M}(s) = \int_0^s M(\tau)d\tau$.

3. THE ASSOCIATED NONHOMOGENEOUS EQUATION WITH KIRCHHOFF NONLINEARITY TERM

In this section, we prove the global existence of solution for the nonhomogeneous equation with Kirchhoff nonlinearity term associated with the problem (1.1).

Definition 3.1. Let $T > 0$ and $f \in L^2(0,T; H^{-1}(\Omega))$. A function

$$u \in C^1([0,T], H_0^1(\Omega))$$

is a weak solution for

$$\begin{cases} u_t + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds = f, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \tag{3.1}$$

if for any $\omega \in H_0^1(\Omega)$ and $t \in [0,T]$,

$$\begin{cases} (u_t, \omega) + M(\|u(t)\|^2)(\nabla u(t), \nabla\omega) + (\nabla u_{tt}(t), \nabla\omega) \\ - \int_0^t g(t-s)(\nabla u(s), \nabla\omega)ds = \langle f, \omega\rangle, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases}$$

is satisfied.

Theorem 3.2. *Assume* (H$_1$), (H$_2$) *and that* $m_0 + l - 1 > 0$. *Let* $f \in L^2(0,T; H^{-1}(\Omega))$ *and* $u_0, u_1 \in H_0^1(\Omega)$. *Then there exists a unique weak solution* u *for the problem* (3.1). *Further,* u_{tt} *belongs to the class* $L^\infty(0,T; H_0^1(\Omega))$.

Proof. Let $(\omega_\nu)_{\nu\in\mathbb{N}} \subset H_0^1(\Omega)\cap H^2(\Omega)$ be a basis of $L^2(\Omega)$ from the eigenvectors of the operator $-\Delta$. It is known that $(\omega_\nu)_{\nu\in\mathbb{N}}$ is a complete and orthonormal system of $H_0^1(\Omega)$. Let $V_m = \text{span}[\omega_1,\dots,\omega_m]$ be the space generated by the first m eigenvector of the system $(\omega_\nu)_{\nu\in\mathbb{N}}$ and $u^m(t) = \sum_{j=1}^m g_{jm}(t)\omega_j$ be a solution in the interval $[0,t_m)$ of the approximated problem

$$M(\|u^m(t)\|^2)(\nabla u^m(t),\nabla\omega) + (\nabla u^m_{tt}(t),\nabla\omega) - \int_0^t g(t-s)(\nabla u^m(s),\nabla\omega)ds \\ +\ (u^m_t,\omega) = \langle f,\omega\rangle, \quad \forall\omega\in V_m, \tag{3.2}$$

$$u^m(0) = u_{0m} \to u_0 \text{ strongly in } H_0^1(\Omega), \tag{3.3}$$

$$u^m_t(0) = u_{1m} \to u_1 \text{ strongly in } H_0^1(\Omega). \tag{3.4}$$

The system (3.2)–(3.4) has a local solution (u^m) in $[0,t_m]$ by virtue of Carathéodory's theorem (see [4]).

3.1. FIRST A PRIORI ESTIMATE

Setting $\omega = u^m_t$ in (3.2) and using (2.4) and notation (2.5), we obtain

$$\frac{d}{dt}\Big\{\frac{1}{2}\hat{M}(\|u^m\|^2) + \frac{1}{2}|\nabla u^m_t|^2 + \frac{1}{2}(g\diamond\nabla u^m) - \frac{1}{2}\Big(\int_0^t g(s)ds\Big)\|u^m\|^2\Big\}$$
$$= -\frac{1}{2}g(t)|\nabla u^m|^2 + \frac{1}{2}(g'\diamond\nabla u^m) - |u^m_t|^2 + \langle f,u^m_t\rangle.$$

We know that $g'<0$ by hypothesis (H_2), which gives us

$$\frac{d}{dt}\Big\{\frac{1}{2}\Big[\hat{M}(\|u^m\|^2) - \Big(\int_0^t g(s)ds\Big)\|u^m\|^2\Big] + \frac{1}{2}|\nabla u^m_t|^2 + \frac{1}{2}(g\diamond\nabla u^m)\Big\}$$
$$\le \langle f,u^m_t\rangle \le \frac{1}{2}\Big(\|f\|^2_{H^{-1}(\Omega)} + \|u^m_t\|^2\Big).$$

Integrating (3.5) from 0 to $t\in[0,T]$ and regarding that $m_0+l-1>0$, it yields

$$(m_0+l-1)\|u^m\|^2 + \frac{1}{2}\|u^m_t\|^2 \le C + \int_0^t \|u^m_t(s)\|^2 ds. \tag{3.5}$$

Employing the Grönwall inequality in (3.5), we find a constant $C_1>0$ such that

$$\|u^m\|^2 + \|u^m_t\|^2 \le C_1. \tag{3.6}$$

We can extend, thereby, the approximated solution $u^m(t)$ to the interval $[0,T]$. It is inferred particularly from (3.6) that

$$(u^m),(u^m_t) \text{ are bounded in } L^\infty(0,T;H_0^1(\Omega)).$$

3.2. SECOND A PRIORI ESTIMATE

Taking now $\omega = u_{tt}^m$ in (3.2), we have

$$\begin{aligned}
\|u_{tt}^m\|^2 &= -(u_t^m, u_{tt}^m) - M(\|u^m(t)\|^2)(\nabla u^m, \nabla u_{tt}^m) \\
&\quad + \int_0^t g(t-s)(\nabla u^m(s), \nabla u_{tt}^m)ds + \langle f, u_{tt}^m\rangle \\
&\quad + \frac{C_p^2}{4\eta}\|u_t^m\|^2 + C_p^2\eta\|u_{tt}\|^2 + \overline{m}\left(\frac{1}{4\eta}\|u^m\|^2 + \eta\|u_{tt}^m\|^2\right) \\
&\quad + \|u_{tt}^m\| \int_0^t |g(t-s)|\|u^m(s)\|ds + \|f\|_{H^{-1}(\Omega)}|u_{tt}^m| \\
&\le \frac{C_p^2}{4\eta}\|u_t^m\|^2 + C_p^2\eta\|u_{tt}^m\|^2 + \overline{m}\left(\frac{1}{4\eta}\|u^m\|^2 + \eta\|u_{tt}^m\|^2\right) \\
&\quad + \eta\|u_{tt}\|^2 + \frac{1}{4\eta}\int_0^t |g(t-s)|\|u^m(s)\|ds \\
&\quad + \frac{1}{4\eta}\|f\|_{H^{-1}(\Omega)}^2 + C_p^2\eta\|u_{tt}^m\|^2.
\end{aligned}$$

The first a priori estimate allows us to find a constant $C > 0$ for that

$$\|u_{tt}\|^2 \le (2C_p^2 + \overline{m} + 1)\eta\|u_{tt}^m\|^2 + \frac{1}{4\eta}\|f\|_{H^{-1}(\Omega)}^2 + C.$$

The constant η can be chosen small enough for that $\eta(2C_p^2 + \overline{m} + 1) \le \frac{1}{2}$. Thus, we find a positive constant L_2 independently on m and t for which

$$\|u_{tt}^m\|^2 \le L_2.$$

Here we conclude the second a priori estimate, from which we have

$$(u_{tt}^m) \text{ is bounded in } L^\infty(0, T; H_0^1(\Omega)). \tag{3.7}$$

3.3. THIRD A PRIORI ESTIMATE

We set $\omega = -\Delta u_t^m$ in (3.2). We have

$$\begin{aligned}
&M(\|u^m\|^2)\frac{d}{dt}|\Delta u^m|^2 + \frac{d}{dt}|\Delta u^m|^2 \\
&\le \frac{1}{2}\|g\|_{L^\infty(0,\infty)}\int_0^t |\Delta u^m(s)|^2 ds \\
&\quad + \frac{1}{2}\|g\|_{L^1(0,\infty)}|\Delta u_t^m|^2 + \|u_t^m\|^2 + \frac{1}{2}\|f\|_{H^{-1}(\Omega)}^2 + \frac{1}{2}|\Delta u_t^m|^2.
\end{aligned}$$

By owing to first and second a priori estimates, we get

$$\begin{aligned} \frac{d}{dt}\left(M(\|u^m\|^2)|\Delta u^m|^2 + |\Delta u_t^m|^2\right) \leq \frac{d}{dt} M(\|u^m\|^2)|\Delta u^m|^2 + \frac{1}{2}\|g\|_{L^1(0,\infty)}|\Delta u_t^m|^2 \\ + \frac{1}{2}\|g\|_{L^\infty(0,\infty)}\int_0^t |\Delta u^m(s)|^2 ds + C. \end{aligned} \tag{3.8}$$

