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# CHARACTERIZATION OF SUPER-RADIAL GRAPHS

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**Abstract**

In a graph $G$, the distance $d(u, v)$ between a pair of vertices $u$ and $v$ is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex $u$ is the distance to a vertex farthest from $u$. The minimum eccentricity is called the radius, $r(G)$, of the graph and the maximum eccentricity is called the diameter, $d(G)$, of the graph. The super-radial graph $R^*(G)$ based on $G$ has the vertex set as in $G$ and two vertices $u$ and $v$ are adjacent in $R^*(G)$ if the distance between them in $G$ is greater than or equal to $d(G) - r(G) + 1$ in $G$. If $G$ is disconnected, then two vertices are adjacent in $R^*(G)$ if they belong to different components. A graph $G$ is said to be a super-radial graph if it is a super-radial graph $R^*(H)$ of some graph $H$. The main objective of this paper is to solve the graph equation $R^*(H) = G$ for a given graph $G$.

**Keywords:** radius, diameter, super-radial graph.

**2010 Mathematics Subject Classification:** 05C12.

## 1. Introduction

The graphs considered are simple, non-trivial, undirected and finite. $G = (V, E)$ is a graph with vertex set $V(G)$ and edge set $E(G)$. In a graph $G$, the *distance* $d(u, v)$ between a pair of vertices $u$ and $v$ is the length of a shortest path joining them. The *eccentricity* $e(u)$ of a vertex $u$ is the distance to a vertex farthest from $u$. The *radius* $r(G)$ of $G$ is defined by $r(G) = \min\{e(u) : u \in V(G)\}$ and the *diameter* $d(G)$ of $G$ is defined by $d(G) = \max\{e(u) : u \in V(G)\}$. A graph $G$ for which $r(G) = d(G)$ is called a *self-centered* graph of radius $r(G)$. A vertex $v$ is called an *eccentric vertex of a vertex* $u$ if $d(u, v) = e(u)$. A vertex $v$ of $G$ is called an *eccentric vertex of* $G$ if it is an eccentric vertex of some vertex of $G$. The concept of antipodal graph was initially introduced by Singleton [21] and was further expanded by Aravamudhan and Rajendran [2, 3]. The *antipodal graph* of a graph $G$, denoted by $A(G)$, is the graph on the same set of vertices as of $G$, two vertices being adjacent if the distance between them is equal to the diameter of $G$ while $G$ is connected and if $G$ is disconnected, then two vertices are adjacent in $A(G)$ if they belong to different components of $G$. A graph $G$ is said to be *antipodal* if it is the antipodal graph of some graph $H$.

Aravamudhan and Rajendran [2, 3] have proved the following theorem. A graph $G$ is an antipodal graph if and only if it is the antipodal graph of its complement $\overline{G}$. In [4] the same authors observed that if $H$ is a connected graph with $diam(H) > 2$, then $A(H) = A(H')$, where $H'$ is the graph on the same vertex set such that two vertices are adjacent in $H'$ if the distance between them in $H$ is less than $diam(H)$. This observation is still true when $diam(H) = 2$ (for then $H' = H$) and when $H$ is disconnected. In this case, the components of $H$ and $H'$ consists of the same vertices and the edges of $A(H)$ and $A(H')$ are exactly the edges joining vertices in different components. This extension leads to another proof of the characterization of antipodal graphs which involves showing that $A(H') = \overline{H'}$ by Johns [9].

Kathiresan and Marimuthu [14] introduced the *radial graph* $R(G)$ of a graph $G$ on the same vertex set as $G$ and two vertices $u$ and $v$ are adjacent in $R(G)$ if and only if the distance between them is equal to the radius. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of $G$. A graph $G$ is called a *radial graph* if $R(H) = G$ for some graph $H$. Kathiresan and Marimuthu [15] characterized graphs $G$ with specified radius for its radial graph.

In paper [20], the author defines a metric operator $X_{\mathcal{P}}$ which unifies every known digraph operator related to a distance property $\mathcal{P}$. In Theorem 1 [20] the author characterizes those digraphs $G$ such that $X_{\mathcal{P}}(G) = H$ for some digraph $G$ when $\mathcal{P}$ is both unitary and vertex free distance property. In particular, the characterization of both antipodal and radial graphs arises from it.

Kathiresan *et al.* [16] defined a graph $G$ to be *periodic* if $R^m(G) = G$ for some $m$. If $p$ is the least positive integer with this property, then $G$ is called a *periodic graph with iso-period* $p$. A graph $G$ is said to be an *eventually periodic graph* if there exist positive integers $m$ and $k > 0$, such that $R^{m+i}(G) = R^i(G)$, for all $i \geq k$. They proved that every graph is either periodic or eventually periodic. In their paper they characterized all periodic graphs.

Akiyama *et al.* [1] defined the *eccentric graph* $G_e$ of $G$ on the same set of vertices, by joining two vertices if and only if one of the two vertices has the maximum possible distance from the other, that is $d(u,v) = \min\{e(u), e(v)\}$. Iqbalunnisa *et al.* [10] defined the *super-eccentric graph* $J(G)$ of a graph $G$ on the same set of vertices of $G$ and the adjacency relation between vertices is defined by $d(u,v) \geq rad(G)$ while $G$ is connected and when $G$ is disconnected, two vertices are adjacent in $J(G)$ if they belong to different components of $G$. Kathiresan *et al.* [18] have given a characterization of super-eccentric graphs.

For a digraph $D$, the *antipodal digraph* $A(D)$ of $D$ is the digraph which $V(A(D)) = V(D)$ and $E(A(D)) = \{(u,v) : u, v \in V(D) \text{ and } d_D(u,v) = d(D)\}$. Johns and Sleno [8] obtained a characterization of antipodal digraphs. A digraph $D$ is *self-antipodal* if $A(D)$ is isomorphic to $D$.

Kathiresan and Sumathi [17] extended the definition of radial graph to a digraph $D$ where the arc $(u,v)$ is included in $R(G)$ if $d(u,v)$ is the radius of $D$. According to them a digraph $D$ is called a *radial digraph* if $R(H) = D$ for some digraph $H$.

Buckley [6] defined the *eccentric digraph* $ED(G)$ of graph $G$ to be the digraph that has the same vertex set as $G$ such that there is an arc from $v$ to $u$ provided that $u$ is an eccentric vertex of $v$. He examined eccentric digraphs of graphs.

Gimbert *et al.* [12] considered the behaviour of an iterated sequence of eccentric graphs or digraphs of a graph or a digraph. They concluded with several open problems. Boland *et al.* [11] defined the eccentric digraph of a digraph. They examined eccentric digraphs of digraphs for various families of digraphs and they considered the behaviour of an iterated sequence of eccentric digraphs of a digraph.

Huilgol *et al.* [19] considered an open problem, which is found in [11]. They characterized graphs with specified maximum degree such that $ED(G) = G$.

Gimbert *et al.* [13] presented a characterization of eccentric digraphs, which in the undirected case says that a graph $G$ is eccentric if and only if its complement graph $\overline{G}$ is either self-centered of radius two or it is the union of complete graphs.

