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DENSE ARBITRARILY PARTITIONABLE GRAPHS

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Abstract

A graph G of order n is called *arbitrarily partitionable* (AP for short) if, for every sequence $(n_1, \ldots, n_k)$ of positive integers with $n_1 + \cdots + n_k = n$, there exists a partition $(V_1, \ldots, V_k)$ of the vertex set $V(G)$ such that V_i induces a connected subgraph of order n_i for $i = 1, \ldots, k$. In this paper we show that every connected graph G of order $n \geq 22$ and with $\|G\| > \binom{n-4}{2} + 12$ edges is AP or belongs to few classes of exceptional graphs.

Keywords: arbitrarily partitionable graph, Erdős-Gallai condition, traceable graph, perfect matching.

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1. Introduction and Main Result

We use standard notation of graph theory (cf. [8]). In particular, $|G|$ and $\|G\|$ will stand for the order and the size of a graph G, respectively. The minimum degree of a vertex in a graph G will be denoted by $\delta(G)$. By $c(G)$ we denote the *circumference* of a graph G, i.e., the length of a longest cycle. If G and H are two graphs with disjoint vertex sets, then the *join* of G and H is the graph, denoted by $G \vee H$, with the vertex set $V(G \vee H) = V(G) \cup V(H)$ and the edge set

$$E(G \vee H) = E(G) \cup E(H) \cup \{xy \,:\, x \in V(G), y \in V(H)\}.$$

A sequence $(n_1, \ldots, n_k)$ of positive integers is called *admissible* for a graph $G = (V, E)$ of order n if $n_1 + \cdots + n_k = n$. An admissible sequence is said to be *realizable in* G if there exists a partition of V into k parts $(V_1, \ldots, V_k)$ such that $|V_i| = n_i$ and the subgraph $G[V_i]$ induced by V_i is connected, for every $i = 1, \ldots, k$. Such a partition is called a *realization* of the sequence $(n_1, \ldots, n_k)$ in G. Note that in fact the ordering of $(n_1, \ldots, n_k)$ is irrelevant, i.e., if this sequence is realizable in G, then it is also realizable after any permutation of its elements. We say that G is *arbitrarily partitionable* (AP for short) if every admissible sequence is realizable in G.

A simple example of an arbitrarily partitionable graph is a path P_n. Two obvious and well-known facts play a key role in this paper.

Proposition 1. *If G has a spanning subgraph which is AP, then G is AP itself.*

Proposition 2. *Every traceable graph is AP.*

The following easy observation sometimes makes proofs shorter and allows us to assume throughout the paper that every admissible sequence has all elements greater than 1.

Proposition 3 [15]**.** *A graph G is AP if and only if every admissible sequence $(n_1, \ldots, n_k)$ with $n_i \geq 2$ for $i = 1, \ldots, k$ is realizable in G.*

The notion of AP graphs was introduced by Barth, Baudon and Puech [1] (and independently by Horňák and Woźniak [13]) to model a problem in the design of computer networks (see [1] for details). The concept of arbitrarily partitionable graphs, sometimes also called *arbitrarily vertex decomposable* or *fully decomposable* or just *decomposable*, has spawned numerous papers. Some of them investigate AP graphs within some classes of graphs (e.g., [1, 2, 9, 7, 13], KPWZ1). Horňák, Tuza and Woźniak [14] introduced the notion of *on-line arbitrarily partionable* graphs, and then a few other definitions strengthening the condition for AP graphs appeared (e.g., [5, 6, 3, 16]). Here we present only those previous results on AP graphs we make use of in the paper.

A sequence $(d, \ldots, d)$ of length λ will be denoted by $(d)^\lambda$. A caterpillar with three leaves is denoted by $\mathrm{Cat}(a, b)$ if it is obtained from the star $K_{1,3}$ by substituting two of its edges by paths of orders a and b, respectively (see Figure 1). As $b = n - a$, we will later also use a shorter notation $\mathrm{Cat}(a)$. The following result was proved by Barth *et al.* [1], and independently by Horňák and Woźniak [13].

Theorem 4. *The caterpillar $Cat(a, b)$, with $2 \leq a \leq b$, is AP if and only if a and b are relatively prime. Moreover, each admissible and nonrealizable sequence is of the form $(d)^k$, where $a \equiv b \equiv 0 \pmod d$ and $d > 1$.*

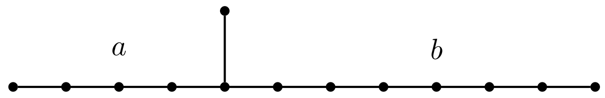


Figure 1. $\mathrm{Cat}(a, b)$ with $a = 5, b = 8$.

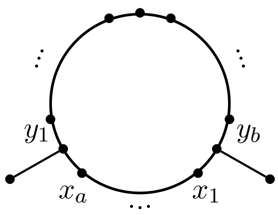


Figure 2. $\mathrm{Sun}(a, b)$.

A *sun with r rays* is a graph of order $n \geq 2r$ with r pendant vertices $u_1, \ldots, u_r$ whose deletion yields a cycle C_{n-r}, and each vertex v_i on C_{n-r} adjacent to u_i is of degree three. If the sequence of vertices v_i is situated on the cycle C_{n-r} in such a way that there are exactly $a_i \geq 0$ vertices, each of degree two, between v_i and v_{i+1}, $i = 1, \ldots, r$ (the indices taken modulo r), then this sun is denoted by $\mathrm{Sun}(a_1, \ldots, a_r)$. Suns with two and three rays are presented in Figures 2 and 3, respectively. Kalinowski, Pilśniak, Woźniak and Zioło characterized all AP suns with at most three rays.

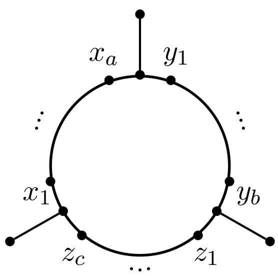


Figure 3. $\mathrm{Sun}(a, b, c)$.

Theorem 5 [15]**.** *A sun with two rays* $\mathrm{Sun}(a, b)$ *is AP if and only if at most one of the numbers a and b is odd. Moreover,* $\mathrm{Sun}(a, b)$ *of order n is not AP if and only if* $(2)^{n/2}$ *is the unique admissible and nonrealizable sequence.*

Theorem 6 [15]**.** *A sun with three rays* $\mathrm{Sun}(a, b, c)$ *is AP if and only if none of the following three conditions is fulfilled:*

(1) *at most one of the numbers* a, b, c *is even,*

(2) $a \equiv b \equiv c \equiv 0 \pmod 3$,

(3) $a \equiv b \equiv c \equiv 2 \pmod 3$.