Since $\{\|u^m(t)\|\}$ is uniformly bounded for $t \in [0,T]$ and $m \in \mathbb{N}$, and $M \in C^1[0,\infty)$, there exists $\tilde{M} > 0$ such that $\frac{d}{dt}(M(\|u\|^2)) \leq \tilde{M}$. Integrating from 0 to t inequality (3.8) we obtain

$$\begin{aligned} M(\|u^m\|^2)|\Delta u^m|^2 + |\Delta u_t^m|^2 &\leq \tilde{M}\int_0^t |\Delta u^m(s)|^2 ds + \frac{1}{2}\|g\|_{L^1(0,\infty)}\int_0^t |\Delta u_t^m(s)|^2 ds \\ &\quad + \frac{1}{2}\|g\|_{L^\infty(0,\infty)} T\int_0^t |\Delta u^m(s)|^2 ds + C \\ &= C_1\int_0^t |\Delta u^m(s)|^2 ds + C_2\int_0^t |\Delta u_t^m(s)|^2 ds + C \\ &\leq C_3\int_0^t (m_0|\Delta u^m(s)|^2 + |\Delta u_t^m(s)|^2) ds + C. \end{aligned}$$

Hence,

$$m_0|\Delta u^m|^2 + |\Delta u_t^m|^2 \leq C_3\int_0^t (m_0|\Delta u^m(s)|^2 + |\Delta u_t^m(s)|^2) ds + C.$$

From the Grönwall inequality it follows there is a constant $L_3 > 0$ independently on m and t such that

$$m_0|\Delta u^m|^2 + |\Delta u_t^m|^2 \leq L_3.$$

Particularly,

$$|\Delta u^m|^2 \text{ is bounded in } L^\infty(0,T;L^2(\Omega)).$$

Combining the above boundedness with the first estimate a priori, we obtain

$$(u^m) \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{3.9}$$

$$(u_t^m) \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)). \tag{3.10}$$

3.4. PASSAGE TO THE LIMIT

In this subsection, the reader interested in further details on the compactness argument involving Aubin Lion's Lemma used in the passage to limit in the related Kirchhoff nonlinearity, we strongly recommend [2, 8].

From the boundedness (3.7) and (3.9), it follows from the Banach–Alaoglu–Bourbaki Theorem (see [2]) that

$$\nabla u^m \overset{*}{\rightharpoonup} \nabla u \text{ weakly star in } L^\infty(0,\tau;L^2(\Omega)), \tag{3.11}$$

$$\nabla u_{tt}^m \overset{*}{\rightharpoonup} \nabla u_{tt} \text{ weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.12}$$

with $0<\tau<T$.

Since

$$L^\infty(0,T;L^2(\Omega)) = (L^1(0,T;L^2(\Omega)))',$$

from (3.12) we have, for every $\varphi \in L^1(0,T;L^2(\Omega))$, that

$$\int_0^T\int_\Omega \nabla u_{tt}^m(t,x)\varphi(t)dxdt \to \int_0^T\int_\Omega \nabla u_{tt}(t,x)\varphi(t)dxdt. \tag{3.13}$$

Taking, in particular, $\varphi(t)(x) = \nabla\omega(x)\theta(t)$ in (3.13), where $\omega \in H_0^1(\Omega)$ and $\theta \in D(0,T)$, we obtain

$$\int_0^T (\nabla u_{tt}^m(t),\nabla\omega)\theta(t)dt \to \int_0^T (\nabla u_{tt}(t),\nabla\omega)\theta(t)dt. \tag{3.14}$$

Similarly, since

$$L^\infty(0,\tau;L^2(\Omega)) = (L^1(0,\tau;L^2(\Omega)))',$$

then, for every $\psi \in L^1(0,\tau;L^2(\Omega))$, we have

$$\int_0^\tau\int_\Omega \nabla u^m(s,x)\psi(t)dxds \to \int_0^\tau\int_\Omega \nabla u(s,x)\psi(t)dxds.$$

Hence, putting $\psi(s)(x) = g(\tau-s)\nabla\omega(x)$, it comes

$$\int_0^\tau g(\tau-s)(\nabla u^m(s),\nabla\omega)ds \to \int_0^\tau g(\tau-s)(\nabla u(s),\nabla\omega)ds, \tag{3.15}$$

for all $\omega \in H_0^1(\Omega)$.

Multiplying (3.15) by $\theta(\tau)$, it follows from the Dominated Convergence Theorem that

$$\int_0^T\int_0^\tau g(\tau-s)(\nabla u^m(s),\nabla\omega)ds\theta(\tau)d\tau \to \int_0^T\int_0^\tau g(\tau-s)(\nabla u(s),\nabla\omega)ds\theta(\tau)d\tau. \tag{3.16}$$

From the boundedness (3.10) and by the continous injections

$$L^{\infty}(0,T;H_0^1(\Omega)) \hookrightarrow L^2(0,T;H_0^1(\Omega)),$$

there exists, from Kakutani's Theorem (see [2]), a subsequence of (u^m), also denoted by (u^m), such that

$$u_t^m \rightharpoonup u_t \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{3.17}$$

The convergence (3.17) means that, for every $\alpha \in L^2(0,T;H^{-1}(\Omega))$, the convergence

$$\int_0^T \int_\Omega \alpha(t) u_t^m(t) dx dt \to \int_0^T \int_\Omega \alpha(t) u_t(t) dx dt \tag{3.18}$$

holds. Taking, in particular, $\alpha(t)(x) = \omega(x)\theta(t)$, with $\omega \in H_0^1(\Omega)$ and $\theta \in D(0,T)$, it follows that

$$\int_0^T (u_t^m(t), \omega)\theta(t) dt \to \int_0^T (u_t(t), \omega)\theta(t) dt \tag{3.19}$$

for all $\omega \in H_0^1(\Omega)$ and $\theta \in D(0,T)$.

We need to consider the variational limit corresponding to the nonlinearity M.

Note that from (3.9), (3.10) we have

$$(u^m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)), \tag{3.20}$$

$$(u_t^m) \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{3.21}$$

Thus putting, in the notations of Lemma 2.5, $B_0 = H_0^1(\Omega) \cap H^2(\Omega)$, $B = H_0^1(\Omega)$, $B_1 = L^2(\Omega)$, $p_0 = p_1 = 2$, and

$$W = \{u : u \in L^{p_0}(0,T,B_0), u_t \in L^{p_1}(0,T,B_1)\}$$

equipped with the norm

$$\|w\| = \|u\|_{L^{p_0}(0,T,B_0)} + \|u_t\|_{L^{p_1}(0,T,B_1)}$$

results from (3.20) and (3.21) that

$$(u^m) \text{ is bounded in } W.$$

Then, by the Aubin–Lions Lemma (Lemma 2.5) we obtain a subsequence of (u_m) for which we continue to denote in the same way, such that

$$u^m \to u \text{ strongly in } L^2(0,T;H_0^1(\Omega)). \tag{3.22}$$

Hence, using

$$\Big| \|u^m\| - \|u\| \Big|^2 \le \|u^m - u\|^2,$$

we have

$$\|u^m\| \to \|u\| \text{ strongly in } L^2(0,T).$$

It follows from Lemma 2.4 that, up to a subsequence, that

$$\|u^m\| \to \|u\| \text{ a.e. in } [0,T].$$

Since M is continuous, using the formula

$$\int_0^T |M(\|u^m(s)\|^2) - M(\|u(s)\|^2)|^2 ds$$
$$= \int_0^T M(\|u^m(s)\|^2)^2 ds - 2\int M(\|u^m(s)\|^2) M(\|u(s)\|^2) ds + \int_0^T M(\|u(s)\|^2)^2 ds$$

it follows from the Dominated Convergence Theorem that

$$M(\|u^m\|^2) \to M(\|u\|^2) \text{ strongly in } L^2(0,T). \tag{3.23}$$

Therefore,

$$(M(\|u^m\|^2)\nabla u^m, w) \to (M(\|u\|^2)\nabla u, w) \text{ strongly in } L^1(0,T) \tag{3.24}$$

for every $w \in H_0^1(\Omega)$. Indeed,

$$\begin{aligned}
&\int_0^T |(M(\|u^m(s)\|^2)\nabla u^m(s), w) - (M(\|u(s)\|^2)\nabla u(s), w)|_{\mathbb{R}} ds \\
&\leq \int_0^T |M(\|u^m(s)\|^2)\nabla u^m(s) - M(\|u^m(s)\|^2)\nabla u(s)||\omega| ds \\
&\quad + \int_0^T |M(\|u^m(s)\|^2)\nabla u(s) - M(\|u(s)\|^2)\nabla u(s)||\omega| ds \\
&\leq \|M(\|u^m(s)\|^2)\|_{L^2(0,T)} \|u^m(s) - u(s)\|_{L^2(0,T;H_0^1(\Omega))} |\omega| \\
&\quad + \|M(\|u^m(s)\|^2) - M(\|u(s)\|^2)\|_{L^2(0,T)} \|u\|_{L^2(0,T;H_0^1(\Omega))} |\omega|.
\end{aligned}$$

By the convergences (3.22) and (3.23), we obtain (3.24). Particularly the convergence in (3.24) is also valid in $D'(0,T)$ for every $w \in H_0^1(\Omega)$, i.e.