In [5], the $k^{th}$ power $G^k$ of the graph $G$ has the same vertex set as $G$ and vertices $u$ and $v$ are adjacent in $G^k$ if the distance between them in $G$ is at most $k$.

Motivated by these works, we introduce a new concept called *super-radial graph* $R^*(G)$ of a graph $G$ on the same vertex set of $G$ and two vertices $u$ and $v$

are adjacent in $R^*(G)$ if and only if the distance between them is greater than or equal to $d(G) - r(G) + 1$. If $G$ is disconnected, then two vertices are adjacent in $R^*(G)$ if they belong to different components of $G$. A graph $G$ is said to be a *super-radial graph* if there exists a graph $H$ such that $R^*(H) = G$. In this paper, we have given a characterization for a graph to be a super-radial graph.

The following notation can be found in [14].

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}, F_3$ denote the set of all connected graphs $G$ for which $r(G) = d(G) = 1, r(G) = 1$ and $d(G) = 2$, $r(G) = d(G) = 2, r(G) = 2$ and $d(G) = 3, r(G) = 2$ and $d(G) = 4, r(G) \geq 3$, respectively. $F_4$ denote the set of all disconnected graphs. For graph theoretic terminology we follow [5], which is devoted entirely to the area of distance in graphs.

The following results will be used throughout this article.

**Theorem A** [5]. *If $G$ is a simple graph with diameter at least 3, then $\overline{G}$ has diameter at most 3.*

**Theorem B** [5]. *If $G$ is a simple graph with diameter at least 4, then $\overline{G}$ has diameter at most 2.*

**Theorem C** [5]. *If $G$ is a simple graph with radius at least 3, then $\overline{G}$ has radius at most 2.*

**Theorem D** [23]. *If $G$ is a selfcentred graph with radius at least 3, then $\overline{G}$ is a self centered graph of radius 2.*

From the above theorems, we have the following.

If $G \in F_{11}$, then $\overline{G}$ is a totaly disconnected graph and if $G \in F_{12}$, then $\overline{G}$ has at least one isolated vertex. If $G \in F_{22}$, then $\overline{G}$ is a member of $F_{22} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. If $G \in F_{23}$, then $\overline{G}$ is a member of $F_{22} \cup F_{23}$. If $G \in F_{24}$, then $\overline{G}$ is a member of $F_{22}$. If $G \in F_3$, then $\overline{G} \in F_{22}$. If every component of $G$ is non-trivial, then $\overline{G} \in F_{22}$. If $G$ has at least one isolated vertex, then $\overline{G}$ is a member of $F_{12}$.

**Lemma E** [23]. *Let $u, v$ be two vertices of a graph $G$. Then $d_{G^k}(u, v) = \left\lceil \frac{d_G(u,v)}{k} \right\rceil$.*

## 2. The Relation Between the Super-radial Operator and the Complement Operator

In this section we find a graph $G$ for which $R^*(G) = H$ for a given graph $H$.

**Proposition 1.** *For any graph $G$ on $p$ vertices, $R^*(G) = K_p$ if and only if either $G$ is self-centered or $G = \overline{K_p}$.*

***Proof.*** If either $G$ is self-centered or $G = \overline{K_p}$, then the result follows from the definition of $R^*(G)$. Suppose that $G$ is connected and $r(G) \neq d(G)$. This shows that $d(G) - r(G) + 1 \geq 2$. Therefore $R^*(G) \subseteq \overline{G}$. This is a contradiction to the fact that $R^*(G) = K_p$. If $G$ is a disconnected graph in which $|V(G_i)| = 2$, for some $i^{th}$ component $G_i$ of $G$, then $uv \notin E(R^*(G))$ whenever $u$ and $v$ belong to $V(G_i)$. This implies that $R^*(G) \neq K_p$. ■

**Proposition 2.** *For any graph $G$ with $p \geq 3$ vertices, $R^*(G) = K_{1,p-1}$ if and only if $G$ is disconnected with exactly two components out of which one is an isolated vertex.*

***Proof.*** If $G$ is disconnected with exactly two components out of which one is an isolated vertex, then by the definition of $R^*(G), R^*(G) = K_{1,p-1}$.

Let $v_1$ be the vertex of degree $p-1$ and $v_2, v_3, \ldots, v_p$ be the pendant vertices of $R^*(G)$. If $G$ is connected, then $d_G(v_1, v_i) \geq d(G) - r(G) + 1$ for all $i \neq 1$ and hence $d_G(v_1, v_i) \geq 2$. This is a contradiction to the fact that $R^*(G) = K_{1,p-1}$. If $G$ is disconnected with more than two nontrivial components, then we arrive at a contradiction to the fact that $R^*(G) = K_{1,p-1}$. If $G$ has exactly two nontrivial components, then $R^*(G)$ is a complete bipartite graph.

Therefore the above argument forces us to conclude that $G$ is a disconnected graph with exactly two components out of which one is an isolated vertex. ■

**Proposition 3.** *If $G$ is a graph with $d(G) \geq r(G) + 1$, then $R^*(G) \subseteq \overline{G}$.*

***Proof.*** By the definition of $R^*(G)$ and $\overline{G}$, we have $V(R^*(G)) = V(\overline{G}) = V(G)$. $d(G) \geq r(G) + 1$ implies that $d(G) - r(G) + 1 \geq 2$. This shows that $R^*(G) \subseteq \overline{G}$. ■

**Lemma 4.** *Let $G$ be a graph of order $p$. Then $R^*(G) = \overline{G}$ if and only if $G$ is a graph with $d(G) = r(G) + 1$ or $G$ is disconnected in which each component is complete.*

***Proof.*** If $d(G) = r(G)+1$, then $d(G)-r(G)+1 = 2$. Therefore $R^*(G) \subseteq \overline{G}$. Also, any two adjacent vertices in $G$ are not adjacent in $R^*(G)$. Therefore $\overline{G} \subseteq R^*(G)$. Thus $R^*(G) = \overline{G}$.

If $G$ is disconnected with each component complete, then by the definition, $R^*(G) = \overline{G}$.

If $d(G) < r(G) + 1$, then $G$ is self-centred and by Proposition 1, $R^*(G) = \overline{G} = K_p$. As a consequence $G = \overline{K_p}$, which is a contradiction to the fact that $G$ is connected. This implies that $R^*(G)$ is a complete graph.

If $d(G) > r(G) + 1$, then $d(G) - r(G) + 1 \geq 3$ and hence $R^*(G) \subset \overline{G}$. Thus $d(G) = r(G) + 1$.

Suppose that $G$ has a non-complete component, say $G_1$. Then $G_1$ has two non-adjacent vertices $u$ and $v$. It follows from the definitions that $uv \in E(\overline{G})$ and $uv \notin E(R^*(G))$. ■

**Corollary 5.** *If* $G \in F_{12}$, *then* $R^*(G) = \overline{G}$.

***Proof.*** Since $G \in F_{12}, d(G) = r(G) + 1$, by Lemma 4, $R^*(G) = \overline{G}$. ■

**Lemma 6.** *If* $G \in F_3$ *with* $r(G)+2 \leq d(G) \leq 2r(G)-1$, *then* $R^*(G) \in F_{22} \cup F_{23}$ *and* $\overline{R^*(G)} \in F_{tt+1}$ *for some* $t \geq 2$.