Moreover, if $\mathrm{Sun}(a, b, c)$ *is not AP, then at least one of the following three sequences* $(2)^{n/2}$, $(3)^{n/3}$, $(3, (2)^{(n-3)/2})$ *is admissible and nonrealizable.*

In this paper we consider the following question. How many edges in a connected graph G guarantee that a graph is AP or belongs to few families of exceptional graphs?

Dense AP graphs were already investigated in another context. This was initiated by Marczyk who proved in [18], [19] some Ore-type sufficient conditions for a graph to be AP. The best result in this direction is due to Horňák, Marczyk, Schiermeyer and Woźniak.

Theorem 7 [12]**.** *Every connected graph* G *of order* $n \geq 20$ *such that the degree sum of each pair of nonadjacent vertices is at least* $n - 5$ *is AP if and only if* G *admits a perfect matching or a quasi-perfect matching* (*i.e., a matching omitting exactly one vertex*)*.*

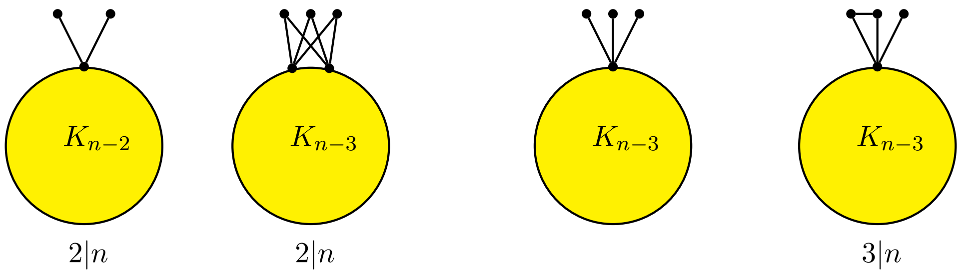


Figure 4. Four graphs such that every non-AP graph G with $\|G\| > \binom{n-4}{2} + 12$ is a spanning subgraph of one of them (below each graph, requirements on the order n are given).

Let us formulate now our main result.

Theorem 8. *If* G *is a connected graph of order* $n \geq 22$ *and size*

$$\|G\| > \binom{n-4}{2} + 12,$$

then G is AP unless G is a spanning subgraph of one of the graphs depicted in Figure 4.

It is easily seen that none of four graphs in Figure 4 is AP whenever its order n meets the divisibility condition given below the graph. By Proposition 1, every spanning subgraph is non-AP, as well. Observe also that the first two graphs have circumference $c(G) = n-2$ and the other two have $c(G) = n-3$.

It has to be noted that for $n < 22$, there are more graphs of order n and size greater than $\binom{n-4}{2} + 12$ that are not AP. For example, the graph $G = K_{(n-2)/2} \vee \overline{K}_{(n+2)/2}$ has no perfect matching, and its size $\|G\| = \frac{1}{2}[\frac{n-2}{2}(n-1) + \frac{n+2}{2} \cdot \frac{n-2}{2}]$ is greater than $\binom{n-4}{2} + 12$ for every even $n = 10, \ldots, 20$. Another example is the graph $G = K_{(n-3)/2} \vee \overline{K}_{(n+3)/2}$ which has no realization of the sequence $(3, (2)^{\frac{n-3}{2}})$, and its size $\|G\| = \frac{1}{2}[\frac{n-3}{2}(n-1) + \frac{n+3}{2} \cdot \frac{n-3}{2}]$ is greater than $\binom{n-4}{2} + 12$ for every odd $n = 11, \ldots, 17$.

2. Preliminary Results

This section contains an initial stage of the proof of Theorem 8. We will make use of some classical sufficient conditions for the existence of long cycles in a graph.

Theorem 9 (Erdős, Gallai [11]). *Let G be a graph of order n. If $\|G\| > \frac{c}{2}(n-1)$, then $c(G) > c$.*

Theorem 9 has been extended by Woodall.

Theorem 10 (Woodall [20]). *Let G be a graph of order $n = t(c-1) + p$, where $c \geq 2$, $t \geq 0$ and $1 \leq p \leq c$. If*

$$\|G\| > t\binom{c}{2} + \binom{p}{2},$$

then $c(G) > c$.

Taking $t = 1, c = n - \delta$ and $p = \delta + 1$, we obtain the following

Corollary 11. *If $n = |G|, \delta = \delta(G)$ and*

$$\|G\| > \binom{n-\delta}{2} + \binom{\delta+1}{2},$$

then $c(G) > n - \delta$.

The next theorem is the well-known Erdős sufficient condition for hamiltonicity depending on the size and minimum degree.

Theorem 12 (Erdős [10]). *Let G be a graph of order n and with minimum degree δ. Denote*

$$f(n,\delta) = \max\left\{\binom{n-\delta}{2} + \delta^2, \binom{n-\lfloor\frac{n-1}{2}\rfloor}{2} + \left\lfloor\frac{n-1}{2}\right\rfloor^2\right\}.$$

If $\delta \geq \frac{n}{2}$ or $\|G\| > f(n,\delta)$, then G is Hamiltonian.

We can use Theorem 12 for traceability as follows. Let $H = G \vee K_1$. Then H is Hamiltonian if and only if G is traceable. Denote $g(n,\delta) = f(n+1,\delta+1) - n$. Thus

$$g(n,\delta) = \max\left\{\binom{n-\delta}{2} + (\delta+1)^2 - n, \binom{n+1-\lfloor\frac{n}{2}\rfloor}{2} + \left\lfloor\frac{n}{2}\right\rfloor^2 - n\right\}.$$

As $\binom{n-\delta}{2} + (\delta+1)^2 - n = \binom{n-\delta-1}{2} + \delta(\delta+1)$, this justifies the following result.

Corollary 13. *Let G be a graph of order n and with minimum degree δ. If $\delta \geq \frac{n-1}{2}$ or*

$$\|G\| > \max\left\{\binom{n-\delta-1}{2} + \delta(\delta+1), \binom{n+1-\lfloor\frac{n}{2}\rfloor}{2} + \left\lfloor\frac{n}{2}\right\rfloor^2 - n\right\},$$

then G is traceable, and hence AP.