$$\int_0^T (M(\|u^m(t)\|^2)\nabla u^m(t), w)\theta(t) dt \to \int_0^T (M(\|u(t)\|^2)\nabla u(t), w)\theta(t) dt. \tag{3.25}$$

Multiplying (3.2) by $\theta \in \mathcal{D}(0;T)$ and integrating the obtained result over $(0;T)$, it holds that

$$\begin{aligned}&\int_0^T M(\|u^m(t)\|^2)(\nabla u^m(t),\nabla\omega)\theta(t)dt + \int_0^T (\nabla u^m_{tt}(t),\nabla\omega)\theta(t)dt\\ &-\int_0^T\int_0^\tau g(\tau-s)(\nabla u^m(s),\nabla\omega)\theta(\tau)dsd\tau \\ &+\int_0^T (u^m_t,\omega)\theta(t)dt = \int_0^T (f(t),\omega)\theta(t)dt, \quad \forall\omega\in V_m.\end{aligned} \tag{3.26}$$

Taking the limit in (3.26) with $m\to\infty$ by using the convergences (3.14), (3.19), (3.15) and (3.25), for all $w\in\bigcup_{m\in\mathbb{N}} V_m$, yields

$$\begin{aligned}&\int_0^T M(\|u(t)\|^2)(\nabla u(t),\nabla\omega)\theta(t)dt + \int_0^T (\nabla u_{tt}(t),\nabla\omega)\theta(t)dt\\ &\quad -\int_0^T\int_0^\tau g(\tau-s)(\nabla u(s),\nabla\omega)\theta(\tau)dsd\tau + \int_0^T (u_t,\omega)\theta(t)dt\\ &= \int_0^T (f(t),\omega)\theta(t)dt.\end{aligned}$$

This means that

$$M(\|u\|^2)(\nabla u,\nabla\omega)+(\nabla u_{tt},\nabla\omega)-\int_0^t g(t-s)(\nabla u(s),\nabla\omega)ds+(u_t,\omega) = (f(t),\omega) \tag{3.27}$$

holds in $\mathcal{D}'(0,T)$ for all $\omega\in\bigcup_{m\in\mathbb{N}} V_m$.

As $\bigcup_{m\in\mathbb{N}} V_m$ is dense in $H^1_0(\Omega)$, (3.27) is valid for all ω in $H^1_0(\Omega)$. Since we proved $u, u_t, u_{tt} \in L^2(0,T;H^1_0(\Omega))$, then owing to Proposition 2.3, results that $u\in C^1([0,T],H^1_0(\Omega))$.

3.5. VERIFICATION OF INITIAL DATA

Recalling the boundedness of $\{u^m_t\}$ in $L^2(0,T;H^1_0(\Omega))$ and the compact embedding $L^2(0,T;H^1_0(\Omega)) \subset\subset L^2(0,T;H^{-1}(\Omega))$, we can get, up to a subsequence, that

$$u^m_t \to u_t \text{ strongly in } L^2(0,T;H^{-1}(\Omega)). \tag{3.28}$$

Hence, the convergences (3.22) with (3.28) and Proposition 2.3 yield

$$u^m \to u \text{ strongly in } C(0,T;H^1_0(\Omega)).$$

Hence, $u(0)$ makes sense and

$$u^m(0) \to u(0) \text{ in } H_0^1(\Omega).$$

Also, we assumed in the Faedo–Galerkin's method that

$$u^m(0) \to u_0 \text{ in } H_0^1(\Omega).$$

Hence, $u(x,0) = u_0(x)$.

Next, recall that

$$(u_t^m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{3.29}$$

$$(u_{tt}^m) \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{3.30}$$

Thus, (3.29) and (3.30) and Lemma 2.5 yield to obtain, up to a subsequence,

$$u_t^m \to u_t \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.31}$$

Similarly as before, by the boundedness of $\{u_{tt}^m\}$ in $L^2(0,T;H_0^1(\Omega))$ and the compact embedding $L^2(0,T;H_0^1(\Omega)) \subset\subset L^2(0,T;H^{-1}(\Omega))$, we have, up to a subsequence, that

$$u_{tt}^m \to u_{tt} \ \text{ strongly in } L^2(0,T;H^{-1}(\Omega)). \tag{3.32}$$

Therefore, the convergences (3.31) and (3.32) and Proposition 2.3 yield

$$u_t^m \to u_t \ \text{ strongly in } C(0,T;L^2(\Omega)).$$

Hence $u_t(0)$ makes sense and

$$u_t^m(0) \to u_t(0) \text{ in } L^2(\Omega).$$

Since in Faedo–Galerking's method we assumed that

$$u_t^m(0) \to u_1 \text{ in } H_0^1(\Omega),$$

we also obtain $u_t(x,0) = u_0(x)$.

3.6. UNIQUENESS

We shall prove first the continuous dependence of initial conditions for the problem (3.1) and then the uniqueness of the solution follows as a consequence. Let us consider u, v two weak solutions of (3.1) corresponding to the initial condition $u(0) = u_0$, $u_t(0) = u_1$ and $v(0) = v_0$, $v_t(0) = v_1$, respectively. Setting $U = u - v$, we see that U solves the equation

$$\begin{aligned}&(\nabla U_{tt}, \nabla\omega) - \int_0^t g(t-s)(\nabla U(s), \nabla\omega)ds + (U_t, \omega)\\ &= (M(|\nabla u|^2)\nabla v - M(|\nabla v|^2)\nabla v, \nabla\omega) \quad \text{for all } \omega \in H_0^1(\Omega).\end{aligned} \tag{3.33}$$

Setting $\omega = U_t$ in (3.33), and next using (2.4), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[|\nabla U_t|^2 + \hat{M}(|\nabla U|^2) - \int\limits_0^t g(s)ds|\nabla U|^2 + g\diamond\nabla U\Big] + |U_t|^2 \\ &= \Big(M(|\nabla v|^2)\nabla v - M(|\nabla u|^2)\nabla u + M(|\nabla U|^2)\nabla U, \nabla U_t\Big) \\ &\quad + \frac{1}{2}(g'\diamond\nabla U)(t) - \frac{1}{2}g(t)|\nabla U|^2 \le G(t)\end{aligned} \tag{3.34}$$

where

$$G(t) = \Big(M(|\nabla v|^2)\nabla v - M(|\nabla u|^2)\nabla u + M(|\nabla U|^2)\nabla U, \nabla U_t\Big).$$

Next we estimate $G(t)$. By the Mean Value Theorem, there exists $\mu \in (0,1)$ such that

$$M(\|u\|^2) - M(\|v\|^2) = M'(\|u\|^2 + \mu(\|v\|^2 - \|u\|^2))(\|u\|^2 - \|v\|^2).$$

We have

$$\begin{aligned}|M(\|v\|^2)\nabla v - M(\|u\|^2)\nabla u| &\le \Big|M(\|v\|^2)\nabla v - M(\|v\|^2)\nabla u\Big| \\ &\quad + \Big|M(\|u\|^2)\nabla u - M(\|v\|^2)\nabla u\Big| M(\|v\|^2)\|u-v\| \\ &\quad + \Big|M'(\|u\|^2 + \mu(\|v\|^2 - \|u\|^2))|_{\mathbb{R}}|\|u\| - \|v\||_{\mathbb{R}}\Big|\nabla u| \\ &\le \tilde{m}\|u - v\|\end{aligned} \tag{3.35}$$

where $\tilde{m}$ is obtained from the boundedness of M and M' in $[0,T]$, and the fact that $u, v \in L^\infty(0,T;H_0^1(\Omega))$.