***Proof.*** Suppose $R^*(G) \in F_{11}$. Then by Proposition 1, either $G$ is self-centered or $G$ is totally disconnected. This is a contradiction to $G \in F_3$ with $r(G) + 2 \leq d(G) \leq 2r(G) - 1$. Suppose $R^*(G) \in F_{12}$. Then $R^*(G)$ has at least one vertex $u$ of eccentricity one. Then $d(u, v) \geq d(G) - r(G) + 1 \geq 3$ in $G$ for all $u \in V(G) - \{u\}$. Since $G$ is connected, $u$ has at least one adjacent vertex $w$ in $G$. Therefore $d(u, w) = 1$ in $G$. Then $u$ is not adjacent to $w$ in $R^*(G)$. Which is a contradiction to $R^*(G) \in F_{12}$. Therefore $R^*(G) \notin F_{12}$. Now we claim that $R^*(G)$ has at least one vertex of eccentricity two. Let $u$ be any peripheral vertex. Then there exists a vertex $v$ in $G$ such that $d(u, v) = d(G)$ in $G$. Therefore $u$ and $v$ are adjacent in $R^*(G)$.

Consider the set $\overline{N}(u) = \{w : d(u, w) \leq d(G) - r(G)\}$ in $G$. Clearly in $R^*(G), u$ is not adjacent to any vertex of $\overline{N}(u)$.

Let $w \in \overline{N}(u)$. Then $d(u, w) \leq d(G) - r(G)$ for all $w \in \overline{N}(u)$. Now $d(u, v) \leq d(u, w) + d(w, v)$ in $G$. Therefore $d(G) \leq d(G) - r(G) + d(w, v)$ in $G$. Hence

$$d(w, v) \geq r(G) \text{ in } G. \tag{1}$$

Futher $r(G) + 2 \leq d(G) \leq 2r(G) - 1$, which implies,

$$d(G) - r(G) + 1 \leq r(G) \text{ in } G. \tag{2}$$

From (1) and (2),

$$d(w, v) \geq r(G) \geq d(G) - r(G) + 1 \text{ in } G.$$

Hence by the definition, $v$ is adjacent to all the vertices of $\overline{N}(u)$ in $R^*(G)$. Let $d$ be the distance in $R^*(G)$. Therefore, $d(u, w) = d(u, v) + d(v, w) = 1 + 1 = 2$ for all $w \in \overline{N}(u)$. Thus, $R^*(G)$ has a vertex of eccentricity two. Hence $R^*(G) \in F_{22} \cup F_{23} \cup F_{24}$. Let $S = \{w : e(w) = d(G) \text{ in } G\}$. Clearly, $e(w) = 2$ for all $w \in S$ in $R^*(G)$. Let $x \in V(G) - S$. Let $\overline{N}(x) = \{y : d(x, y) \leq d(G) - r(G)$ in $G\}$. Clearly, $x$ is not adjacent to any vertex of $\overline{N}(x)$ in $R^*(G)$. Since $d(x, u) \geq d(G) - r(G) + 1, d(x, u) = 1$ in $R^*(G)$ for all $u \notin \overline{N}(x)$. That is $xu \in E(R^*(G))$.

Let $v' \in S$. Then there exists a vertex $v'' \in S$ such that

$$d(v', v'') = d(G) \text{ in } G. \tag{3}$$

Clearly, $v'v'' \in E(R^*(G))$. Suppose both $v'$ and $v''$ are in $\overline{N}(x)$ in $G$. Since $r(G) + 2 \le d(G) \le 2r(G) - 1$,

$$\begin{aligned} d(v', v'') &\le d(v', x) + d(x, v'') \\ &\le d(G) - r(G) + d(G) - r(G), \\ d(v', v'') &\le 2(d(G) - r(G)) < d(G) \text{ since } d < 2n. \end{aligned}$$

Therefore, $d(v', v'') < d(G)$ in $G$ which is a contradiction to (3).

Hence among $v'$ and $v''$ at most one vertex can be in $\overline{N}(x)$ in $G$. Without loss of generality, $v' \notin \overline{N}(x)$ in $G$. $xv' \in E(R^*(G))$. Let $w \in \overline{N}(x)$ in $G$. In $R^*(G), d(x, w) \le d(x, v') + d(v', w) \le 1 + 2$ (because $e(v') = 2$). That is $d(x, w) \le 3$ for all $w \in \overline{N}(x)$.

Suppose both $v', v'' \notin \overline{N}(x)$. Then $d(x, w) \le 3$ in $R^*(G)$ for all $w \in \overline{N}(x)$ in $G$. This is true for all $x \in V(G) - S$. Therefore $2 \le e(u) \le 3$ in $R^*(G)$ for all $u \in V(R^*(G))$. That is $R^*(G) \notin F_{24}$ and $R^*(G) \in F_{22} \cup F_{23}$.

**Claim.** $\overline{R^*(G)} \in F_{tt+1}$ *where* $t \ge 2$.

By the definition of the $k^{th}$ power of a graph $G$, we have $d_{G^k}(u, v) = \left\lceil \frac{d_G(u,v)}{k} \right\rceil$. Hence $G^k = \overline{R^*(G)}$ where $k = d(G) - r(G)$. $r(G) \le e(u) \le d(G)$ for all $u$ in $G$ implies $\left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil \le e_{\overline{R^*(G)}}(u) \le \left\lceil \frac{d(G)}{d(G)-r(G)} \right\rceil$ for all $u \in V(\overline{R^*(G)})$. Since $\frac{d(G)}{d(G)-r(G)} = 1 + \frac{r(G)}{d(G)-r(G)}, \left\lceil \frac{d(G)}{d(G)-r(G)} \right\rceil = 1 + \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil$.

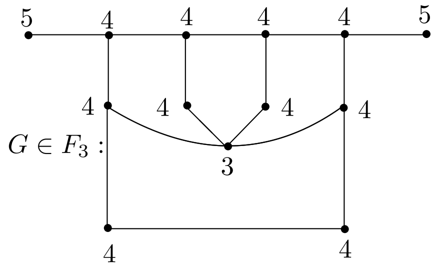
Let $t = \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil$, since $r(G) \ge 3$ and $r(G) + 2 \le d(G) \le 2r(G) - 1, t \ge 2$. Therefore $t \le e_{\overline{R^*(G)}}(u) \le 1 + t$ for all $u \in V(\overline{R^*(G)})$. Suppose $u$ and $v$ are antipodal vertices of $G$. Then $d(u, v) = d(G)$.

