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IRREDUCIBLE NO-HOLE $L(2,1)$-COLORING OF EDGE-MULTIPLICITY-PATHS-REPLACEMENT GRAPH

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Abstract

An $L(2,1)$-*coloring* (or *labeling*) of a simple connected graph G is a mapping $f : V(G) \to Z^+ \cup \{0\}$ such that $|f(u) - f(v)| \geq 2$ for all edges uv of G, and $|f(u) - f(v)| \geq 1$ if u and v are at distance two in G. The *span of an* $L(2,1)$-*coloring* f, denoted by span(f), of G is $\max\{f(v) : v \in V(G)\}$. The *span of* G, denoted by $\lambda(G)$, is the minimum span of all possible $L(2,1)$-colorings of G. For an $L(2,1)$-coloring f of a graph G with span k, an integer l is a *hole* in f if $l \in (0,k)$ and there is no vertex v in G such that $f(v) = l$. An $L(2,1)$-coloring is a *no-hole coloring* if there is no hole in it, and is an *irreducible coloring* if color of none of the vertices in the graph can be decreased and yield another $L(2,1)$-coloring of the same graph. An *irreducible no-hole coloring*, in short *inh-coloring*, of G is an $L(2,1)$-coloring of G which is both irreducible and no-hole. For an inh-colorable graph G, the *inh-span* of G, denoted by $\lambda_{inh}(G)$, is defined as $\lambda_{inh}(G) = \min\{\text{span}(f) : f$ is an inh-coloring of $G\}$. Given a function $h : E(G) \to \mathbb{N} - \{1\}$, and a positive integer $r \geq 2$, the *edge-multiplicity-paths-replacement graph* $G(rP_h)$ of G is the graph obtained by replacing every edge uv of G with r paths of length $h(uv)$ each. In this paper we show that $G(rP_h)$ is inh-colorable except possibly the cases $h(e) \geq 2$ with equality for at least one but not for all edges e and (i) $\Delta(G) = 2$, $r = 2$ or (ii) $\Delta(G) \geq 3$, $2 \leq r \leq 4$. We find the exact value of $\lambda_{inh}(G(rP_h))$ in several cases and give upper bounds of the same in the remaining. Moreover, we find the value of $\lambda(G(rP_h))$ in most of the cases which were left by Lü and Sun in [$L(2,1)$-*labelings of the edge-multiplicity-paths-replacement of a graph*, J. Comb. Optim. 31 (2016) 396–404].

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1. Introduction

The channel assignment problem is to assign frequencies to a given group of radio transmitters so that interfering transmitters are assigned frequencies with at least a minimum allowed separation. Griggs and Yeh [4] mentioned that in 1988, Roberts (in a private communication to Griggs) proposed the problem of efficiently assigning radio channels to transmitters at several locations, using nonnegative integers to represent channels, so that close locations receive different channels, and channels for very close locations are at least two apart. Motivated by this problem, Griggs and Yeh [4] proposed the $L(2,1)$-coloring problem of a graph as follows. The *$L(2,1)$-coloring* of a simple connected graph G is a vertex coloring (or labeling) $f : V(G) \to Z^+ \cup \{0\}$ such that $|f(u) - f(v)| \geq 2$ for all edges uv of G, and $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$, where $d(u,v)$ is the distance between vertices u and v in G. The *span of an $L(2,1)$-coloring* f of a graph G, denoted by $\text{span}(f)$, is equal to $\max\{f(v) : v \in V(G)\}$. The *span of a graph* G, denoted by $\lambda(G)$, is equal to $\min\{\text{span}(f) : f \text{ is an } L(2,1)\text{-coloring of } G\}$. An $L(2,1)$-coloring whose span is equal to the span of the graph is called a *span coloring*.

Throughout the paper we consider simple connected graphs only. The maximum degree of a graph G is denoted by $\Delta(G)$ or simply Δ if no confusion arises. Now we state a result by Chang and Lu [2] which will be used in the sequel.