We have from Hölder and Young's inequalities

$$\Big(M(|\nabla v|^2)\nabla v - M(|\nabla u|^2)\nabla u, \nabla U_t\Big) \le \tilde{m}\Big(\frac{1}{2}|\nabla U|^2 + \frac{1}{2}|\nabla U_t|^2\Big). \tag{3.36}$$

Now combining (3.34) with (3.36), and then integrating over $[0,t]$, it follows

$$\begin{aligned}\frac{1}{2}\Big[|\nabla U_t|^2 + \hat{M}(|\nabla U|^2) - \int\limits_0^t g(s)ds|\nabla U|^2\Big] &\le \frac{|\nabla U_1|^2 + \hat{M}(|\nabla U_0|^2)}{2} \\ &\quad + \frac{\tilde{m}}{2}\int\limits_0^t \left(|\nabla U(s)|^2 + |\nabla U_t(s)|^2\right)ds,\end{aligned} \tag{3.37}$$

where $U_0 = u_0 - v_0$ and $U_1 = u_1 - v_1$. The condition $m_0 + l - 1 > 0$ implies $\hat{M}(|\nabla U|^2) - g(t)|\nabla U|^2 > 0$. Thus,

$$\begin{aligned}\frac{m_0 + l - 1}{2}|\nabla U|^2 + \frac{1}{2}|\nabla U_t|^2 &\le \frac{|\nabla U_1|^2 + \hat{M}(|\nabla U_0|^2)}{2} \\ &\quad + \frac{\tilde{m}}{2}\int\limits_0^t \left(|\nabla U(s)|^2 + |\nabla U_t(s)|^2\right)ds.\end{aligned}$$

From the Grönwall inequality it follows that

$$|\nabla U|^2 + |\nabla U_t|^2 \le L\frac{|\nabla U_1|^2 + \hat{M}(|\nabla U_0|^2)}{2}. \tag{3.38}$$

This proves the continuous dependence on the initial condition and particularly the uniqueness of the solution. □

4. LOCAL WEAK SOLUTION

Definition 4.1. We say that $u \in C^1([0,T], H_0^1(\Omega))$ is a weak solution for

$$\begin{cases} |u_t|_{\mathbb{R}}^{\rho} u_{tt} + M(\|u\|^2)(-\Delta u) - \Delta u_{tt} + \int\limits_0^t g(t-s)\Delta u(s)ds + u_t = u|u|_{\mathbb{R}}^{p-2}\ln|u|_{\mathbb{R}}^k, \\ u = 0 \text{ on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x) \text{ in } \Omega, \\ u_t(x,0) = u_1(x) \text{ in } \Omega, \end{cases}$$

on $[0,T]$ if for any $\omega \in H_0^1(\Omega)$ and $t \in [0,T]$, we have

$$\begin{cases} \frac{d}{dt}(|u_t(t)|^{\rho} u_t(t), \omega) + M(\|u(t)\|^2)(\nabla u(t), \nabla\omega) + (\nabla u_{tt}(t), \nabla\omega) \\ -\int\limits_0^t g(t-s)(\nabla u(s), \nabla\omega)ds + (u_t, \omega) = (u(t)|u(t)|^{p-2}\ln|u(t)|^k, \omega), \\ u(0) = u_0, \;\; u_t(0) = u_1. \end{cases} \tag{4.1}$$

For our purposes hereafter, let us define

$$\mathbf{W} := \Big\{ w : w, w_t \in C(0,T; H_0^1(\Omega)), w_{tt} \in L^\infty(0,T; H_0^1(\Omega)) \Big\}$$

equipped with the norm

$$\|w\|_{\mathbf{W}}^2 := \alpha\|w\|_{L^\infty(0,T;H_0^1(\Omega))}^2 + \delta\|w_t\|_{L^\infty(0,T;H_0^1(\Omega))}^2 + \gamma\|w_{tt}\|_{L^2(0,T;H_0^1(\Omega))}^2,$$

where $\alpha := \frac{m_0+l-1}{2}$ and δ and γ are large and left to be defined later.

It is easy to check that $\mathbf{W}$ is a Banach spaces with the norm $\|\cdot\|_{\mathbf{W}}$.

The next lemma states that for a weaker condition than a contraction map, a fixed point existence is still valid, but not necessarily the uniqueness. The proof of the lemma can be done in a very similar way as the Banach fixed point Theorem, i.e. by showing that any sequence generated by iterations of such map starting from any initial point is a Cauchy sequence.

Lemma 4.2. *Let (X, d) be a complete metric space and $F : X \to X$ an application satisfying $d(F(x), F(y)) \leq q_1 d(x, y)^\rho + q_2 d(x, y)$, for all $x, y \in X$, for which $0 < q_1 < 1$, $0 < q_2 < 1$, $\rho > 0$. Then F admits a fixed point $x^* \in X$.*

Theorem 4.3. *Let $u_0, u_1 \in H_0^1(\Omega)$ and assume that H1–H2, $m_0 + l - 1 > 0$ and (2.1) are valid. Then the problem (1.1) has a local weak solution u in $\mathbf{W}$ for T small enough, which is unique if $\rho \in I_\rho \cap \{\rho \geq 1\}$.*

Proof. Let $M > 0$ sufficiently large and $T > 0$ and denote $\mathbf{Z}(M, T)$ the class of functions w belonging to $\mathbf{W}$, satisfying $w(0) = u_0$, $w_t(0) = u_1$ and $\|w\|_{\mathbf{W}} \leq M$. Notice that $\mathbf{Z}(M, T)$ in nonempty since $w(t) = u_0 + tu_1$ is an element of this set for some $M > 0$. Let us consider the application $A : \mathbf{Z}(M, T) \to \mathbf{W}$ defined in the following way. For each $v \in \mathbf{Z}(M, T)$, take $u := A(v)$ as the unique solution of the problem (3.1) with $f = v|v|^{p-2} \ln |v|^k - |v_t|^\rho v_{tt}$. We shall prove that A is a contraction from $\mathbf{Z}(M, T)$ from itself. In order to prove that $u \in \mathbf{Z}(M, T)$, we use the inequality obtained from the energy of the equation (3.1). For our goal we are multiplying this equation by u_t and integrate over Ω and obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}\Big[\hat{M}(\|u\|^2) - \int_0^t g(s)ds\|u\|^2\Big] + \frac{1}{2}|\nabla u_t|^2 + \frac{1}{2}(g \diamond \nabla u)(t)\Big\} + |u_t|^2 \\
&= -\int_\Omega |v_t|^\rho v_{tt} u_t dx + \int_\Omega v|v|^{p-2} \ln |v|^k u_t dx.
\end{aligned} \tag{4.2}$$

By using the generalized Hölder's inequality regarding that

$$\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$$

and Young's inequality, we estimate the first term of the right side of (4.2) as

$$-\int_\Omega |v_t|^\rho v_{tt} u_t dx \leq \frac{1}{2}(\|v_t\|_{2(\rho+1)}^\rho \|v_{tt}\|_{2(\rho+1)})^2 + \frac{1}{2}|u_t|^2. \tag{4.3}$$

In the second term, we employ Young's inequality and notice the elemental logarithmic inequality

$$\Big|\xi|\xi|^{p-2} \ln \xi\Big| \leq a_0(|\xi| + |\xi|^p), \quad \xi \in \mathbb{R} \setminus \{0\}, \tag{4.4}$$

which leads us to

$$\int_\Omega v|v|^{p-2} \ln |v|^k u_t dx \leq \frac{a_0^2}{2} \int_\Omega \Big(|v|_{\mathbb{R}} + |v|_{\mathbb{R}}^p\Big)^2 dx + \frac{1}{2} \int_\Omega |u_t|^2 dx. \tag{4.5}$$

Denoting

$$\Gamma(t) \coloneqq \frac{1}{2}\Big[\hat{M}(\|u\|^2) - \int\limits_0^t g(s)ds\|u\|^2\Big] + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(g \diamond \nabla u)(t)$$

and using (4.3) and (4.5) in (4.2) and then integrating (4.2) from 0 to T, we obtain

$$\begin{aligned}\Gamma(t) &\leq \tilde{a}_0 T\Big(\|v\|^2_{L^\infty(0,T,H_0^1(\Omega))} + \|v\|^{p+1}_{L^\infty(0,T,H_0^1(\Omega))} + \|v\|^{2p}_{L^\infty(0,T,H_0^1(\Omega))}\Big)\\ &\quad + \|v_t\|^{2\rho}_{L^\infty(0,T,H_0^1(\Omega))}\|v_{tt}(s)\|^2_{L^2(0,T;H_0^1(\Omega))},\end{aligned}$$

where $\tilde{a}_0$ is the aftermath constant obtained from the embeddings $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{2p}(\Omega)$.

Taking the essential supremum in $t \in [0,T]$ in the inequality (4.6), we obtain

$$\begin{aligned}&\frac{m_0+l-1}{2}\|u\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \delta\|u_t\|^2_{L^\infty(0,T;H_0^1(\Omega))}\\ &\leq \tilde{a}_0 T\Big(\|v\|^2_{L^\infty(0,T,H_0^1(\Omega))} + \|v\|^{p+1}_{L^\infty(0,T,H_0^1(\Omega))} + \|v\|^{2p}_{L^\infty(0,T,H_0^1(\Omega))}\Big)\\ &\quad + \|v_t\|^{2\rho}_{L^\infty(0,T,H_0^1(\Omega))}\|v_{tt}(s)\|^2_{L^2(0,T;H_0^1(\Omega))}\\ &\quad +2\delta\Bigg\{\tilde{a}_0 T\Big(\|v\|^2_{L^\infty(0,T,H_0^1(\Omega))} + \|v\|^{p+1}_{L^\infty(0,T,H_0^1(\Omega))} + \|v\|^{2p}_{L^\infty(0,T,H_0^1(\Omega))}\Big)\\ &\qquad + \Big(\frac{M}{\sqrt{\delta}}\Big)^{2\rho}\Big(\frac{M}{\sqrt{\gamma}}\Big)^2\Bigg\}.\end{aligned} \tag{4.6}$$

Hence, choosing $T>0$ small and γ large enough, we obtain

$$\frac{m_0+l-1}{2}\|u\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \delta\|u_t\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq \frac{M^2}{2}. \tag{4.7}$$