$d_{G^k}(u, v) = \left\lceil \frac{d_G(u,v)}{k} \right\rceil = \left\lceil \frac{d(G)}{d(G)-r(G)} \right\rceil = 1 + \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil = 1 + t, t \ge 2.$

That is $d_{G^k}(u, v) = 1 + t, t \ge 2$. Suppose $e(u) = 1 + t$, $t \ge 2$. $w$ is any central vertex of $G$. Then $d(w, u) = r(G) = d(w, v)$

$d_{G^k}(w, u) = \left\lceil \frac{d_G(w,u)}{d(G)-r(G)} \right\rceil = \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil = t.$

That is $\overline{R^*(G)} \in F_{tt+1}$ where $t \ge 2$. Hence the proof. ■



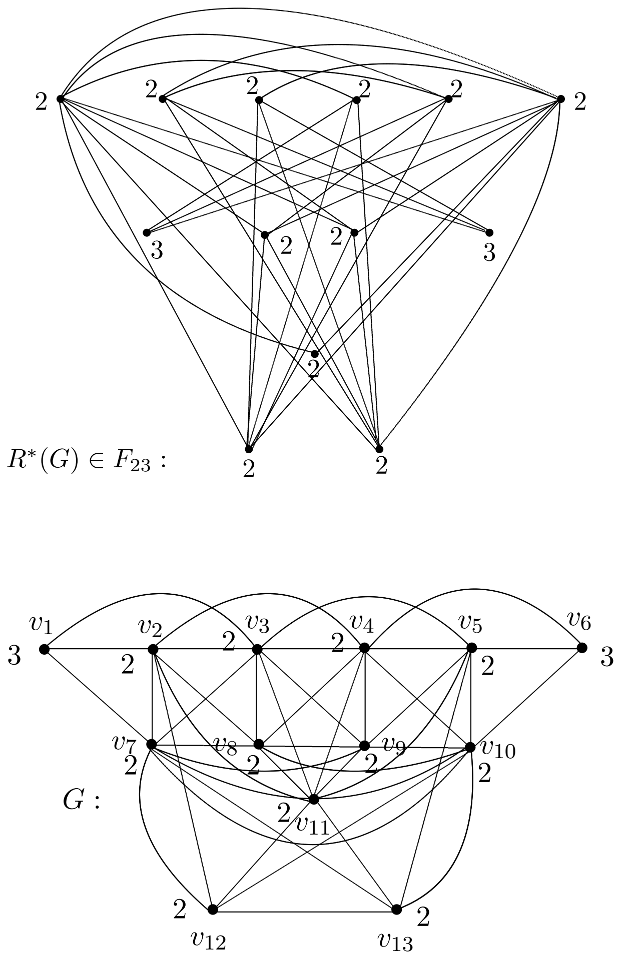



Figure 1. A graph $G$, its super-radial graph $R^*(G)$ and its complement $\overline{R^*(G)}$ with eccentricities.

Note that there is no characterization of $G$ for which $R(G) = G$. But we have the following.

## 3. Characterization of Super-radial Graphs

The concept of super-radial graph does not fall into any one of the cases in the metric operator $X_p$ defined by [20]. The property defined by the super-radial graph operator is vertex free but no unitary, so it does not fall into Theorem 1 in [20]. This motivate us to characterize all super-radial graphs.

**Proposition 7.** *For any graph $G, R^*(G) = G$ if and only if either $G \in F_{11}$ or $G \in F_{23}$ with $G = \overline{G}$.*

***Proof.*** If $G \in F_{11}$, then $R^*(G) = G$. If $G \in F_{23}$ with $G = \overline{G}$, then by Lemma 4, $R^*(G) = \overline{G}$. $G = \overline{G}$ implies that $R^*(G) = G$. Suppose $R^*(G) = G$. If $G \in F_{23}$ with $G \neq \overline{G}$, then by Lemma 4, $R^*(G) = \overline{G}$, but by our assumption $R^*(G) = G$ implies $G = \overline{G}$, which is a contradiction to $G \neq \overline{G}$.

Now let $G \in \mathcal{A} = F_{12} \cup F_{22} \cup F_{24} \cup F_3 \cup F_4$. If $G \in F_{12} \cup F_{22} \cup F_{24}$, then by Proposition 1, Proposition 3 and Corollary 5, $R^*(G) = \overline{G}$ or $R^*(G) = K_p$ or $R^*(G) \in F_4$. Since by assumption $R^*(G) = G$, either $G = K_p$ or $G \in F_4$, which is a contradiction to $G \in F_{12} \cup F_{22} \cup F_{24}$. If $G \in F_3$ with $G$ being a self-centered graph, then $R^*(G) = K_p$. That is $G = K_p$, which is a contradiction to $G \in F_3$. If $G \in F_3$ with $d(G) = r(G) + 1$, then by Lemma 4, $R^*(G) = \overline{G}$. But by our assumption $R^*(G) = G, G = \overline{G}$. Since $G \in F_3, d(\overline{G}) \leq 2$, which is contradiction to $G = \overline{G}$,

Suppose $G \in F_3$ with $r(G) + 2 \leq d(G) \leq 2r(G) - 1$, then by Lemma 6, $R^*(G) \in F_{22} \cup F_{23}$. Since by our assumption $R^*(G) = G, G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_3$. Suppose $G \in F_3$ with $d(G) = 2r(G)$. Then by definition the center vertex in $G$ is isolated in $R^*(G)$. Therefore $R^*(G) \in F_4$. By our assumption $R^*(G) = G, G \in F_4$, which is a contradiction to $G \in F_3$. Suppose $G \in F_4$. Then $R^*(G) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(G) = G, G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_4$. Therefore if $R^*(G) = G$ then either $G \in F_{11}$ or $G \in F_{23}$ with $G = \overline{G}$. ■

Motivated by the above proposition we state the following open problem.

**Problem 8.** Discuss the behaviour of the iterated sequence $G, R^*(G), R^*(R^*(G)), \ldots$ .

**Corollary 9.** *A self-centered graph $G$ is self super-radial if and only if $G \in F_{11}$.*

***Proof.*** Let $G$ be a self-centered graph. Suppose $G \in F_{11}$. Then $R^*(G) = K_p = G$. Therefore $G$ is self super-radial graph. Conversely, suppose $G$ is self super-radial graph. Then there exists a graph $G$ such that $R^*(G) = G$. Now we claim that $G \in F_{11}$. Suppose $G \in F_{ii}$ where $i \geq 2$. Then by definition, $R^*(G) = K_p$, also by assumption $R^*(G) = G, G = K_p$, which is a contradiction to $G \in F_{ii}, i \geq 2$. Hence $G \in F_{11}$. ■

**Lemma 10.** *If $G$ is a disconnected graph, then each component of $\overline{R^*(G)}$ is complete.*

***Proof.*** Since $G$ is a disconnected graph, by definition $R^*(G)$ is connected. Suppose $u$ and $v$ are two vertices of a component $G_i$ of $G$. If $uv \in E(G_i)$, then $uv \notin E(R^*(G))$ and $uv \in E(\overline{R^*(G)})$.

Also, if $uv \notin E(G_i)$, then $uv \notin E(R^*(G))$ and $uv \in E(\overline{R^*(G)})$.