Proposition 1 (Proposition 1, [2]). *For any graph G, $\lambda(G) \geq \Delta + 1$. Further, if $\lambda(G) = \Delta + 1$, then in any span coloring of G the maximum degree vertices must be colored with 0 (or $\Delta + 1$) and its neighbors must be colored with $2 + i$ (or i), $i = 0, 1, \ldots, \Delta - 1$.*

Fishburn and Roberts [3] introduced no-hole coloring of graphs. For a graph G and an $L(2,1)$-coloring f of it with span k, an integer l is called a *hole* in f, if $l \in (0,k)$ and there is no vertex v in G such that $f(v) = l$. An $L(2,1)$-coloring of a graph is a *no-hole coloring* if there is no hole in it. Since frequencies are typically used in a block, one may want to use all available frequencies in that block. This is assured by a no-hole coloring. An $L(2,1)$-coloring f of a graph G is called *reducible* if there exists another $L(2,1)$-coloring g of G such that $g(u) \leq f(u)$ for all vertices $u \in V(G)$ and there exists a vertex $v \in V(G)$ such that $g(v) < f(v)$. An $L(2,1)$-coloring is *irreducible* if it is not reducible. An irreducible no-hole coloring is referred as *inh-coloring* and a graph is called *inh-colorable* if there exists an inh-coloring of it. For an inh-colorable graph G the *lower inh-span* or simply *inh-span* of G, denoted by $\lambda_{inh}(G)$, is defined as $\lambda_{inh}(G) = \min\{\text{span}(f) : f \text{ is an inh-coloring of } G\}$. Laskar and Villalpando [10] introduced inh-coloring and studied some properties of it. Further, they obtained upper and lower bounds of inh-span of unicyclic graphs and triangular lattices.

Laskar *et al.* [9] proved that every tree T different from a star is inh-colorable with $\lambda_{inh}(T) = \lambda(T)$. Jacob *et al.* [7] studied irreducible no-hole coloring of bipartite graphs and Cartesian product of graphs.

Given a graph G and a function $h : E(G) \to \mathbb{N} - \{1\}$ the *h-subdivision* of G, denoted by $G_{(h)}$, is the graph obtained from G by replacing each edge uv in G with a path of length $h(uv)$. If $h(e) = c$ for all $e \in E(G)$, then we refer $G_{(h)}$ as $G_{(c)}$. Further, if $r \geq 2$ is an integer, the *edge-multiplicity-paths-replacement graph* $G(rP_h)$ of G is obtained by replacing every edge uv of G with r paths of length $h(uv)$ each. In particular, if $h(e) = c$ for all edges $e \in E(G)$, we denote $G(rP_h)$ simply by $G(rP_c)$. The vertices of G in $G_{(h)}$ or $G(rP_h)$ are called *nodes*.

Throughout the paper we follow some notations as given below.

Notation 2. For any graph G we take h as a function from $E(G)$ to $\mathbb{N} - \{1\}$. The path of length k in $G_{(h)}$ which replaces the edge uv in G is denoted by $ux^1_{uv}x^2_{uv}\cdots x^{k-1}_{uv}v$. The r paths of length k each in $G(rP_h)$ which replace the edge uv in G are denoted by $P^i_h = ux^{i_1}_{uv}x^{i_2}_{uv}\cdots x^{i_{k-1}}_{uv}v$, $1 \leq i \leq r$.

The $L(2,1)$-colorings of $G_{(2)}$, for any graph G, are studied by Whittlesey *et al.* [15], and Havet and Yu [5, 6]. The $L(2,1)$-colorings of subdivisions of graphs are studied by Lü [11], Karst *et al.* [8] and Chang *et al.* [1]. Moreover, Mandal and Panigrahi [13] have studied inh-coloring of subdivision graphs. An $L(2,1)$-coloring f of $G_{(h)}$ is said to be a *λ-perfect labeling* if $f(u) = 0$ for all nodes u and $\text{span}(f) = \Delta(G) + 1$ [1]. We state the following proposition by Chang *et al.* [1] which will be used in the sequel.

Proposition 3 (Theorem 12, [1]). *If G is a graph with $\Delta(G) \geq 4$, then $G_{(3)}$ has a λ-perfect labeling.*

Lü and Sun [12] studied the $L(2,1)$-coloring of the edge-multiplicity-paths-replacement graph $G(rP_c)$ of a graph G. The main results of them are given in Table 1. They found the exact value of $\lambda(G(rP_c))$ in the following cases: $\Delta(G) \leq 2$; $c \geq 3$, $\Delta(G) \geq 4$ is even; and $c \geq 5$, $\Delta(G) \geq 3$ is odd. For the remaining cases they gave upper bounds to $\lambda(G(rP_c))$. From Proposition 1 we get the following result.