Next, we multiply both sides of (3.1) by u_{tt} and integrate it over Ω, and we obtain

$$\begin{aligned}\int\limits_\Omega |v_t|^\rho v_{tt}u_{tt}dx + \|u_{tt}\|^2 &= -(u_t,u_{tt}) - M(\|u\|^2)(\nabla u, \nabla u_{tt})\\ &\quad + \int\limits_0^t g(t-s)(\nabla u(s), \nabla u_{tt})ds\\ &\quad + (v|v|^{p-2}\ln|v|^k, u_{tt}).\end{aligned}$$

Hence,

$$
\begin{aligned}
\|u_{tt}\|^2 \leq{}& C_p\|v_t\|^{\rho}_{2(\rho+1)}\|v_{tt}\|_{2(\rho+1)}\|u_{tt}\| + C_p^2\|u_t\|\|u_{tt}\| \\
&+ \overline{m}\|u\|\|u_{tt}\| + \frac{1}{4\eta}\int_0^t g(t-s)\|u(s)\|^2 ds + \eta\int_0^t g(t-s)ds\|u_{tt}\|^2 \\
&+ \frac{a_0^2}{4\eta}\int_\Omega (|v|_{\mathbb{R}} + |v|^p_{\mathbb{R}})^2 dx + \eta C_p^2\|u_{tt}\|^2 \\
\leq{}& C_p C_*^{\rho+1}\|v_t\|^\rho\|v_{tt}\|^2\|u_{tt}\| + \eta C_p^4\|u_{tt}\|^2 + \frac{1}{4\eta}\|u_t\|^2 + \eta\overline{m}^2\|u_{tt}\|^2 \\
&+ \frac{1}{4\eta}\|u\|^2 + \frac{1}{4\eta}\int_0^t g(t-s)\|u(s)\|^2 ds + \eta(1-l)\|u_{tt}\|^2 \\
&+ \frac{\tilde{a}_0}{4\eta}(\|u\|^2 + \|v\|^{p+1} + \|v\|^{2p}) + \eta C_p^2\|u_{tt}\|^2.
\end{aligned}
$$

Gathering the terms with $\|u_{tt}\|^2$, we obtain

$$
\begin{aligned}
&\Big[1 - \eta\Big(C_p^4 + \overline{m} + (1-l) + C_p^2\Big)\Big]\|u_{tt}\|^2 \\
&\leq C_p C_*^{\rho+1}\|v_t\|^\rho_{L^\infty(0,T;H_0^1(\Omega))}\|v_{tt}\|\|u_{tt}\| \\
&\quad + \frac{1}{4\eta}\|u_t\|_{L^\infty(0,T;H_0^1(\Omega))} + \frac{1}{4\eta}\|u\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \frac{1-l}{4\eta}\|u\|^2_{L^\infty(0,T;H_0^1(\Omega))} \\
&\quad + \frac{\tilde{a}_0}{4\eta}\Big(\|v\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \|v\|^{p+1}_{L^\infty(0,T;H_0^1(\Omega))} + \|v\|^{2p}_{L^\infty(0,T;H_0^1(\Omega))}\Big).
\end{aligned}
\tag{4.8}
$$

Choosing η small enough for that $1 - \eta\Big(C_p^4 + \overline{m} + (1-l) + C_p^2\Big) \geq \frac{1}{2}$, and next integrating both side of inequality (4.8) from 0 to T, it follows

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \|u_{tt}(s)\|^2 ds \\
&\leq C_p C_*^{\rho+1}\Big(\frac{M}{\sqrt{\delta}}\Big)^\rho \int_0^T \|v_{tt}(s)\|\|u_{tt}(s)\| ds \\
&\quad + T\Big\{\frac{1}{4\eta}\frac{M^2}{\delta} + \frac{1}{4\eta}\frac{M^2}{\alpha} + \frac{1-l}{4\eta}\frac{M^2}{\alpha} + \frac{\tilde{a}_0}{4\eta}\Big(\frac{M^2}{\alpha} + \frac{M^{p+1}}{\alpha^{\frac{p+1}{2}}} + \frac{M^{2p}}{\alpha^p}\Big)\Big\} \\
&\leq C_p C_*^{\rho+1}\Big(\frac{M}{\sqrt{\delta}}\Big)^\rho\Big(\frac{1}{2}\int_0^T \|v_{tt}(s)\|^2 ds + \frac{1}{2}\int_0^T \|u_{tt}(s)\|^2 ds\Big) \\
&\quad + T\Big\{\frac{M^2}{4\eta}\Big(\frac{1}{\delta} + \frac{2-l}{\alpha}\Big) + \frac{\tilde{a}_0}{4\eta}\Big(\frac{M^2}{\alpha} + \frac{M^{p+1}}{\alpha^{\frac{p+1}{2}}} + \frac{M^{2p}}{\alpha^p}\Big)\Big\}.
\end{aligned}
$$

Rearranging similar terms, we have

$$\begin{aligned}
&\frac{1}{2}\Big(1 - C_p C_*^{\rho+1}\Big(\frac{M}{\sqrt{\delta}}\Big)^{\rho}\Big)\int_0^T \|u_{tt}(s)\|^2 ds \\
&\le \frac{1}{2} C_p C_*^{\rho+1}\Big(\frac{M}{\sqrt{\delta}}\Big)^{\rho}\frac{M^2}{\gamma} \\
&\quad + T\Big\{\frac{M^2}{4\eta}\Big(\frac{1}{\delta} + \frac{2-l}{\alpha}\Big) + \frac{\tilde{a}_0}{4\eta}\Big(\frac{M^2}{\alpha} + \frac{M^{p+1}}{\alpha^{\frac{p+1}{2}}} + \frac{M^{2p}}{\alpha^p}\Big)\Big\}.
\end{aligned} \tag{4.9}$$

The constants γ and δ can be chosen even smaller for that it satisfies

$$1 - C_p C_*^{\rho+1}\Big(\frac{M}{\sqrt{\delta}}\Big)^{\rho} \ge \frac{1}{2}.$$

Now we multiply the both sides of (4.9) by 4γ, and then choose δ and γ large enough and T small for that finally obtain

$$\gamma \int_0^T \|u_{tt}(s)\|^2 ds \le \frac{M^2}{2}. \tag{4.10}$$

The inequalities (4.7) and (4.10) leads us to $\|u\|_{\mathbf{W}} \le M$. Therefore we have completely proved that $A : \mathbf{Z}(M,T) \to \mathbf{Z}(M,T)$ is well defined.

Next, we verify that, for suitable values of the constants M, δ, γ and T, the application A is a contraction if $\rho \in I_\rho \cap \{\rho \ge 1\}$, and satisfies Lemma (4.2) if $\rho \in I_\rho \cap \{\rho < 1\}$. Let $U = u - \tilde{u}$ and $V = v - \tilde{v}$, where $u = A(v)$ and $\tilde{u} = A(\tilde{v})$. It is easy to check that U satisfies the problem

$$\begin{cases}
-M(\|U\|)\Delta U + \int_0^t g(t-s)\Delta U(s)ds - \Delta U_{tt} + U_t \\
= M(\|u\|^2)\Delta u - M(\|\tilde{u}\|^2)\Delta \tilde{u} - M(\|U\|)\Delta U \\
\quad + v|v|^{p-2}\ln|v|^k - \tilde{v}|\tilde{v}|^{p-2}\ln|\tilde{v}|^k - |v_t|^\rho v_{tt} + |\tilde{v}_t|^\rho \tilde{v}_{tt}, \\
U(0) = U_t(0) = 0 \text{ in } \Omega, \\
U = 0 \text{ on } [0,T) \times \partial\Omega.
\end{cases} \tag{4.11}$$

Multiplying (4.11) by U_t and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}\Big[\hat{M}(\|U\|^2) - \int_0^t g(s)ds\|U\|^2\Big] + \frac{1}{2}\|U_t\|^2\Big\} + |U_t|^2 \\
&= \int_\Omega \Big\{M(\|u\|^2)\Delta u - M(\|\tilde{u}\|^2)\Delta\tilde{u} - M(\|U\|)\Delta U \\
&\quad + v|v|^{p-2}\ln|v|^k - \tilde{v}|\tilde{v}|^{p-2}\ln|\tilde{v}|^k - |v_t|^\rho v_{tt} + |\tilde{v}_t|^\rho \tilde{v}_{tt}\Big\}U_t dx.
\end{aligned} \tag{4.12}$$

By (3.35), we find $\tilde{\tilde{m}} > 0$ such that

$$\int_\Omega [M(\|u\|^2)\Delta u - M(\|\tilde{u}\|^2)\Delta\tilde{u} - M(\|U\|)\Delta U]U_t dx \leq \tilde{\tilde{m}}\|U\|\|U_t\|.$$