Therefore for any two vertices in a component $G_i$ of $G$ that are either adjacent or nonadjacent in $G$, that vertices are not adjacent in $R^*(G)$. But in $\overline{R^*(G)}$, the above two vertices are adjacent. This is true for any pair of vertices in the component $G_i$ of $G$. Hence $G_i$ is complete in $\overline{R^*(G)}$. ■

**Lemma 11.** *Let $G \in F_{12}$.*

(i) *If each component of $\overline{G}$ is complete, then $G$ is super-radial.*

(ii) *If at least one component of $\overline{G}$ is not complete, then $G$ is not super-radial.*

***Proof.*** (i) Since each component of $\overline{G}$ is complete, by Lemma 4, $R^*(G) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Therefore $G$ is super-radial.

(ii) Since $G \in F_{12}$ by Corollary 5, $R^*(G) = \overline{G}$, $\overline{G}$ is disconnected. Suppose $\overline{G}$ has at least one component which is not complete. Then by definition of super-radial $R^*(\overline{G}) \subset G$. Therefore neither $R^*(G) = G$ nor $R^*(\overline{G}) = G$. Let $H$ be a graph such that $R^*(H) = G$, which is not isomorphic to $G$ and $\overline{G}$.

Suppose $H$ is a self-centered graph, then by Proposition 1, $R^*(H) = K_p, G = K_p$, which is a contradiction to $G \in F_{12}$. Suppose $H \in F_{23} \cup F_{24}$. Then $R^*(H) \in F_{22} \cup F_{23} \cup F_4$. By our assumption $R^*(H) = G, G \in F_{22} \cup F_{23} \cup F_4$, which is a contradiction to $G \in F_{12}$.

Suppose $H \in F_3$ with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$, $G = \overline{H}, \overline{G} = H$. Since $G \in F_{12}, \overline{G}$ is disconnected, $H$ is disconnected which is a contradiction to $H \in F_3$. Suppose $H \in F_3$ with $d(H) = 2r(H)$, then by definition $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$ which is a contradiction to $G \in F_{12}$.

Suppose $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$, by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G$, $G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_{12}$.

Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. If $R^*(H) \in F_{11} \cup F_{22}$, by our assumption $R^*(H) = G, G \in F_{11} \cup F_{22}$, which is a contradiction to $G \in F_{12}$. If $R^*(H) \in F_{12}$, then by Lemma 10, each component of $\overline{R^*(H)}$ is complete. By our assumption $R^*(H) = G, \overline{R^*(H)} = \overline{G}$, each component of $\overline{G}$ is complete, which is a contradiction to our hypothesis $\overline{G}$ has at least one non complete component. Therefore $R^*(H) \notin F_{12}$. By all the above arguments there is no graph $H$ such that $R^*(H) = G$.

Hence $G$ is not super-radial graph. ■

**Lemma 12.** *Let $G \in F_{22}$.*

(i) *If $\overline{G} \in F_{22}$, then $G$ is not a super-radial graph.*

(ii) *If $\overline{G} \in F_{23}$, then $G$ is a super-radial graph.*

(iii) *If $\overline{G} \in F_{24}$, then $G$ is not a super-radial graph.*

(iv) *If* $\overline{G} \in F_3$, *then* $G$ *is a super-radial graph if and only if* $d(\overline{G}) = r(\overline{G}) + 1$.

(v) *If* $\overline{G} \in F_4$, *then* $G$ *is a super-radial graph if and only if each component of* $\overline{G}$ *is complete.*

***Proof.*** (i) Since $\overline{G} \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Let $H$ be a graph such that $R^*(H) = G$, which is not isomorphic to $\overline{G}$. Suppose that $H \in A = F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. If $H \in F_{11}$, then by Proposition 1, $R^*(H) = K_p$. If $H \in F_{12}$, then by Corollary 5, $R^*(H) = \overline{H}$. But $\overline{H}$ is disconnected and $R^*(H) = G, G = \overline{H}$, $G$ is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{12}$.

If $H$ is a self-centered graph, then by Proposition 1, $R^*(H) = K_p$. Let $H \in F_{23}$ with $\overline{H} \in F_{23}$. Suppose $H = \overline{H}$, by Lemma 4, $R^*(H) = \overline{H}$ which implies $R^*(H) = H$. But by assumption $R^*(H) = G, H = G, G \in F_{23}$, which is a contradiction to $G \in F_{22}$.

Suppose $H \neq \overline{H}$, by Lemma 4, $R^*(H) = \overline{H}$. Since $R^*(H) = G, \overline{H} = G$, which implies $H = \overline{G}, \overline{G} \in F_{23}$, which is a contradiction to $\overline{G} \in F_{22}$. Let $H \in F_{23}$ with $\overline{H} \in F_{22}$. Since $H \in F_{23}$, by Lemma 4, $R^*(H) = \overline{H}$. Since by assumption $R^*(H) = G, G = \overline{H}$, implies $\overline{G} = H$, which is a contradiction to our assumption $H \neq \overline{G}$. Therefore $H \notin F_{23}$.

Suppose that $H \in F_{24}$. By Proposition 3, $R^*(H) \subseteq \overline{H}$. Also any vertex $v$ such that $e(v) = 2$ in $H$ is not adjacent to any vertex in $R^*(H)$. Hence $R^*(H)$ is disconnected. Also by assumption $R^*(H) = G$ implies $G$ is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{24}$.

Suppose that $H \in F_3$. If $H \in F_3$ with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption, $R^*(H) = G$ implies $G = \overline{H}, \overline{G} = H$, which is a contradiction to our assumption $H \neq \overline{G}$. If $H \in F_3$ with $d(H) = 2r(H)$ then by Proposition 3, $R^*(H) \subseteq \overline{H}$. Also any vertex $v$ such that $e(v) = r(H)$ in $H$ is not adjacent to any vertex in $R^*(H)$. Hence $R^*(H)$ is disconnected. Also by assumption $R^*(H) = G$ implies $G$ is disconnected which is a contradiction to $G \in F_{22}$.

If $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$, then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. Suppose $R^*(H) \in F_{23}$. Since $R^*(H) = G, G \in F_{23}$, which is a contradiction to $G \in F_{22}$. Therefore $R^*(H) \notin F_{23}$. Suppose $R^*(H) \in F_{22}$. By hypothesis we have $G \in F_{22}$. If $R^*(H) = G$, then $\overline{R * (H)} = \overline{G}$. But by Lemma 6, $\overline{R^*(H)} \notin F_{22}$, which implies $\overline{G} \notin F_{22}$, which is a contradiction to $\overline{G} \in F_{22}$. Hence by the above arguments we conclude that there is no graph $H \in F_3$ such that $R^*(H) = G$.

If $H \in F_4$, then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. If $R^*(H) \in F_{11} \cup F_{12}$,, then by our assumption $R^*(H) = G$, $G \in F_{11} \cup F_{12}$, which is a contradiction to $G \in F_{22}$. Therefore, $R^*(H) \notin F_{11} \cup F_{12}$. If $R^*(H) \in F_{22}$ and by our assumption $R^*(H) = G$, then $\overline{R^*(H)} = \overline{G}$. Since $H \in F_4, \overline{R^*(H)} \in F_4$. Therefore, $\overline{G} \in F_4$, which is a contradiction to $\overline{G} \in F_{22}$. Therefore $H \notin F_4$.