Proposition 4. $\lambda_{inh}(G(rP_h)) \geq \lambda(G(rP_h)) \geq r\Delta(G) + 1$ *where $h(e) \geq 2$ for all e.*

In this paper we show that for any graph G with $h(e) \geq 3$ and $r \geq 2$, $G(rP_h)$ is inh-colorable and for $\Delta(G) \geq 2$, $G(rP_2)$ is inh-colorable. We also prove that if G is a graph with $\Delta(G) \geq 2$, $h(e) \geq 2$ for all e in $E(G)$ and $h(e) = 2$ for at least one e but not for all, and $r \geq 2$, then $G(rP_h)$ is inh-colorable except possibly the following cases: $\Delta(G) = 2$, $r = 2$; and $\Delta(G) \geq 3$, $2 \leq r \leq 4$. We find the exact value of inh-span of some edge-multiplicity-paths-replacement graphs and

G	c	r	$\lambda(G(rP_c))$
$\Delta(G) \geq 3$	≥ 5	≥ 2	$r\Delta(G)+1$
	4	≥ 2	$\leq r\Delta(G)+2$
$\Delta(G) \geq 4$ and $\Delta(G)$ is even	$3, 4$	≥ 2	$r\Delta(G)+1$
$\Delta(G) \geq 3$ and $\Delta(G)$ is odd	3	≥ 2	$\leq r\Delta(G)+r+1$
$\Delta(G) \geq 3$	2	≥ 2	$\leq r\chi'(G)+\chi(G)$
	2	≥ 2	$\leq r(\Delta(G)+1)+\Delta(G)$
Any graph G	3	≥ 2	$\leq \lambda(G)+2r$
	2	≥ 2	$\leq r(\lambda(G_{(2)})+1)+r-2$
$\Delta(G)=2$	≥ 3	≥ 2	$2r+1$
P_n with $3 \leq n \leq 4$	2	≥ 2	$2r+1$
P_n with $n \geq 5$	2	≥ 2	$2r+2$
C_n with even n	2	≥ 2	$2r+2$
C_n with $n \geq 5$ and n is odd	2	2	6
	2	3	8
	2	≥ 4	$3r-2$
C_3	2	2	6
	2	≥ 3	$3r-1$
P_2	≥ 3	≥ 4	$r+1$
	≥ 7	3	4
	$3 \leq c \leq 6$	3	5
	≥ 2	2	4
	2	≥ 2	$r+2$

Table 1. Results in [12] on $\lambda(G(rP_c))$.

for the remaining we give upper bounds to the same. Moreover, we determine the span of $G(rP_h)$ in most of the cases which were not obtained by Lü and Sun [12]. An important point to be noted is that Lü and Sun [12] have considered the graphs $G(rP_c)$ only, that is, all the edges of G are replaced by paths of the same lengths. We have considered the graphs $G(rP_h)$, where different edges of G may be replaced by paths of different lengths. The main results of the paper are given in Tables 2 and 3.

2. Inh-Colorability of Graphs $G(rP_h)$ with $\Delta(G)=1$

We first consider the case $\Delta(G)=1$, and so here the graph G is obviously P_2. In this case we take $r \geq 3$ because for $r=2$, $P_2(rP_h)$ is a cycle. We also take $h(e) \geq 3$ because $P_2(rP_2)$, $r \geq 2$, is a complete bipartite graph, which is not inh-colorable [3].

$\Delta(G)$	$h(e)$	r	$\lambda(G(rP_h))$	Theorem
$\Delta(G)=3$	$h(e)=3$ for all e	≥ 2	$3r+1$	20
	$h(e)\geq 3$ with $h(e)>3$ for at least one edge	≥ 2	$3r+1$	22
$\Delta(G)\geq 4$ and $\Delta(G)$ is odd	$h(e)=3$ for all e	≥ 2	$r\Delta+1$	Corollary 27
$\Delta(G)\geq 4$	$h(e)\geq 3$ with $h(e)>3$ for at least one edge	≥ 2	$r\Delta+1$	28

Table 2. Results of the paper on $\lambda(G(rP_h))$.