Suppose G is a graph with minimum degree δ and with $\|G\| > \binom{n-4}{2} + 12$. It follows from Corollary 13 that G is traceable whenever $\delta \geq \frac{n-1}{2}$ or $g(n,\delta) \leq g(n,3)$. Observe that $\binom{n-\delta-1}{2} + \delta(\delta+1)$ is a quadratic polynomial with respect to δ, so the latter inequality holds unless $g(n,\delta) = \binom{n+1-\lfloor\frac{n}{2}\rfloor}{2} + \lfloor\frac{n}{2}\rfloor^2 - n$ and

$$\binom{n-4}{2} + 12 < \binom{n+1-\lfloor\frac{n}{2}\rfloor}{2} + \left\lfloor\frac{n}{2}\right\rfloor^2 - n.$$

We solve this inequality regarding to the parity of the order n of G. If n is even, then the inequality is equivalent to $n^2 - 30n + 176 < 0$, so it holds only if $9 \leq n \leq 21$. If n is odd, then we have $n^2 - 24n + 175 < 0$, and this does not hold for any n.

Obviously, every connected graph G with $c(G) = n-1$ is traceable, and hence AP.

Thus, Corollary 11 and Corollary 13 for $\delta = 3$ imply that for the proof of our main result we are left with the following situation

$$n \geq 22, \quad \|G\| > \binom{n-4}{2} + 12, \quad 1 \leq \delta(G) \leq 2, \text{ and } n-3 \leq c(G) \leq n-2.$$

The rest of our proof is divided into two parts corresponding to $c(G) = n-2$ (Section 3) and $c(G) = n-3$ (Section 4).

Let us state yet a lemma that follows the approach in [17] and will be used in both sections. First, we introduce some notation. If C is a cycle in a graph $G = (V, E)$, then each vertex of C adjacent to a vertex outside C is called an *attachment vertex*. Fix an orientation of C. For two vertices $x, y \in V(C)$ we denote by $C[x, y]$ the path of C from x to y along this orientation, and by $\overleftarrow{C}[x, y]$ the path from x to y along the reverse orientation of C. For a vertex $x \in V(C)$ we denote by x^+, x^- its successor and its predecessor along the orientation of C. We also denote $d_C(x) = |N(x) \cap V(C)|$. For two sets $A, B \subset V$, let $E(A, B) = \{xy \in E : x \in A, y \in B\}$.

Lemma 14. *Let $G = (V, E)$ be a connected graph of order $n \geq 22$ with $\delta(G) \leq 2$ and $\|G\| > \binom{n-4}{2} + 12$. Let C be a longest cycle in G such that the set $V \setminus V(C)$ is not a clique.*

(1) *If $c(G) = n-3$, then each vertex outside C is of degree one.*

(2) *If $c(G) = n-2$, then each vertex outside C has at most three neighbors on C.*

Proof. Let $k = k(G) = \max\{d_C(u) : u \in V \setminus V(C)\}$, and let u be a vertex outside C with $d_C(u) = k$ and $N(u) \cap V(C) = \{u_1, \ldots, u_k\}$. Fix an orientation of C. Clearly, $k \leq \frac{c(G)}{2}$ and the set $X = \{u_1^+, \ldots, u_k^+\}$ is independent since C is a longest cycle in G. Moreover, for any pair u_i^+, u_j^+ with $i \neq j$ and any $z \in C[u_i^{++}, u_j]$ we have $u_i^+ z^+ \notin E$ or $z u_j^+ \notin E$, otherwise C would not be a longest cycle. Let $C_1 = C[u_i^{++}, u_j], C_2 = C[u_j^{++}, u_i]$. Then a classical counting argument (cf. [8]) shows that

$$\begin{aligned} d_C(u_i^+) + d_C(u_j^+) &= d_{C_1}(u_i^+) + d_{C_1}(u_j^+) + d_{C_2}(u_i^+) + d_{C_2}(u_j^+) \\ &\leq |V(C_1)| + 1 + |V(C_2)| + 1 = |V(C)|. \end{aligned}$$

Summing up this inequality for all $\binom{k}{2}$ possible pairs of vertices and dividing by $k-1$ we obtain

$$\sum\nolimits_{i=1}^{k} d_C(u_i^+) \leq \frac{k}{2} c(G).$$

Now we want to estimate $|\bar{E}(C)|$, i.e., the number of edges within C that are missing in G. Since X is independent, all edges incident to vertices from X are contained in $E(X, V(C) \setminus X)$. Hence $|\bar{E}(C)| \geq |\bar{E}(X, V(C) \setminus X)| + |\bar{E}(G[X])| \geq k(c(G) - k) - \frac{k}{2} c(G) + \binom{k}{2} = \frac{k}{2}(c(G) - k - 1)$. As $V \setminus V(C)$ is not a clique and each vertex of $V \setminus V(C)$ is connected to C by at most k edges, the number $f(k) = \|\bar{G}\|$ of edges missing in the graph G satisfies the inequality

$$f(k) \geq 1 + (n - c(G))\,(c(G) - k) + \frac{k}{2}\,(c(G) - k - 1)\,.$$

However, it is not difficult to see that if $k = k(G)$ and $f(k) = 1+(n-c(G))(c(G)-k) + \frac{k}{2}(c(G) - k - 1)$ for a graph G, then $\delta(G) = d(u) = k + n - c(G) - 2$ where u is a vertex outside C. Note that $k + n - c(G) - 2 \geq k + 1$ if $c(G) \leq n - 3$. But $\delta(G) \leq 2$ by assumption, hence we have to increase $f(k)$ by $k - 1$, so actually

$$f(k) \geq (n - c(G)) (c(G) - k) + \frac{k}{2} (c(G) - k + 1) .$$

Note that $\binom{n}{2} - \binom{n-4}{2} - 12 = 4n-22$, hence $f(k) \leq 4n-23$ since $\|G\| > \binom{n-4}{2}+12$.