Hence by all the above arguments, we conclude that there is no graph $H$ such that $R^*(H) = G$. Therefore $G \in F_{22}$ with $\overline{G} \in F_{22}$, $G$ is not a super-radial graph.

(ii) Since $\overline{G} \in F_{23}$, by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is, $R^*(\overline{G}) = G$. Hence $G$ is a super-radial graph.

(iii) Since $G \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Since $\overline{G} \in F_{24}$, by Proposition 3, $R^*(\overline{G}) \subseteq \overline{\overline{G}} = G$. But any vertex $v$ such that $e(v) = 2$ in $G$ is not adjacent to any vertex in $R^*(\overline{G})$. Hence $R^*(\overline{G})$ is disconnected.

Let $H$ be a graph such that $R^*(H) = G$ which is not isomorphic to $G$ and $\overline{G}$. Suppose that $H \in \mathcal{A} = F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. If $H$ is a self-centered graph, then $R^*(H) = K_p$. By our assumption $R^*(H) = G, G = K_p$, which is a contradiction to $G \in F_{22}$. If $H \in F_{12}$, then by Corollary 5, $R^*(H) = \overline{H}$. But $\overline{H}$ is disconnected and $R^*(H) = G, G = \overline{H}$, $G$ is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{12}$. Suppose $H \in F_{23}$ with $\overline{H} \in F_{23}$. If $H = \overline{H}$, by Lemma 4, $R^*(H) = \overline{H}$ implies $R^*(H) = H$. But by assumption $R^*(H) = G, H = G$. Hence, $H \in F_{23}$ implies $G \in F_{23}$ which is a contradiction to $G \in F_{22}$.

If $H \neq \overline{H}$, then by Lemma 4, $R^*(H) = \overline{H}$. Since $R^*(H) = G$, $\overline{H} = G$ which implies $H = \overline{G}$. Since $H \in F_{23}, \overline{G} \in F_{23}$, which is a contradiction to $\overline{G} \in F_{24}$. Let $H \in F_{23}$ with $\overline{H} \in F_{22}$. Since $H \in F_{23}$, by Lemma 4, $R^*(H) = \overline{H}$. Since by our assumption $R^*(H) = G, G = \overline{H}$ implies $\overline{G} = H$, which is a contradiction to our assumption $\overline{G} \neq H$. Therefore $H \notin F_{23}$.

Suppose that $H \in F_{24}$. By Proposition 3, $R^*(H) \subseteq \overline{H}$. But any vertex $v$ such that $e(v) = 2$ in $H$ is not adjacent to any vertex in $R^*(H)$. That is $R^*(H)$ is disconnected. By our assumption $R^*(H) = G$ implies $G$ is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{24}$.

If $H \in F_3$ with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption, $R^*(H) = G$ implies $G = \overline{H}$, $\overline{G} = H$, which is a contradiction to our assumption $H \neq \overline{G}$. If $H \in F_3$ with $d(H) = 2r(H)$, then by Proposition 3, $R^*(H) \subseteq \overline{H}$. But any vertex $v$ such that $e(v) = r(H)$ in $H$ is not adjacent to any vertex in $R^*(H)$. That is $R^*(H)$ is disconnected. By our assumption $R^*(H) = G$, implies $G$ is disconnected, which is a contradiction to $G \in F_{22}$. If $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$, then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. Suppose $R^*(H) \in F_{23}$. Since $R^*(H) = G, G \in F_{23}$, which is a contradiction to $G \in F_{22}$. Therefore $R^*(H) \notin F_{23}$.

Suppose $R^*(H) \in F_{22}$. By hypothesis we have $G \in F_{22}$. If $R^*(H) = G$ then $\overline{R^*(H)} = \overline{G}$. By Lemma 6, $\overline{R^*(H)} \notin F_{22} \cup F_{24}$, which implies $\overline{G} \notin F_{22} \cup F_{24}$. In particular, $\overline{G} \notin F_{24}$, which is a contradiction to $\overline{G} \in F_{24}$. Hence we conclude that there is no graph $H \in F_3$ such that $R^*(H) = G$.

If $H \in F_4$, then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. If $R^*(H) \in F_{11} \cup F_{12}$, by our assumption $R^*(H) = G$, $G \in F_{11} \cup F_{12}$, which is a contradiction to $G \in F_{22}$. If $R^*(H) \in F_{22}$

and by our assumption $R^*(H) = G$, then $\overline{R^*(H)} = \overline{G}$. By Lemma 10, $\overline{R^*(H)} \in F_4$ and implies $\overline{G} \in F_4$, which is a contradiction to $\overline{G} \in F_{24}$. Therefore $H \notin F_4$.

Hence by all the above arguments, we conclude that there is no graph $H$ such that $R^*(H) = G$. Therefore, if $G \in F_{22}$ with $\overline{G} \in F_{24}$ is not a super-radial graph.

(iv) Suppose $\overline{G} \in F_3$ with $d(\overline{G}) = r(\overline{G}) + 1$. By Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Therefore $G$ is a super-radial graph. Conversely, suppose $G$ is a super-radial graph. Then there exists a graph $H$ such that $R^*(H) = G$. Suppose $H$ is self-centered graph, then $R^*(H) = K_p$. Suppose $H \in F_{12} \cup F_{23}$, then by Lemma 4, $R^*(H) = \overline{H}$.

By our assumption $R^*(H) = G$, implies $G = \overline{H}$ implies $\overline{G} = H$. Therefore $\overline{G} \in F_{12} \cup F_{23}$, which is a contradiction to $\overline{G} \in F_3$. Suppose $H \in F_{24}$, then $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$, which is a contradiction to $G \in F_{22}$. Suppose $H \in F_3$ with $d(H) = r(H)+1$. Then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$ implies $\overline{H} = G$ implies, $H = \overline{G}$. That is $R^*(\overline{G}) = G$ with $d(\overline{G}) = r(\overline{G}) + 1$.

Suppose $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$. By Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G$ implies $G \in F_{22} \cup F_{23}$. Assuming $R^*(H) \in F_{23}$ implies $G \in F_{23}$, which is a contradiction to $G \in F_{22}$. If $R^*(H) \in F_{22}$ then by Lemma 6, $\overline{R^*(H)} \in F_{tt+1}$. Since $R^*(H) = G, \overline{R^*(H)} = \overline{G}, \overline{G} \in F_{tt+1}$. That is $d(\overline{G}) = r(\overline{G}) + 1$ which is a contradiction to our assumption $r(\overline{G}) + 2 \leq d(\overline{G}) \leq 2r(\overline{G}) - 1$. Therefore $H \notin F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$.