Since $\|U(t)\| \leq \frac{1}{\sqrt{\alpha}}\|U\|_{\mathbf{W}}$ and $\|U_t(t)\| \leq \frac{1}{\sqrt{\delta}}\|U\|_{\mathbf{W}}$ for all $0 \leq t \leq T$, it follows that

$$\int_\Omega M(\|u\|^2)\Delta u - M(\|\tilde{u}\|^2)\Delta\tilde{u} - M(\|U\|)\Delta U U_t dx \leq C_0\|U\|^2_{\mathbf{W}}. \tag{4.13}$$

We also can find a constant $C_1 > 0$ such that

$$\int_\Omega \Big(v|v|^{p-2}\ln|v|^k - \tilde{v}|\tilde{v}|^{p-2}\ln|\tilde{v}|^k\Big)U_t dx \leq C_1\|V\|\|U_t\|. \tag{4.14}$$

Indeed, since

$$\frac{d}{d\xi}\Big(\xi|\xi|^{p-2}\ln|\xi|^k\Big) = k(p-1)|\xi|^{p-2}\ln|\xi| + k|\xi|^{p-2},$$

from (4.4) we have

$$\Big|\frac{d}{d\xi}\Big(\xi|\xi|^{p-2}\ln|\xi|^k\Big)\Big| \leq C(|\xi| + |\xi|^{p-1} + |\xi|^{p-2}).$$

By the Mean Value Theorem, for each $(x,t) \in \Omega \times (0,\infty)$ fixed, there exists $\theta(x,t) \in (0,1)$ such that, if we denote $\overline{v}(x,t) := v(x,t) + \theta(x,t)(v(x,t) - \tilde{v}(x,t))$, then

$$\Big|v|v|^{p-2}_{\mathbb{R}}\ln|v|^k_{\mathbb{R}} - \tilde{v}|\tilde{v}|^{p-2}_{\mathbb{R}}\ln|\tilde{v}|^k_{\mathbb{R}}\Big|_{\mathbb{R}} \leq C\Big(|\overline{v}|_{\mathbb{R}} + |\overline{v}|^{p-1}_{\mathbb{R}} + |\overline{v}|^{p-2}_{\mathbb{R}}\Big)V.$$

Hence,

$$\begin{aligned}
&\int_\Omega \Big(v|v|^{p-2}_{\mathbb{R}}\ln|v|^k_{\mathbb{R}} - \tilde{v}|\tilde{v}|^{p-2}_{\mathbb{R}}\ln|\tilde{v}|^k_{\mathbb{R}}\Big)U_t \\
&\leq C\int_\Omega \Big(|\overline{v}|_{\mathbb{R}} + |\overline{v}|^{p-1}_{\mathbb{R}} + |\overline{v}|^{p-2}_{\mathbb{R}}\Big)VU_t dx \\
&\leq C\Big(\int_\Omega (|\overline{v}|_{\mathbb{R}} + |\overline{v}|^{p-1}_{\mathbb{R}} + |\overline{v}|^{p-2}_{\mathbb{R}}\Big)^s dx\Big)^{\frac{1}{s}}\Big(\int_\Omega |V|^r dx\Big)^{\frac{1}{r}}\Big(\int_\Omega |U_t|^q dx\Big)^{\frac{1}{q}} \\
&\leq \tilde{C}\Big(\|\overline{v}\|_s + \|\overline{v}\|^{p-1}_{s(p-1)} + \|\overline{v}\|^{p-2}_{s(p-2)}\Big)\|V\|_r\|U_t\|_q,
\end{aligned}$$

where $\frac{1}{s} + \frac{1}{r} + \frac{1}{q} = 1$.

In order to have suitable embeddings for our estimate, we set s, r and q as follows: $s = \frac{p-1}{p-2}$ and $r = q = 2(p-1)$ if $n = 1$ or 2, and $s = r = q = 3$ if $n = 3$. Hence,

$$\int_\Omega \Big(v|v|^{p-2}_{\mathbb{R}}\ln|v|^k_{\mathbb{R}} - \tilde{v}|\tilde{v}|^{p-2}_{\mathbb{R}}\ln|\tilde{v}|^k_{\mathbb{R}}\Big)U_t dx \leq \tilde{\tilde{C}}\Big(\|\tilde{v}\| + \|\tilde{v}\|^{p-1} + \|\tilde{v}\|^{p-2}\Big)\|V\|\|U_t\|.$$

Since $\|v\|, \|\tilde{v}\| \leq \frac{M}{\sqrt{\alpha}}$ for all $t \in [0,T]$, there exists $C_1 > 0$ satisfying (4.14). Or

$$\int_\Omega \Big(v|v|^{p-2}\ln|v|^k - \tilde{v}|\tilde{v}|^{p-2}\ln|\tilde{v}|^k\Big) \leq \tilde{C}_1\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}. \tag{4.15}$$

Next, notice that

$$|v_t|^\rho v_{tt} - |\tilde{v}_t|^\rho \tilde{v}_{tt} = \frac{1}{2}(|v_t|^\rho + |\tilde{v}_t|^\rho)V_{tt} + \frac{1}{2}(|v_t|^\rho - |\tilde{v}_t|^\rho)(v_{tt}+\tilde{v}_{tt}). \tag{4.16}$$

Thus multiplying both sides of (4.16) by U_t and integrating over Ω, it follows

$$\begin{aligned}\int_\Omega \Big(|v_t|_{\mathbb{R}}^\rho v_{tt} - |\tilde{v}_t|_{\mathbb{R}}^\rho \tilde{v}_{tt}\Big)U_t dx \leq \Big(\|v_t\|_{2(\rho+1)}^\rho + \|\tilde{v}_t\|_{2(\rho+1)}^\rho\Big)\|V_{tt}\|_{2(\rho+1)}|U_t| \\ + \int_\Omega \frac{1}{2}(|v_t|_{\mathbb{R}}^\rho - |\tilde{v}_t|_{\mathbb{R}}^\rho)(v_{tt}+\tilde{v}_{tt})U_t dx.\end{aligned} \tag{4.17}$$

Owing to the embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ and recalling that $\|v_t\| \leq \frac{M}{\sqrt{\delta}}$, $\|\tilde{v}_t\| \leq \frac{M}{\sqrt{\delta}}$ and $|U_t| \leq C_p\|U_t\| \leq C_p\frac{1}{\sqrt{\delta}}\|U\|_{\mathbf{W}}$, for all $0 \leq t \leq T$, we obtain from (4.17):

$$\int_\Omega \Big(|v_t|_{\mathbb{R}}^\rho v_{tt} - |\tilde{v}_t|_{\mathbb{R}}^\rho \tilde{v}_{tt}\Big)U_t dx \leq C_2\|V_{tt}\|\|U\|_{\mathbf{W}} + \frac{1}{2}\int_\Omega (|v_t|_{\mathbb{R}}^\rho - |\tilde{v}_t|_{\mathbb{R}}^\rho)(v_{tt}+\tilde{v}_{tt})U_t dx. \tag{4.18}$$

In order to estimate the second term of the right side of (4.18), we consider two cases for ρ.
Case 1. $\rho \in I_\rho \cap \{\rho < 1\}$. Let $r := \frac{2}{1-\rho}$ if $n = 1$ or 2, or $r := 6$ if $n = 3$. Notice that

$$\frac{1}{\frac{1}{\rho}} + \frac{1}{\frac{r}{r(1-\rho)-1}} + \frac{1}{r} = 1.$$

From the generalized Hölder inequality we have

$$\frac{1}{2}\int_\Omega (|v_t|_{\mathbb{R}}^\rho - |\tilde{v}_t|_{\mathbb{R}}^\rho)(v_{tt}+\tilde{v}_{tt})U_t dx \leq \frac{1}{2}\Big[\int_\Omega \Big||v_t|_{\mathbb{R}}^\rho - |\tilde{v}_t|_{\mathbb{R}}^\rho\Big|^{\frac{1}{\rho}} dx\Big]^\rho \|v_{tt}+\tilde{v}_{tt}\|_{\frac{r}{r(1-\rho)-1}}|U_t|.$$

It is easy to check that the function

$$\xi \in \mathbb{R}_+ \setminus \{1\} \mapsto \frac{|\xi^\rho - 1|}{|\xi - 1|^\rho}$$

is bounded. Also we can use the embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{r}{r(1-\rho)-1}}(\Omega)$. Hence,

$$\begin{aligned}\frac{1}{2}\int_\Omega (|v_t|_{\mathbb{R}}^\rho - |\tilde{v}_t|_{\mathbb{R}}^\rho)(v_{tt}+\tilde{v}_{tt})U_t dx &\leq C_3\Big[\int_\Omega |V_t|_{\mathbb{R}} dx\Big]^\rho \|v_{tt}+\tilde{v}_{tt}\|\|U_t\| \\ &\leq \tilde{C}_3\|V_t\|^\rho\|v_{tt}+\tilde{v}_{tt}\|\|U_t\| \\ &\leq \tilde{\tilde{C}}_3\frac{\|V\|_{\mathbf{W}}^\rho}{\sqrt{\delta}^\rho}\|v_{tt}+\tilde{v}_{tt}\|\frac{\|U\|_{\mathbf{W}}}{\sqrt{\delta}}.\end{aligned} \tag{4.19}$$