Consider first the case $c(G) = n-3$. Then $f(k) \leq 3(n-k-3)+\frac{k}{2}(n-k-2)$. Suppose, contrary to the claim, that $k \geq 2$. We search for the smallest value of $f(k)$. The derivative $f'(k) = \frac{n}{2} - k - 4$ is nonnegative for $2 \leq k \leq \frac{n}{2} - 4$. Hence $f(k)$ is increasing for $2 \leq k \leq \frac{n}{2}-4$, and decreasing for $\frac{n}{2}-4 \leq k \leq \frac{n-3}{2}$. We have $f(2) = 4n - 19 > 4n - 23$. Also, $f(\frac{n-3}{2}) = \frac{n^2+8n-33}{8}$, and $f(\frac{n-3}{2}) - (4n - 23) = \frac{1}{8}(n^2 - 24n + 151) > 0$ for any $n \geq 14$. Thus $f(k) > 4n - 23$ for every k with $2 \leq k \leq \frac{n-3}{2}$, a contradiction.

Now, let $c(G) = n - 2$. Thus $f(k) = 2(n - k - 2) + \frac{k}{2}(n - k - 1)$ and $f'(k) = \frac{n-5}{2} - k$. Hence $f(k)$ is increasing for $2 \leq k \leq \frac{n-5}{2}$, and decreasing for $\frac{n-5}{2} \leq k \leq \frac{n-2}{2}$. Note that $f(4) = \frac{4}{n} - 22 > 4n - 23$, and $f(\frac{n-2}{2}) = \frac{1}{8}(n^2 + 6n - 16) > 4n - 23$ because $\frac{1}{8}(n^2 + 6n - 16) - (4n - 23) = \frac{1}{8}(n^2 - 26n + 168) > 0$. Therefore $k \leq 3$. ■

In most cases considered in the next two sections, we apply the following strategy. To prove that a graph $G = (V, E)$ satisfying certain conditions has no more than $\binom{n-4}{2} + 12$ edges, we choose a graph G_0 such that $V(G_0) = V$, $\|G_0\| \leq \binom{n-4}{2} + 12$, and there exists an injective mapping of $E \setminus E(G_0)$ into $E(G_0) \setminus E$, whence $\|G\| \leq \|G_0\|$.

3. Proof for Circumference $n - 2$

To prove that Theorem 8 holds for graphs with circumference $n-2$, it is enough to justify the following.

Proposition 15. *If $G = (V, E)$ is a connected graph of order $n \geq 22$ with $c(G) = n - 2$ and $\|G\| > \binom{n-4}{2} + 12$, then G is AP unless n is even and G is a spanning subgraph of one of two graphs of even order shown in Figure* 5.

Proof. Let C be a longest cycle in G and let u, v be the two vertices outside C. Clearly, G is traceable if $uv \in E$. Then assume $uv \notin E$.

First suppose that there is only one attachment vertex. If n is even, then the sequence $(2)^{n/2}$ is not realizable, G is not AP and is a spanning subgraph of the first graph in Figure 5 when $\|G\| > \binom{n-4}{2} + 12$ what is possible for $n \geq 10$.

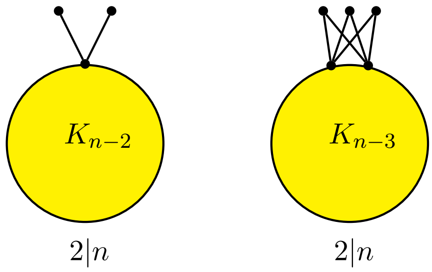


Figure 5. Exceptional supergraphs with circumference $n-2$.

If n is odd, then an admissible sequence contains an element $n_i \geq 3$. We take a part V_i containing u, v and their common neighbour, and the remaining graph is traceable, so G is AP.

Now assume that there are at least two attachment vertices. For every pair of independent edges uu', vv' with $u', v' \in V(C)$, the deletion of u', v' from C yields two paths of orders a and b such that $a+b = n-4$ and $0 \leq a \leq b \leq n-4$. Thus $\mathrm{Sun}(a, b)$ is a spanning subgraph of G. By Theorem 5, the graph G is AP when at most one of the numbers a, b is odd (in particular when n is odd). Henceforth, we assume that n is even and both a and b are odd for any pair of independent edges uu', vv'. Again, Theorem 5 implies that to prove that G is AP, it suffices to show that the sequence $(2)^{n/2}$ is realizable in G, i.e., G admits a perfect matching. Choose edges uu', vv' such that a is as large as possible (and not greater than b), and denote the vertices of C by $u', x_1, \ldots, x_a, v', y_1, \ldots, y_b$ according to the orientation of C. Suppose that G is not AP.

Case $a = 1$. Suppose first that there are only two attachment vertices. Then $d(u) \leq 2$ and $d(v) \leq 2$. Let $d(x_1) = 2$. If n is even, then G has no perfect matching and is a spanning subgraph of the second graph in Figure 5 whenever $\|G\| > \binom{n-4}{2} + 12$, and the latter inequality may hold for $n \geq 12$.

Then assume that $d(x_1) \geq 3$, i.e., C has at least one chord incident to x_1. We will show that in this case there does not exist a non-AP graph satisfying our assumptions. Indeed, suppose there exists such a graph G. First observe that x_1 cannot be adjacent to any vertex y_{2l-1} since otherwise G would have a perfect matching: $\{uu', vv', x_1y_{2l-1}\} \cup \{x_{2i-1}x_{2i} : i = 1, \ldots, l-1\} \cup \{x_{2i}x_{2i+1} : i = l, \ldots, \frac{b-1}{2}\}$. Suppose l is the smallest positive integer such that $x_1y_{2l} \in E$. Without loss of generality, we may assume that $2l < \frac{b}{2}$ (we can change the orientation of C, if necessary), i.e., $l \leq \frac{n-4}{4}$. For any $i \leq l$ and $j \geq l$, an edge

$y_{2i-1}y_{2j+1}$ would give a perfect matching if it appeared in G. The number of these edges equals $l(\frac{n-4}{2}-l) \geq l^2$, and they are missing in G. Moreover, for every $p > l$, an edge $x_1y_{2p} \in E$ creates a new missing edge $y_{2p-1}y_{2p+1}$, and an edge $y_1y_{2p} \in E$, except for $p = \frac{n-6}{2}$, creates another missing edge $y_{2p-1}y_{2p+3}$. Hence $\|G\| \leq \binom{n-4}{2} + 8 + 2l - 2 - l(\frac{n-4}{2} - l) + 1 \leq \binom{n-4}{2} + 9 + 2l - l^2 \leq \binom{n-4}{2} + 9$, a contradiction.