If $H \in F_3$ with $d(H) = 2r(H)$, then $R^*(H) \in F_4$. By our assumption $R^*(H) = G$ implies $G \in F_4$, which is a contradiction to $G \in F_{22}$. Suppose $H \in F_4$ then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. Since $G \in F_{22}$ and $R^*(H) = G$ implies $R^*(H) \in F_{22}$. Then $\overline{R^*(H)} \in F_4$, which implies $\overline{G} \in F_4$, which is a contradiction to $\overline{G} \in F_3$. By all the above argument, there is no graph $H$ such that $R^*(H) \in G$. Therefore, if $\overline{G} \in F_3$, then $G$ is a super-radial graph if and only if $d(\overline{G}) = r(\overline{G})+1$.

(v) Since $G \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Suppose $\overline{G} \in F_4$ with each component of $\overline{G}$ is complete. Then by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Hence $G$ is a super-radial graph. Conversely, suppose $G$ is a super-radial graph. Then there exists a graph $H$ such that $R^*(H) = G$. Since $G \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Therefore $H \neq G$.

Suppose $\overline{G} \in F_4$ with at least one non complete componen, then $R^*(\overline{G}) \subset G$. Therefore $H \neq \overline{G}$. Suppose $\overline{G} \in F_4$ with each component of $\overline{G}$ is complete. Then by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R(\overline{G}) = G$. By our assumption $R^*(H) = G$ implies $R^*(H) = R^*(\overline{G})$ implies $H = \overline{G}$.

Suppose $H$ is a self-centered graph. Then by Proposition 1, $R^*(H) = K_p, G = K_p$, which is a contradiction to $G \in F_{22}$. Suppose $H$ satisfies $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$, $G = \overline{H}$ implies $\overline{G} = H$. But $\overline{G}$ is disconnected, $H$ is disconnected, which is a contradiction to

$d(H) = r(H) + 1$.

Suppose $H$ satisfies $d(H) = 2r(H)$. Then $R^*(H) \in F_4$. By our assumption $G \in F_4$ which is a contradiction to $G \in F_{22}$. Suppose $H$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. If $R^*(H) \in F_{23}$, then by our assumption $G \in F_{23}$ which is a contradiction to $G \in F_{22}$. If $R^*(H) \in F_{22}$, then by Lemma 6, $\overline{R^*(H)} \in F_{tt+1}$. By our assumption $R^*(H) = G$ implies $\overline{R^*(H)} = \overline{G}$. Therefore $\overline{G} \in F_{tt+1}$ which is a contradiction to $\overline{G} \in F_4$.

Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. Since by our assumption, $R^*(H) = G$, $G \in F_{11} \cup F_{12} \cup F_{22}$. By hypothesis $G \in F_{22}$ which implies $G \notin F_{11} \cup F_{12}$. Suppose $R^*(H) \in F_{22}$ and $G \in F_{22}$. But $H \neq \overline{G}$ implies $\overline{H} \neq G$. That is $R^*(H) \neq \overline{H}$. By Lemma 4, $d(H) \neq r(H) + 1$ or $H$ is disconnected in which at least one component is non complete. Therefore, if $\overline{G} \in F_4$ then each component of $\overline{G}$ is complete if and only if $G$ is a super-radial graph. ■

**Lemma 13.** *Let $G \in F_{23}$.*

(i) *If $\overline{G} \in F_{22}$, then $G$ is not a super-radial graph.*

(ii) *If $\overline{G} \in F_{23}$, then $G$ is a super-radial graph.*

***Proof.*** (i) Since $G \in F_{23}$, by Lemma 4, $R^*(G) = \overline{G}$. Since $\overline{G} \in F_{22}$, by Proposition 1, $R^*(\overline{G}) = K_p$. Let $H$ be a graph such that $R^*(H) = G$, which is not isomorphic to $G$ and $\overline{G}$. If $H$ is a self-centered graph then by Proposition 1, $R^*(H) = K_p, G = K_p$, which is a contradiction to $G \in F_{23}$. Suppose $H$ with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G, G = \overline{H}$ implies $\overline{G} = H$ which is a contradiction to $H \neq \overline{G}$.

Suppose $H$ with $d(H) = 2r(H)$. Then $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$, which is a contradiction. Suppose $H$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G, G \in F_{22} \cup F_{23}$. If $R^*(H) \in F_{22}$, then $G \in F_{22}$, a contradiction to $G \in F_{23}$. If $R^*(H) \in F_{23}$, then $G \in F_{23}$. Suppose $R^*(H) = G$ implies $\overline{R^*(H)} = \overline{G}$. Since by Lemma 6, $\overline{R^*(H)} \in F_{tt+1}$ implies $\overline{G} \in F_{tt+1}$, which is a contradiction to $\overline{G} \in F_{22}$. Therefore $H \notin F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$.

Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. Since by our assumption, $R^*(H) = G$, which implies $G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_{23}$. Hence there is no graph $H$ such that $R^*(H) = G$. Therefore if $G \in F_{23}$ with $\overline{G} \in F_{22}$, then $G$ is not a super-radial graph.

(ii) Since $G \in F_{23}$, by Lemma 4, $R^*(G) = \overline{G}$. Since $\overline{G} \in F_{23}$, by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. Hence $G$ is a super-radial graph. ■

**Lemma 14.** *If $G \in F_{24}$, then $G$ is not a super-radial graph.*

***Proof.*** Since $G \in F_{24}, \overline{G} \in F_{22}$. Since $G \in F_{24}$, by definition $R^*(G) \in F_4$. Let $H$ be a graph such that $R^*(H) = G$, which is not isomorphic to $G$. Suppose $H$ is a self-centered graph. Then by Proposition 1, $R^*(H) = K_p$ and by our assumption $G = K_p$, which is a contradiction to $G \in F_{24}$. Suppose $H$ with $d(H) = r(H) + 1$. Then by Lemma 4, $R^*(H) = \overline{H}$ and by our assumption $G = \overline{H}$ it implies $\overline{G} = H$. Since $d(H) = r(H) + 1$ implies $d(\overline{G}) = r(\overline{G}) + 1$, which is a contradiction to $\overline{G} \in F_{22}$.

Suppose $H$ with $d(H) = 2r(H)$. Then $R^*(H) \in F_4$ and by our assumption $G \in F_4$, which is a contradiction to $G \in F_{24}$. Suppose $H$ with $r(H)+2 \leq d(H) \leq 2r(H)-1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption, $R^*(G) = G$ implies $G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_{24}$. Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(H) = G, G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_{24}$. Hence there is no graph $H$ such that $R^*(H) = G$. Therefore $G \in F_{24}$ is not a super-radial graph. ■

**Lemma 15.** *If $G \in F_3$, then $G$ is not a super-radial graph.*

***Proof.*** Suppose $G \in F_3$ is a super-radial graph. Then there exists a graph $H$ such that $R^*(H) = G$. If $H \in F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24}$, then by previous argument $R^*(H) \in F_{11} \cup F_{22} \cup F_{23} \cup F_4$. By our assumption $R^*(H) = G, G \in F_{11} \cup F_{22} \cup F_{23} \cup F_4$, which is a contradiction to $G \in F_3$. Therefore $H \notin F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24}$. Suppose $H \in F_3$ with $d(H) = r(H) + 1$. Then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$ implies $G = \overline{H}$, which implies $\overline{G} = H$. Since $H \in F_3$ with $d(H) = r(H)+1, d(H) \geq 4$ and by Theorem B, $d(\overline{H}) \leq 2$. Since $G = \overline{H}, d(G) \leq 2$, which is a contradiction to $G \in F_3$.