Theorem 5. *For $r\geq 3$, $\lambda_{inh}(P_2(rP_3))=r+2$.*

Proof. Let $P_2=uv$. We give an $L(2,1)$-coloring f to $P_2(rP_3)$ as follows. If $r=3$ then $f(u)=4, f(v)=5$, $f\left(x_{uv}^{1_1}\right)=0$, $f\left(x_{uv}^{2_1}\right)=2$, $f\left(x_{uv}^{3_1}\right)=1$, $f\left(x_{uv}^{1_2}\right)=2$, $f\left(x_{uv}^{2_2}\right)=0$, and $f\left(x_{uv}^{3_2}\right)=3$. If $r\geq 4$ then $f(u)=0, f(v)=r+2$, $f\left(x_{uv}^{i_1}\right)=i+1$ for $1\leq i\leq r$, $f\left(x_{uv}^{1_2}\right)=r$, and $f\left(x_{uv}^{i_2}\right)=i-1$ for $2\leq i\leq r$. We check that f is an inh-coloring of $P_2(rP_3)$. Thus $\lambda_{inh}(P_2(rP_3))\leq r+2$.

Now we prove that $\lambda_{inh}(P_2(rP_3))\geq r+2$. We know that $\lambda(P_2(rP_3))=r+1$ [12]. Suppose $\lambda_{inh}(P_2(rP_3))=r+1$ and g is an inh-coloring of $P_2(rP_3)$ with span $r+1$. If both the nodes are colored with 0 then 1 is a hole, and if both the nodes are colored with $r+1$ then r is a hole. Hence one node, say u, is colored with 0 and the other node, say v, is colored with $r+1$. Then for some i, $1\leq i\leq r$, $g\left(x_{uv}^{i_1}\right)=r+1$. This is a contradiction since $d\left(x_{uv}^{i_1},v\right)=2$. Thus $\lambda_{inh}(P_2(rP_3))\geq r+2$ and we get $\lambda_{inh}(P_2(rP_3))=r+2$. ■

In the next three theorems we show that inh-span of $P_2(rP_k)$, $k\geq 4$, $r\geq 3$, coincides with its span as computed by Lü and Sun [12].

Theorem 6. *For $k\geq 4$, $\lambda_{inh}(P_2(4P_k))=5$.*

Proof. Let $P_2=uv$. We first take $k\equiv 1 \pmod 3$. We give an $L(2,1)$-coloring f_1 to $P_2(4P_k)$ as follows: $f_1(u)=f_1(v)=0$, $f_1\left(x_{uv}^{1_1}\right)=2$, $f_1\left(x_{uv}^{1_2}\right)=5$, $f_1\left(x_{uv}^{1_3}\right)=3$, $f_1\left(x_{uv}^{1_j}\right)=0$, 5 or 3 according as $j\equiv 1$, 2 or 0 (mod 3) for $4\leq j\leq k-1$; $f_1\left(x_{uv}^{2_1}\right)=3$, $f_1\left(x_{uv}^{2_2}\right)=5$, $f_1\left(x_{uv}^{2_3}\right)=2$, $f_1\left(x_{uv}^{2_j}\right)=0$, 4 or 2 according as $j\equiv 1$, 2 or 0 (mod 3) for $4\leq j\leq k-1$; $f_1\left(x_{uv}^{3_1}\right)=4$, $f_1\left(x_{uv}^{3_2}\right)=1$, $f_1\left(x_{uv}^{3_3}\right)=5$, for $f_1\left(x_{uv}^{3_j}\right)=0$, 2 or 5 according as $j\equiv$ 1,2 or 0 (mod 3) for $4\leq j\leq k-1$; $f_1\left(x_{uv}^{4_1}\right)=5$, $f_1\left(x_{uv}^{4_2}\right)=1$, $f_1\left(x_{uv}^{4_3}\right)=4$, $f_1\left(x_{uv}^{4_j}\right)=0$, 2 or 4 according as $j\equiv 1$, 2 or 0 (mod 3) for $4\leq j\leq k-1$. It can be checked that f_1 is an $L(2,1)$-coloring. We reduce f_1 until we arrive at an irreducible coloring f_1'. In the coloring f_1', u is colored with 0, its neighbors are colored with 2, 3, 4, 5, and $x_{uv}^{3_2}$ is colored with 1. Hence f_1' is an inh-coloring with span 5.