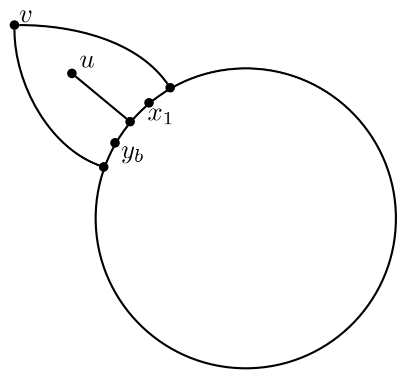


Figure 6. Three attachment vertices for $a = 1$.

It is easily seen that the number of attachment vertices can be at most three as $a = 1$ was chosen greatest possible. Then one of the vertices outside C, say u, is a pendant vertex and v is adjacent to y_{b-1} (see Figure 6). Suppose that G satisfies our assumptions and has no perfect matching. Then clearly, x_1y_b, as well as x_1y_{2i+1} and y_by_{2i+1} cannot belong to E. Consider a graph G_0 of size $\binom{n-4}{2} + 10$ such that $V(G_0) = V$, the set $V \setminus \{u, v, x_1, y_b\}$ is a clique, and $E(G_0)$ contains also the edges $uu', vv', vu', vy_{b-1}, v'x_1, x_1u', u'y_b, y_by_{b-1}, x_1y_{b-1}, y_bv'$. For every $l = 1, \ldots, \frac{b-3}{2}$, whenever x_1y_{2l} belonged to E, the edge $y_{2l-1}y_{2l+1}$ would create a perfect matching in G, thus it is missing in G, and whenever $y_by_{2l} \in E$ (except $2l = b-3$), then $y_{2l-1}y_{2l+3}$ is missing in G. Therefore $\|G\| \leq \|G_0\| + 1 = \binom{n-4}{2} + 11 < \binom{n-4}{2} + 12$, a contradiction.

Case $a \geq 3$. It follows from Lemma 14 that the vertices u, v are of degree at most three, since the total number of vertices incident to them cannot be greater than six. Let G_1 be a graph of size $\binom{n-4}{2} + 12$ containing these six edges, six edges $x_1x_2, x_1u', x_1v', x_ax_2, x_au', x_av'$ and $\binom{n-4}{2}$ edges of the clique $V \setminus \{u, v, x_1, x_a\}$. If $a = 3$, we set $G_0 = G_1$. For $a \geq 5$, we define G_0 as follows. We add to G_1 the edges x_1x_{2j}, x_ax_{2j}, $j = 2, \ldots, \frac{a-1}{2}$, and delete the edges $x_{2i-1}y_{2j-1}$, $i = 2, \ldots, \frac{a-1}{2}, j = 1, \ldots, \frac{b+1}{2}$. Thus the number of added edges equals $a - 3$, and the number of deleted ones equals $\frac{a-1}{2} \cdot \frac{b+1}{2}$ and is not smaller than $\frac{a^2-1}{4}$ since $b \geq a$. Therefore $\|G_0\| < \|G_1\|$. Observe that the set $\{x_{2i} : i =$

$1, \ldots, \frac{a-1}{2}\} \cup \{y_i : i = 1, \ldots, b\} \cup \{u', v'\}$ forms a clique in G_0, and the only edges that may appear in G and not in G_0 are of the form $x_\nu x_{2j-1}$ or $x_\nu y_{2l}$, where $\nu \in \{1, a\}$.

For any $i = 1, \ldots, \frac{a-1}{2}$, if $x_1 x_{2i+1} \in E$, then $y_1 x_{2i} \notin E$, and if $x_a x_{2i-1} \in E$, then $y_b x_{2i} \notin E$, otherwise G has a perfect matching. For any $j = 1, \ldots, \frac{b-1}{2}$, if $x_1 y_{2j} \in E$, then $y_1 y_{2j+1} \notin E$, and if $x_a y_{2j} \in E$, then $y_b y_{2j-1} \notin E$. It is easy to see that we have just defined an injective mapping of $E \setminus E(G_0)$ into $E(G_0) \setminus E$ unless the edge $y_1 y_b$ was counted twice as a missing edge in E. This means that either $\|G\| \le \|G_0\| \le \binom{n-4}{2} + 12$ or $\|G\| = \|G_0\| + 1 = \binom{n-4}{2} + 13$. But it is easy to see that in the latter case $\delta(G) = 3$, so G is traceable by Corollary 13. We thus obtained a contradiction in both cases. ■

4. Proof for Circumference $n-3$

In this section we accomplish the proof of our main result by showing that Theorem 8 holds for graphs with circumference $n-3$. Let us introduce some additional notation first.

For any two vertices x and y of a cycle C of a sun S with a fixed orientation, we denote by xCy the caterpillar consisting of a path $C[x, y]$ together with the leaves of S if the corresponding attachment vertex belongs to $C[x, y]$. By $y\overleftarrow{C}x$ we denote the same caterpillar but in the reverse order.

Let $G = (V, E)$ be a connected graph of size $\|G\| > \binom{n-4}{2} + 12$, and let C be a longest cycle of G of length $n - 3$. Lemma 14 states that each of three vertices u, v, w outside C has at most one neighbor on C. The attachment vertices of C adjacent to u, v, w are denoted by u', v', w', respectively (some of the vertices u', v', w' may coincide or do not exist if there are less than three attachment vertices). If C has three attachment vertices, denote the vertices of C by $u', x_1, \ldots, x_a, v', y_1, \ldots, y_b, w', z_1, \ldots, z_c$ according to a fixed orientation of C. Let $X = \{x_i : i = 1, \ldots, a\}$, $Y = \{y_i : i = 1, \ldots, b\}$, $Z = \{z_i : i = 1, \ldots, c\}$. If C has only two attachment vertices, then we assume that Z is empty, and C is the sequence $u', x_1, \ldots, x_a, v', y_1, \ldots, y_b$.

If $a \ge 1$ and $b \ge 2$, then two edges of the form $x_1 y_{i+1}, x_a y_i$ are said to be a *good couple* from X to Y. The case $a = 1$ is allowed. Analogously we define good couples from Y to X, from X to Z and so on (see Figure 7).