Suppose $H \in F_3$ with $H$ is self-centered graph. Then $R^*(H) = K_p$. By our assumption $R^*(H) = G, G = K_p$, which is a contradiction to $G \in F_3$. Suppose $H \in F_3$ with $d(H) = 2r(H)$. Then $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$, which is a contradiction to $G \in F_3$. Suppose $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G, G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_3$. Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(H) = G, G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_3$. By all the above arguments, there exists no graph $H$ such that $R^*(H) = G$. Hence $G \in F_3$ is not a super-radial graph. ■

**Lemma 16.** *If $G \in F_4$ and $\overline{G} \in F_{11} \cup F_{22}$, then $G$ is not a super-radial graph.*

***Proof.*** Since $\overline{G} \in F_{11} \cup F_{22}$, then $R^*(\overline{G}) = K_p$. Suppose there exists a graph $H$ such that $R^*(H) = G$, which is not isomorphic to $\overline{G}$.

*Case* (i). Suppose $H$ is a self-centered graph. Then by Proposition 1, $R^*(H) = K_p$. By our assumption $R^*(H) = G, G = K_p$, which is a contradiction to $G \in F_4$.

*Case* (ii). Suppose $H$ with $d(H) = r(H)+1$. Then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$, $G = \overline{H}$ implies $\overline{G} = H$. By hypothesis $\overline{G} \in F_{11} \cup F_{22}$ implies $H \in F_{11} \cup F_{22}$, which is a contradiction to $d(H) = r(H) + 1$.

*Case* (iii). Suppose $H$ with $r(H)+2 \leq d(H) \leq 2r(H)-1$. Then by Lemma 6, $R^*H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G$ implies $G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_4$.

*Case* (iv). Suppose $H$ with $d(H) = 2r(H)$. Then $d(H) - r(H) + 1 = 2r(H) - r(H) + 1 = r(H) + 1$. Clearly, every vertex with eccentricity $r(H)$ in $H$ is isolated vertex in $R^*(H)$. Therefore, $R^*(H) \in F_4$.

In $\overline{R^*(H)}$, every isolated vertex in $R^*(H)$ is adjacent to all the vertices of $R^*(H)$. Therefore, $\overline{R^*(H)} \in F_{12}$. By our assumption $R^*(H) = G$. $\overline{R^*(H)} = \overline{G}$ implies $\overline{G} \in F_{12}$, which is a contradiction to $\overline{G} \in F_{11} \cup F_{22}$.

*Case* (v). Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(H) = G$ implies $G \in F_{11} \cup F_{12} \cup F_{22}$ which is a contradiction to $G \in F_4$.

Hence by all the above arguments, $G \in F_4$ and $\overline{G} \in F_{11} \cup F_{22}$ is not a super-radial graph. ■

**Theorem 17.** *A connected graph $G$ is super-radial graph if and only if $G$ has any one of the following properties.*

(i) $G \in F_{11}$,

(ii) $G \in F_{12}$ *with each component of* $\overline{G}$ *being complete,*

(iii) $G \in F_{22}$ *with* $\overline{G} \in F_{23}$,

(iv) $G \in F_{22}$ *and* $\overline{G} \in F_3$ *with* $d(\overline{G}) = r(\overline{G}) + 1$,

(v) $G \in F_{22}$ *and* $\overline{G} \in F_4$ *with each component of* $\overline{G}$ *being complete,*

(iv) $G \in F_{23}$ *with* $\overline{G} \in F_{23}$.

***Proof.*** As the following table exhausts all the possibilities, we get the theorem.

| | $G$ | $\overline{G}$ | **By Lemma/ Proposition** | $G$ **is super-radial** |
|---|---|---|---|---|
| 1 | $F_{11}$ | $F_4$ | 8 | Yes |
| 2 | $F_{12}$ | Each component of $\overline{G}$ is complete. | 12(i) | Yes |
| | | At least one component of $\overline{G}$ is not complete. | 12(ii) | No |
| 3 | $F_{22}$ | $F_{22}$ | 13(i) | No |
| | | $F_{23}$ | 13(ii) | Yes |
| | | $F_{24}$ | 13(iii) | No |
| | | $F_3$ with $d(\overline{G}) = r(\overline{G}) + 1$ | 13(iv) | Yes |
| | | $F_3$ with $d(\overline{G}) \neq r(\overline{G}) + 1$ | 13(iv) | No |

| | | | | |
|---|---|---|---|---|
| | | $F_4$ with each component of $\overline{G}$ beingcomplete | 13(v) | Yes |
| | | $F_4$ with at least one component of $\overline{G}$ being non complete | 13(v) | No |
| 4 | $F_{23}$ | $F_{22}$ | 14(i) | No |
| | | $F_{23}$ | 14(ii) | Yes |
| 5 | $F_{24}$ | $F_{22}$ | 15 | No |
| 6 | $F_3$ | | 16 | No |

■

**Theorem 18.** *A disconnected graph $G$ is a super-radial graph if and only if $\overline{G} \in F_{12}$.*

***Proof.*** Since $G$ is disconnected, $\overline{G} \in F_{11} \cup F_{12} \cup F_{22}$. If $\overline{G} \in F_{11} \cup F_{22}$, then by Lemma 16, $G$ is not a super-radial graph. If $\overline{G} \in F_{12}$, then by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Hence $G$ is a super-radial graph. ■

The following examples show that the notion of super-radial graph is independent of radial graph, antipodal graph, eccentric graph and super-eccentric graph.

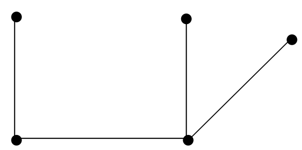


Figure 2. Super-radial graph but not antipodal graph.



Figure 3. Antipodal graph but not super-radial graph.

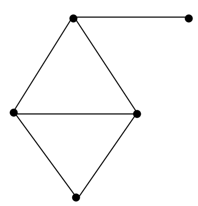


Figure 4. Super-radial graph but not eccentric graph.

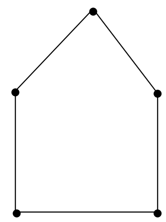


Figure 5. Eccentric graph but not super-radial graph.

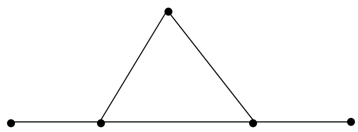


Figure 6. Super-radial graph but not radial graph.

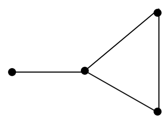


Figure 7. Radial graph but not super-radial graph.

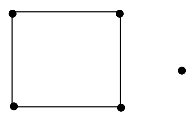


Figure 8. Super-radial graph but not super-eccentric graph.

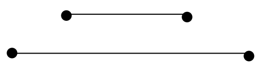


Figure 9. Super-eccentric graph but not super-radial graph.

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