Case 2. $\rho \in I_\rho \cap \{\rho \geq 1\}$. By the Mean Value Theorem there exists a function $\lambda : (0,T) \to (0,1)$ such that

$$\Big| |v_t|^\rho_{\mathbb{R}} - |\tilde{v}_t|^\rho_{\mathbb{R}} \Big| \leq \rho |v_t + \lambda(v_t - \tilde{v})_t|^{\rho-1}_{\mathbb{R}} |v_t - \tilde{v}_t|_{\mathbb{R}}.$$

Using the generalized Hölder inequality with

$$\frac{1}{\frac{r}{\rho-1}} + \frac{1}{\frac{3r}{r-(\rho-1)}} + \frac{1}{\frac{3r}{r-(\rho-1)}} + \frac{1}{\frac{3r}{r-(\rho-1)}} = 1,$$

where $r \geq \rho$, if $n = 1$ or 2, or $r = 6$ if $n = 3$, and the embeddings $H^1_0(\Omega) \hookrightarrow L^r(\Omega)$ and $H^1_0(\Omega) \hookrightarrow L^{\frac{3r}{r-(\rho-1)}}(\Omega)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (|v_t|^\rho_{\mathbb{R}} - |\tilde{v}_t|^\rho_{\mathbb{R}})(v_{tt} + \tilde{v}_{tt})U_t dx \\
&\leq \frac{\rho}{2}\|v_t + \lambda(v_t - \tilde{v}_t)\|^{\rho-1}_{L^r(\Omega)} \|v_t - \tilde{v}_t\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)} \|v_{tt} + \tilde{v}_{tt}\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)} \|U_t\|_{L^{\frac{3r}{r-(\rho-1)}}(\Omega)} \\
&\leq \tilde{\tilde{C}}_4 \|v_t + \lambda(v_t - \tilde{v}_t)\|^{\rho-1} \|V_t\| \|v_{tt} + \tilde{v}_{tt}\| \|U_t\| \qquad (4.20)\\
&\leq \tilde{\tilde{C}}_4 \Big(\frac{\|v + \lambda(v - \tilde{v})\|_{\mathbf{W}}}{\sqrt{\delta}}\Big)^{\rho-1} \frac{\|V\|_{\mathbf{W}}}{\sqrt{\delta}} \|v_{tt} + \tilde{v}_{tt}\| \frac{\|U\|_{\mathbf{W}}}{\sqrt{\delta}} \\
&\leq \tilde{\tilde{\tilde{C}}}_4 \frac{1}{\sqrt{\delta}^{\rho+1}} \|V\|_{\mathbf{W}} \|v_{tt} + \tilde{v}_{tt}\| \|U\|_{\mathbf{W}}.
\end{aligned}$$

Thus, if $\rho \in I_\rho$, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (|v_t|^\rho_{\mathbb{R}} - |\tilde{v}_t|^\rho_{\mathbb{R}})(v_{tt} + \tilde{v}_{tt})U_t dx \\
&\leq C_5 \frac{1}{\sqrt{\delta}^{\rho+1}} \Big(k_0\|V\|^\rho_{\mathbf{W}} + (1-k_0)\|V\|_{\mathbf{W}}\Big) \|v_{tt} + \tilde{v}_{tt}\| \|U\|_{\mathbf{W}},
\end{aligned} \qquad (4.21)$$

where $k_0 = 1$ if $\rho \in I_\rho \cap \{\rho \leq 1\}$ and $k_0 = 0$ if $\rho \in I_\rho \cap \{\rho > 1\}$.

Integrating (4.12) from 0 to T, and use (4.13),(4.15), (4.18) and (4.21), we obtain

$$
\begin{aligned}
&\frac{m_0+l-1}{2}\|U\|^2+\frac{1}{2}\|U_t\|^2 \\
&\leq TC_0\|U\|^2_{\mathbf{W}}+T\tilde{C_1}\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}+C_2\int\limits_0^T\|V_{tt}(s)\|\|U\|_{\mathbf{W}}ds \\
&+C_5\frac{1}{\sqrt{\delta}^{\rho+1}}\int\limits_0^T\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|v_{tt}+\tilde{v}_{tt}\|\|U\|_{\mathbf{W}}ds \qquad (4.22)\\
&\leq TC_0\|U\|^2_{\mathbf{W}}+T\tilde{C_1}\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}+C_2\frac{\sqrt{T}}{\sqrt{\gamma}}\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}} \\
&+2MC_5\frac{1}{\sqrt{\delta}^{\rho+1}}\frac{\sqrt{T}}{\sqrt{\gamma}}\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|U\|_{\mathbf{W}}.
\end{aligned}
$$

Taking the essential supremum in t over $[0,T]$ in (4.22), it leads us to

$$
\begin{aligned}
&\frac{m_0+l-1}{2}\|U\|^2_{L^\infty(0,T;H^1_0(\Omega))}+\delta\|U_t\|^2_{L^\infty(0,T;H^1_0(\Omega))} \\
&\leq(2\delta+1)\Big\{TC_0\|U\|^2_{\mathbf{W}}+T\tilde{C_1}\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}+C_2\frac{\sqrt{T}}{\sqrt{\gamma}}\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}} \qquad (4.23)\\
&+2MC_5\frac{1}{\sqrt{\delta}^{\rho+1}}\frac{\sqrt{T}}{\sqrt{\gamma}}\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|U\|_{\mathbf{W}}\Big\}.
\end{aligned}
$$

The next step is to multiply (4.11) by U_{tt} and integrate over Ω, and then to apply similar estimates that led to the estimates (4.13),(4.15), (4.18) and (4.21). Following this way we have

$$
\begin{aligned}
\|U_{tt}\|^2 &\leq \frac{1}{2}\int\limits_0^t g(t-s)\|U(s)\|^2ds+\frac{1}{2}\int\limits_0^t g(t-s)ds\|U_{tt}\|^2 \\
&+C_6\|U\|\|U_{tt}\|+C_7\|V\|\|U_{tt}\| \\
&+C_8\frac{1}{\sqrt{\delta}^{\rho}}\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|v_{tt}+\tilde{v}_{tt}\|\|U_{tt}\| \\
&\leq \frac{1}{2\alpha}\|U\|^2_{\mathbf{W}}(1-l)+\frac{1-l}{2}\|U_{tt}\|^2+C_6\Big(\frac{\|U\|^2}{4\eta}+\eta\|U_{tt}\|^2\Big) \\
&+C_7\frac{1}{\sqrt{\alpha}}\|V\|_{\mathbf{W}}\|U_{tt}\| \\
&+C_8\frac{1}{\sqrt{\delta}^{\rho}}\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|v_{tt}+\tilde{v}_{tt}\|\|U_{tt}\|,
\end{aligned}
$$

or yet

$$\begin{aligned}\Big(1-\frac{(1-l)}{2}-\eta C_6\Big)\|U_{tt}\|^2 \le{}& \frac{1}{2\alpha}\|U\|^2_{\mathbf{W}}(1-l)+C_6\frac{\|U\|^2}{4\eta}\\ &+C_7\frac{1}{\sqrt{\alpha}}\|V\|_{\mathbf{W}}\|U_{tt}\|\\ &+C_8\frac{1}{\sqrt{\delta}^{\rho}}\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|v_{tt}\\ &+\tilde{v}_{tt}\|\|U_{tt}\|.\end{aligned}\tag{4.24}$$

Integrating the both sides of (4.24) from 0 to T, it follows

$$\begin{aligned}&\Big(1-\frac{(1-l)}{2}-\eta C_6\Big)\int_0^T\|U_{tt}\|^2ds\\ &\le \frac{T}{2\alpha}\|U\|^2_{\mathbf{W}}(1-l)+C_6T\frac{\|U\|^2}{4\eta}\\ &\quad+C_7\frac{\sqrt{T}}{\sqrt{\alpha}}\|V\|_{\mathbf{W}}\Big(\int_0^T\|U_{tt}(s)\|^2ds\Big)^{\frac12}\\ &\quad+C_8\frac{1}{\sqrt{\delta}^{\rho}}\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\\ &\qquad\times\Big(\int_0^T\|v_{tt}(s)+\tilde{v}_{tt}(s)\|^2ds\Big)^{\frac12}\Big(\int_0^T\|U_{tt}(s)\|^2ds\Big)^{\frac12}\\ &\le \frac{T}{2\alpha}\|U\|^2_{\mathbf{W}}(1-l)+C_6T\frac{\|U\|^2_{\mathbf{W}}}{4\alpha\eta}+C_7\frac{\sqrt{T}}{\sqrt{\alpha\gamma}}\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}\\ &\quad+2C_8\frac{1}{\sqrt{\delta}^{\rho}}\frac{M}{\gamma}\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|U\|_{\mathbf{W}}.\end{aligned}\tag{4.25}$$