Lemma 16. *Let C be a longest cycle of a connected graph $G = (V, E)$ of size $\|G\| > \binom{n-4}{2} + 12$ and circumference $c(G) = n - 3$. If the number of attachment vertices of C is two and their distance on C is at least three, then there exists a good couple of edges in G.*

Proof. Using the notation from the beginning of this section, we may assume without loss of generality that w is a vertex adjacent to v or v'. Then $G - w$ is

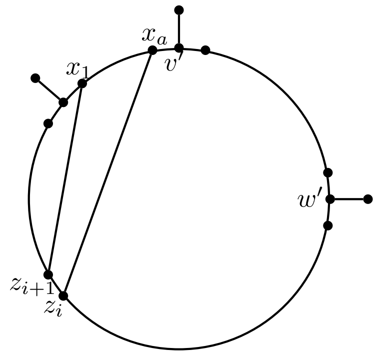


Figure 7. A good couple of edges from X to Z.

spanned by $\mathrm{Sun}(a,b)$ where $2 \le a \le b$ and $a+b=n-5$. There are three or four edges outside C, therefore the number of chords of C missing in the graph G is less than $n+12$. Indeed, $\binom{n-3}{2}+4-\binom{n-4}{2}-12=n-12$.

Suppose that there is no good couple of edges in G. Then for every $i=1,\ldots,b-1$, if $x_1y_{i+1}\in E$, then $x_ay_i\notin E$. It follows that the number of missing edges between $\{x_1,x_a\}$ and Y is at least $b-1$. Moreover, for every $i=2,\ldots,a-1$, if $y_1x_{i+1}\in E$, then $y_bx_i\notin E$. Therefore the total number of missing chords between X and Y is at least $a-3+b-1=n-9>n-12$, a contradiction. ■

Lemma 17. *Let C be a longest cycle of a connected graph $G=(V,E)$ of size $\|G\|>\binom{n-4}{2}+12$ and circumference $c(G)=n-3$. If C has three attachment vertices and no two of them are consecutive vertices on C, then there exists a good couple of edges in G.*

Proof. By assumptions, the graph G is spanned by $\mathrm{Sun}(a,b,c)$ where $1\le a\le b\le c$ and $a+b+c=n-6$. Lemma 14 implies that there are exactly three edges outside C. Hence, there are less than $n-12$ chords of C missing in G. Assume that G has no good couple of edges.

Suppose first that $a=b=1$. Then $x_1z_{i+1}\in E$ implies $x_1z_i\notin E$, $i=1,\ldots,c-1$, otherwise these two edges would be a good couple. Therefore, there are at least $\frac{c-1}{2}$ missing edges from x_1 to Z. Analogously, the number of missing edges between y_1 and Z is not less than $\frac{c-1}{2}$. Altogether, we get at least $c-1=n-9>n-12$ missing chords of C, a contradiction.

Suppose now that $a=1$ and $b\ge 2$. We analogously infer that there are at least $\frac{c-1}{2}$ missing edges from x_1 to Z. For any $i=1,\ldots,c-1$, whenever y_1z_{i+1} is an edge in G, then y_bz_i is not, for, otherwise these two edges would be a good couple. Thus there are at least $c-1$ chords of C between $\{y_1,y_b\}$ and Z missing in G. Furthermore, $z_1y_{i+1}\in E$ implies $z_cy_i\notin E$ for $i=2,\ldots,b-1$, so we get additional $b-3$ missing edges between Y and Z. Hence there are at least

$\frac{c-1}{2} + c - 1 + b - 3 \leq \frac{3}{2}c + b - 5 = n - 12 + \frac{c}{2} > n - 12$, again a contradiction.

Finally, let $a \geq 2$. Denote by ρ the number of edges joining x_a with the set Z. Then $c - \rho$ edges between x_a and Z are missing. Moreover, for each edge $x_a z_i$, the edge $x_1 z_{i+1}$ is missing. Therefore, at least ρ edges between x_1 and Z are missing. So, since there is no good couple from X to Z, at least $(c - \rho) + \rho = c$ edges joining the vertices x_1 and x_a with Z are missing. Analogously, since there is no good couple from X to Y as well as Y to Z we can show that there are at least b missing edges joining the vertices x_1 and x_a with Y and at least c missing edges joining the vertices y_1 and y_b with Z. Therefore, there are at least $c+(b+c)$ missing edges, and since $b + c \geq \frac{2}{3}(n-6)$ and $c \geq \frac{1}{3}(n-6)$, we have at least $n - 6 > n - 12$ missing edges, a contradiction. ■

Let C have only one attachment vertex u'. If the subgraph $G[\{u, v, w\}]$ induced by u, v, w is traceable, then G is traceable itself. If all three vertices u, v, w are pendant in G, then G is a spanning subgraph of the third graph in Figure 4 and is not AP for any n because either $(2)^{n/2}$ or $(3, (2)^{(n-3)/2})$ is an admissible and nonrealizable sequence. Otherwise, $G[\{u, v, w\}]$ has exactly one edge, say uv, and G is a spanning subgraph of the fourth graph in Figure 4. Then G is not AP if and only if the order n of G is a multiple of three since the sequence $(3)^{n/3}$ cannot be realized. For any other n, every admissible sequence $(n_1, \ldots, n_k)$ either has an element $n_i = 2$ or $n_i \geq 4$. If $n_i = 2$ we take a corresponding part $V_i = \{u, v\}$ and if $n_i \geq 4$ we take $V_i \supseteq \{u, v, w, u'\}$. Then $G - V_i$ is traceable, and hence AP.

Suppose that C has two attachment vertices u', v' with $uu', vv' \in E$. As before, we assume that w is adjacent v or v'. Observe that the subgraph $G' = G - w$ of size $\|G'\| \geq \binom{n-4}{2} + 11$ is spanned by a sun with two rays $\mathrm{Sun}(a, b)$ with $0 \leq a \leq b$. We will first show that G' is traceable. This is clear for $a = 0$. If $a \geq 2$ then G has a good couple of edges. Without loss of generality, we may assume that $x_1 y_{i+1}, x_a y_i$ is a good couple. Then $vv'y_1 \cdots y_i x_a \cdots x_1 y_{i+1} \cdots y_b u'u$ is a Hamiltonian path of G'. If $a = 1$, suppose that G' is not traceable and consider the graph G_0 such that $V(G_0) = V(G')$ and $E(G_0)$ consists of $\binom{n-4}{2} + 4$ edges: $uu', vv', u'x_1, v'x_1$ and all edges of the clique induced by $V(C) \setminus \{x_1\}$. Hence G' has at least seven chords incident to x_1. However, if $E(G')$ contained $x_1 y_1$ or $x_1 y_b$, then it is easy to see that G' would be traceable. Moreover, for $i = 2, \ldots, b-1$, if $x_1 y_i \in E(G')$ then $y_{i-1} y_b \notin E(G')$ since otherwise $vv'y_1 \cdots y_{i-1} y_b \cdots y_i x_1 u'u$ would be a Hamiltonian path of G'. Thus $\|G'\| \leq \|G_0\| < \binom{n-4}{2} + 11$, a contradiction.

It follows that G is traceable whenever $vw \in E$. Then assume $vw \notin E$. Let $(n_1, \ldots, n_k)$ be an admissible sequence for G ordered decreasingly: $n_1 \geq \cdots \geq n_k \geq 2$. If $n_1 \geq 3$, then we put w, v, v' to V_1 and continue a partition of V along the Hamiltonian path of G'. Otherwise, $(n_1, \ldots, n_k) = (2)^{n/2}$ and n is even.

Then G is a spanning subgraph of the second graph in Figure 4 without a perfect matching.

To end the proof of Theorem 8 it suffices to settle the case when a longest cycle C has three attachment vertices.

Lemma 18. *Let $G = (V, E)$ be a graph of size $\|G\| > \binom{n-4}{2} + 12$ and circumference $c(G) = n - 3$. If a longest cycle C has three attachment vertices, then G is AP.*

Proof. It follows from Lemma 14, that there are exactly three independent edges outside C, namely uu', vv', ww', and G is spanned by $\mathrm{Sun}(a, b, c)$ where $0 \le a \le b \le c \le n - 6$. To show that G is AP, we consider three cases depending on admissible sequences.

Case 1: Sequence $(2)^{n/2}$. Suppose that the sequence $(2)^{n/2}$ is admissible but not realizable in the graph G. Hence exactly two of the numbers a, b, c are odd, say a and b with $a \le b$.

Let $a = 1$. Consider a graph G_0 of size $\|G_0\| = \binom{n-4}{2} + 6$ containing all edges of the clique $V \setminus \{u, v, w, x_1\}$ and the edges $uu', vv', ww', u'x_1, x_1v', x_1w'$. It follows that x_1 is adjacent to at least seven vertices of $Y \cup Z$. However, any edge of the form x_1y_{2l-1} would give a perfect matching in G. Moreover, if $x_1y_{2l} \in E$, then $y_1y_{2l+1} \notin E$. Furthermore, if $x_1z_{2l} \in E$, then $y_1z_{2l-1} \notin E$, and if $x_1z_{2l-1} \in E$, then $y_1z_{2l} \notin E$, otherwise G would have a perfect matching. Therefore $\|G\| \le \|G_0\|$, a contradiction.

Let $a \ge 3$. Here we argue similarly as in Section 3 for $a \ge 3$. Let G_1 be a graph of size $\binom{n-5}{2} + 11$ containing the edges $uu', vv', ww', u'x_1$, x_1x_2, x_1v', x_1w', $x_ax_2, x_au', x_av', x_aw'$ and all edges of the clique formed by $V \setminus \{u, v, w, x_1, x_a\}$. If $a = 3$, we set $G_0 = G_1$. For $a \ge 5$, we define G_0 as follows. We add to G_1 the edges x_1x_{2j}, x_ax_{2j}, $j = 2, \ldots, \frac{a-1}{2}$, and delete the edges $x_{2i-1}y_{2j-1}$, $i = 2, \ldots, \frac{a-1}{2}, j = 1, \ldots, \frac{b+1}{2}$. Thus the number of added edges equals $a - 3$, and the number of deleted ones equals $\frac{a-1}{2} \cdot \frac{b+1}{2}$ and is not smaller than $\frac{a^2-1}{4}$ since $b \ge a$. Therefore $\|G_0\| < \|G_1\|$. To avoid a perfect matching, the only edges that may appear in G and are not in G_0 are of the form $x_\nu x_{2i-1}$ or $x_\nu y_{2j}$ or $x_\nu z_l$ where $\nu \in \{1, a\}$.

For any $i = 1, \ldots, \frac{a-1}{2}$, if $x_1x_{2i+1} \in E$, then $y_1x_{2i} \notin E$, and if $x_ax_{2i-1} \in E$, then $y_bx_{2i} \notin E$, otherwise G admits a perfect matching. For any $j = 1, \ldots, \frac{b-1}{2}$, if $x_1y_{2j} \in E$, then $y_1y_{2j+1} \notin E$, and if $x_ay_{2j} \in E$, then $y_by_{2j-1} \notin E$ (here the edge y_1y_b may be counted twice as missing in E). For any $j = 1, \ldots, \frac{c-1}{2}$, if $x_1z_{2j} \in E$, then $y_1z_{2j-1} \notin E$, and if $x_az_{2j} \in E$, then $y_bz_{2j-1} \notin E$. Finally, for any $j = 1, \ldots, \frac{c+1}{2}$, if $x_1z_{2j-1} \in E$, then $y_1z_{2j} \notin E$, and if $x_az_{2j-1} \in E$, then $y_bz_{2j} \notin E$. Whence, $\|G\| \le \|G_0\| + 1 \le \binom{n-4}{2} + 12$, a contradiction.

Case 2: Sequence $(3)^{n/3}$. Suppose that the sequence $(3)^{n/3}$ is admissible but is not realizable in G. It easily follows from Theorem 6 that either $a \equiv b \equiv c \equiv 0 \pmod 3$ or $a \equiv b \equiv c \equiv 2 \pmod 3$.

Assume first that $a \equiv b \equiv c \equiv 0 \pmod 3$. Let $0 \le a \le b \le c$. Put $V_1 = \{u, u', z_c\}$ and $V_2 = \{w, w', z_1\}$. Let G_0 be a subgraph of G obtained by deleting all chords of C incident to v' except $v'z_1$ and $v'z_c$. Then $\|G_0\| \le \binom{n-4}{2} + 7$, and there are another chords of C incident to v' in G.

Suppose $a = b = 0$. If $v'z_{3l+2}$ was an edge of G for some $l \ge 0$, then the sequence $(3)^{n/3}$ would have a realization in G. Indeed, we put $V_3 = \{v, v', z_{3k+2}\}$ and observe that the cycle C splits into at most four paths of order divisible by 3 after removing the vertices of $V_1 \cup V_2 \cup V_3$. Also, if $v'z_{3l} \in E$ with $1 \le l \le \frac{c}{3} - 1$, then $z_{3l-1}z_{3l+1} \notin E$ otherwise G has a realization of $(3)^{n/3}$. Similarly, if $v'z_{3l+1} \in E$ with $1 \le l \le \frac{c}{3} - 1$, then $z_{3l}z_{3l+2} \notin E$.