Choosing $\eta>0$ small enough for that $1-\frac{(1-l)}{2}-\eta C_6>\frac{1+l}{4}$ and then multiplying (4.25) by 4γ, we get

$$\begin{aligned}\gamma(1+l)\int_0^T\|U_{tt}\|^2ds\le{}&\frac{2T\gamma}{\alpha}\|U\|^2_{\mathbf{W}}(1-l)+C_6T\gamma\frac{\|U\|^2_{\mathbf{W}}}{\alpha\eta}\\ &+4C_7\frac{\sqrt{T}\sqrt{\gamma}}{\sqrt{\alpha}}\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}\\ &+8C_8\frac{1}{\sqrt{\delta}^{\rho}}M\Big(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}}\Big)\|U\|_{\mathbf{W}}.\end{aligned}\tag{4.26}$$

Now we combine (4.23) with (4.26):

$$
\begin{aligned}
&(1+l)\Big(\frac{m_0+l-1}{2}\|U\|^2_{L^\infty(0,T;H_0^1(\Omega))}+\frac{1}{2}\|U_t\|^2_{L^\infty(0,T;H_0^1(\Omega))}+\gamma\int_0^T\|U_{tt}(s)\|^2ds\Big)\\
&\le T\Big[(2\delta+1)(1+l)C_0+\frac{\gamma}{\alpha}\Big(2(1-l)+\frac{C_6}{\eta}\Big)+\frac{C_6\gamma}{\alpha\eta}\Big]\|U\|^2_{\mathbf{W}}\\
&\quad+\Big[(2\delta+1)(1+l)\Big(T\tilde{C}_1+C_2\frac{\sqrt{T}}{\sqrt{\gamma}}\Big)+4C_7\frac{\sqrt{T}\sqrt{\gamma}}{\sqrt{\alpha}}\Big]\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}\\
&\quad+\Big(2MC_5\frac{1}{\sqrt{\delta}^{\rho+1}}\frac{\sqrt{T}}{\sqrt{\gamma}}+8C_8\frac{1}{\sqrt{\delta}^{\rho}}M\Big)(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}})\|U\|_{\mathbf{W}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big\{(1+l)-T\Big[(2\delta+1)(1+l)C_0+\frac{\gamma}{\alpha}\Big(2(1-l)+\frac{C_6}{\eta}\Big)+\frac{C_6\gamma}{\alpha\eta}\Big]\Big\}\|U\|^2_{\mathbf{W}}\\
&\le\Big[(2\delta+1)(1+l)\Big(T\tilde{C}_1+C_2\frac{\sqrt{T}}{\sqrt{\gamma}}\Big)+4C_7\frac{\sqrt{T}\sqrt{\gamma}}{\sqrt{\alpha}}\Big]\|V\|_{\mathbf{W}}\|U\|_{\mathbf{W}}\\
&+\Big(2MC_5\frac{1}{\sqrt{\delta}^{\rho+1}}\frac{\sqrt{T}}{\sqrt{\gamma}}+8C_8\frac{1}{\sqrt{\delta}^{\rho}}M\Big)(k_0\|V\|^{\rho}_{\mathbf{W}}+(1-k_0)\|V\|_{\mathbf{W}})\|U\|_{\mathbf{W}}.
\end{aligned}
\tag{4.27}
$$

Let us now investigate (4.27) for the two possible values for k_0.
Possibility one: $k_0=0$. We have $\rho\in I_\rho\cap\{\rho\ge 1\}$. The inequality (4.27) becomes

$$
\begin{aligned}
&\Big\{(1+l)-T\Big[(2\delta+1)(1+l)C_0+\frac{\gamma}{\alpha}\Big(2(1-l)+\frac{C_6}{\eta}\Big)+\frac{C_6\gamma}{\alpha\eta}\Big]\Big\}\|U\|_{\mathbf{W}}\\
&\le\Big\{\Big[(2\delta+1)(1+l)\Big(T\tilde{C}_1+C_2\frac{\sqrt{T}}{\sqrt{\gamma}}\Big)+4C_7\frac{\sqrt{T}\sqrt{\gamma}}{\sqrt{\alpha}}\Big]\\
&\quad+\Big(2MC_5\frac{1}{\sqrt{\delta}^{\rho+1}}\frac{\sqrt{T}}{\sqrt{\gamma}}+8C_8\frac{1}{\sqrt{\delta}^{\rho}}M\Big)\Big\}\|V\|_{\mathbf{W}}.
\end{aligned}
$$

Now we choose δ and γ large and T small enough for that

$$
\Big\{(1+l)-T\Big[(2\delta+1)(1+l)C_0+\frac{\gamma}{\alpha}\Big(2(1-l)+\frac{C_6}{\eta}\Big)+\frac{C_6\gamma}{\alpha\eta}\Big]\Big\}<l
$$

and

$$
\begin{aligned}
&\Big[(2\delta+1)(1+l)\Big(T\tilde{C}_1+C_2\frac{\sqrt{T}}{\sqrt{\gamma}}\Big)+4C_7\frac{\sqrt{T}\sqrt{\gamma}}{\sqrt{\alpha}}\Big]\\
&+\Big(2MC_5\frac{1}{\sqrt{\delta}^{\rho+1}}\frac{\sqrt{T}}{\sqrt{\gamma}}+8C_8\frac{1}{\sqrt{\delta}^{\rho}}M\Big)<1.
\end{aligned}
$$

This means that the application A is a contraction and therefore the problem (1.1) posses a unique local weak solution u in $\mathbf{W}$.
Possibility two: $k_0 = 1$. This case corresponds to $\rho \in I_\rho \cap \{\rho < 1\}$. Thus from (4.27) we have

$$\begin{aligned}
&\Big\{(1+l) - T\Big[(2\delta+1)(1+l)C_0 + \frac{\gamma}{\alpha}\Big(2(1-l) + \frac{C_6}{\eta}\Big) + \frac{C_6\gamma}{\alpha\eta}\Big]\Big\}\|U\|_{\mathbf{W}} \\
&\leq \Big[(2\delta+1)(1+l)\Big(T\tilde{C}_1 + C_2\frac{\sqrt{T}}{\sqrt{\gamma}}\Big) + 4C_7\frac{\sqrt{T}\sqrt{\gamma}}{\sqrt{\alpha}}\Big]\|V\|_{\mathbf{W}} \\
&\quad + \Big(2MC_5\frac{1}{\sqrt{\delta}^{\rho+1}}\frac{\sqrt{T}}{\sqrt{\gamma}} + 8C_8\frac{1}{\sqrt{\delta}^{\rho}}M\Big)\|V\|^{\rho}_{\mathbf{W}}.
\end{aligned} \tag{4.28}$$

We then choose δ and γ large and T small enough in order to the application A satisfies the Lemma 4.2. Hence, if $\rho \in I_\rho \cap \{\rho < 1\}$, the problem (1.1) admits a local (not necessarily unique!) weak solution u in $\mathbf{W}$. $\square$

Remark 4.4. In the proof of Theorem 4.3, the choices of the constants γ, δ and T in order to obtain the required estimates can be made setting $\delta := \frac{1}{\sqrt{T}}$, $\gamma := \frac{1}{\sqrt[4]{T}}$ and taking T small.

5. CONCLUSIONS

It is worth mentioning a blow up result for the problem (1.1) considered in [11], without a formal proof of local existence.

A natural investigation concerning (1.1) lies in its extension considering variable exponents and unbounded domain, particularly,

$$\begin{aligned}
&|u_t|^\rho u_{tt} - \nabla\cdot(|\nabla u|^{\gamma(x)-2}\nabla u) - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds \\
&+ b(x,t)u_t|u_t|^{m(x)-2} = u|u|^{p(x)-2}\ln|u|^{p(x)} \quad \text{on } \mathbb{R}^n \times [0,T]
\end{aligned}$$

which is in course to obtain the first results by the authors.

Throughout the paper, the reader will realize the recurrent mention of Mean Value Theorem (MVT), for which the use demanded diferentiability of real functions involving the nonlinear terms, namely, $\xi|\xi|^{p-2}\ln|\xi|$, $|\xi|^\rho$ and $M(\xi)$. Consequently, the derivative in the formula of MVT depends on objects which also depends on t. However, as a function of t, such derivatives remains bounded due to their arguments varies in a ball of $\mathbf{W}$. This shows the Lipschitz character of the nonlinearities, but the same not happens for $|\cdot|^\rho$ near zero when $\rho < 1$. We recommend [6] for a study of well-posedness of a kind of equation with $|u_t|^\rho u_{tt}$ and $\rho < 1$.

Alternatively to the Mean Value Theorem used for nonlinear terms $\xi|\xi|^{p-2}\ln|\xi|$, $|\xi|^\rho$ and $M(\xi)$, we can employ an argument involving a property of Gâteaux derivative as in [27, Lemma 4].

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