Suppose $b > 0$. If $v'y_{3k} \in E$, then $y_{3k-1}z_2 \notin E$, since otherwise we would have a realization of $(3)^{n/3}$ by taking $V_3 = \{y_{3k-2}, y_{3k-1}, z_2\}$. Analogously, if $v'y_{3k+1} \in E$, then $y_{3k+2}z_2 \notin E$ because we could take $V_3 = \{y_{3k+2}, y_{3k}, z_2\}$. Again, if both edges $v'y_{3k+2}$ and $y_{3k+1}z_1$ appeared in G, then we could redefine $V_2 = \{w, w', y_b\}$ and put $V_3 = \{v, v', y_{3k+2}\}$, $V_4 = \{y_{3k+1}, z_1, z_2\}$ to obtain a realization of $(3)^{n/3}$.

If also $a > 0$, then the same arguments as in the previous paragraph for $b > 0$ can be applied to justify the assertion: every new chord from $E \setminus E(G_0)$ causes the absence of another chord in G. It follows that $\|G\| \le \|G_0\| < \binom{n-4}{2} + 12$.

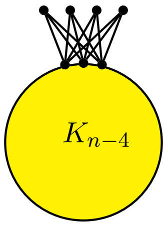
Now, assume that $a \equiv b \equiv c \equiv 2 \pmod 3$. Let $V_1 = \{u, u', x_1\}, V_2 = \{v, v', x_a\}, V_3 = \{w, w', y_b\}$. Remember that the number of chords of C missing in G is at most $n - 12$. For every $i = 1, \ldots, \frac{b}{3}$ and $j = 1, \ldots, \frac{c}{3}$, the vertex y_{3i-2} cannot be a neighbor neither of z_{3j-2} nor of z_{3j-1} because then we could take $V_4 = \{y_{3i-2}, z_{3j-2}, z_{3j-1}\}$, and C would split into paths of orders being multiples of three after removing $V_1 \cup V_2 \cup V_3 \cup V_4$. Analogously, $y_{3i-1}z_{3j-2}, y_{3i-1}z_{3j-1} \notin E$. Thus, there are $4 \cdot \frac{b}{3} \cdot \frac{c}{3}$ missing chords of C. As $b + c \ge \frac{2}{3}(n-6)$, we have $\frac{4}{9}bc \ge \frac{4}{81}(n-6)^2 > n - 12$ for any n, a contradiction.

Case 3: Sequences different from $(2)^{n/2}$ and $(3)^{n/3}$. Consider first the case $a \ge 1$. Then, we can apply Lemma 17. Without loss of generality, we may assume that the good couple is from X to Z, i.e., there is an i such that $x_1z_{i+1}, x_az_i \in E$ and $1 \le i \le c$. Then, observe that the subgraph of G induced by the vertex sequence $x_0 \overleftarrow{C} z_{i+1}x_1Cx_az_i\overleftarrow{C}v'$ contains a caterpillar $\mathrm{Cat}(b+3)$. So, by Theorem 4, we are able to realize all admissible sequences except, maybe, for sequences of the form $(d)^{n/d}$ for $d|(b+3)$. If $a = 1$, then a part of such a sequence could be realized on the caterpillar $\mathrm{Cat}(2) = x_1Cy_b$ of order $b + 3$ because $d \ne 2$ by assumption, and the rest of it on the path $ww'Cu'u$. If $a \ge 2$, then a part of this sequence could be realized either on $\mathrm{Cat}(2) = x_1Cy_b$ or on $\mathrm{Cat}(2) = y_1Cz_1$, and the rest of the sequence either on $\mathrm{Cat}(a) = x_{a-1}\overleftarrow{C}w'$ or on $\mathrm{Cat}(a+3) = v'\overleftarrow{C}z_2$, respectively. If none of the two latter caterpillars admits a realization of the sequence $(d)^{n/d}$

that means that $d|a$ and $d|(a+3)$. This implies that $d = 3$.

Let $a = 0$ and $b \geq 1$. Then G contains a caterpillar $\mathrm{Cat}(b+3) = vv'\overleftarrow{C}u'u$. So, by Theorem 4, any admissible and nonrealizable sequence should be of the form $(d)^{n/d}$ for $d|(b+3)$. As $d \neq 2$, then a part of such a sequence could be realized on $\mathrm{Cat}(2) = y_1Cz_1$, and the rest of the sequence on $\mathrm{Cat}(3) = vv'\overleftarrow{C}z_2$, except for the case where $d = 3$.

If $a = b = 0$ then it is easy to see that G is spanned by a caterpillar $\mathrm{Cat}(3)$, so only the sequence $(3)^{n/3}$ may not be realizable. ■



$2|n$

$n \geq 12$

Figure 8. A non-AP graph of size $\binom{n-4}{2} + 12$.

5. Final Remarks

The following is an easily seen consequence of Corollary 13.

Proposition 19. *If G is a connected graph of order n and size $\|G\| > \binom{n-2}{2} + 2$, then G is traceable.*

Clearly, the bound $\binom{n-2}{2} + 2$ is sharp for every $n \geq 4$ since the first graph shown in Figure 4 (a clique K_{n-2} with two pendant edges attached to it in one vertex) is not traceable. The difference between $\binom{n-2}{2} + 2$ and the lower bound $\binom{n-4}{2} + 12$ in our main result equals $2n - 17$.

Observe that there are quite many connected nontraceable graphs G with more than $\binom{n-4}{2} + 12$ edges, which are AP by Theorem 8. In particular, if the order n of G is not divisible neither by two nor by three, then G is AP unless it is a spanning subgraph of the third graph in Figure 4 (a clique K_{n-3} with three pendant edges attached in one and the same vertex). Moreover, for every n if

$c(G) = n - 3$ and G has three independent pendant edges, then G is AP, and clearly nontraceable.

It has to be noted that if we decrease the bound $\binom{n-4}{2} + 12$ even by one, then we obtain new exceptional graphs that are not AP. For example, the graph in Figure 8 has $\binom{n-4}{2} + 12$ edges and is not AP for even n.

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