$\Delta(G)$		$h(e)$	r	$\lambda_{inh}(G(rP_h))$	Theorem
$\Delta(G)=1$, i.e., $G=P_2$		$h(e)=3$	≥ 3	$r+2$	5
		$h(e)\geq 4$	4	5	6
		$4\leq h(e)\leq 6$	3	5	7
		$h(e)\geq 7$	3	4	7
		$h(e)\geq 4$	≥ 5	$r+1$	8
$\Delta(G)=2$, i.e., $G=P_m$ or C_m, $m\geq 3$	$P_m, 3\leq m\leq 4$	$h(e)=2$	≥ 2	$2r+1$	11
	P_m, $m\geq 5$	$h(e)=2$	≥ 2	$2r+2$	11
	P_m, $m\geq 3$	$h(e)\geq 2$ with $h(e)>2$ for at least one edge but not for all	≥ 3	$\leq$ $3r+3$	12
		$h(e)=3$	≥ 2	$2r+1$	13
	C_3	$h(e)=3$	≥ 2	$2r+2$	14
	C_m, m even	$h(e)=2$	≥ 2	$\leq$ $2r+3$	15
	C_m, m odd	$h(e)=2$	≥ 2	$\leq$ $3r+2$	15
	C_m	$h(e)\geq 2$ with $h(e)>2$ for at least one edge but not for all	≥ 3	$\leq$ $3r+3$	16
	C_m, $m\geq 4$	$h(e)=3$	2	$\leq$ 6	17
		$h(e)=3$	≥ 3	$2r+1$	18
		$h(e)\geq 3$ with $h(e)>3$ for at least one edge	≥ 2	$2r+1$	19
$\Delta(G)=3$		$h(e)=3$	≥ 2	$\leq$ $3r+2$	21
		$h(e)\geq 3$ with $h(e)>3$ for at least one edge	≥ 2	$3r+1$	22
$\Delta(G)\geq 3$		$h(e)=2$	≥ 2	$\leq \begin{cases} \chi(G)+r\chi'(G)+3 & \text{if } G \text{ is a bipartite graph other than a tree} \\ \chi(G)+r\chi'(G), & \text{otherwise} \end{cases}$	24
		$h(e)\geq 2$ with $h(e)>2$ for at least one edge but not for all	≥ 5	$\leq$ $2r\Delta-r+5$	25
$\Delta(G)\geq 4$		$h(e)=3$	≥ 2	$\leq$ $r\Delta+2$ $r\Delta+1$ (with some conditions)	26 26
		$h(e)\geq 3$ with $h(e)>3$ for at least one edge	≥ 2	$r\Delta+1$	28

Table 3. Results of the paper on $\lambda_{inh}(G(rP_h))$.

Let $k\equiv 2 \pmod 3$. We give an $L(2,1)$-coloring f_2 to $P_2(4P_k)$ as follows: $f_2(u)=f_2(v)=0$; $f_2\left(x_{uv}^{1_1}\right)=2$, $f_2\left(x_{uv}^{1_2}\right)=5$, $f_2\left(x_{uv}^{1_3}\right)=1$, $f_2\left(x_{uv}^{1_4}\right)=3$,

$f_2\left(x_{uv}^{1_j}\right) = 0, 5$ or 3 according as $j \equiv 2, 0$ or 1 (mod 3) for $5 \le j \le k-1$; $f_2\left(x_{uv}^{2_1}\right) = 3$, $f_2\left(x_{uv}^{2_2}\right) = 1$, $f_2\left(x_{uv}^{2_3}\right) = 5$, $f_2\left(x_{uv}^{2_4}\right) = 2$, $f_2\left(x_{uv}^{2_j}\right) = 0, 4$ or 2 according as $j \equiv 2, 0$ or 1 (mod 3) for $5 \le j \le k-1$; $f_2\left(x_{uv}^{3_1}\right) = 4$, $f_2\left(x_{uv}^{3_2}\right) = 1$, $f_2\left(x_{uv}^{3_3}\right) = 3$, $f_2\left(x_{uv}^{3_4}\right) = 5$, $f_2\left(x_{uv}^{3_j}\right) = 0, 2$ or 5 according as $j \equiv 2, 0$ or 1 (mod 3) for $5 \le j \le k-1$; $f_2\left(x_{uv}^{4_1}\right) = 5$, $f_2\left(x_{uv}^{4_2}\right) = 3$, $f_2\left(x_{uv}^{4_3}\right) = 1$, $f_2\left(x_{uv}^{4_4}\right) = 4$, $f_2\left(x_{uv}^{4_j}\right) = 0, 2$ or 4 according as $j \equiv 2, 0$ or 1 (mod 3) for $5 \le j \le k-1$. It can be checked that f_2 is an $L(2,1)$-coloring. We reduce f_2 until we arrive at an irreducible coloring f_2'. In the coloring f_2', u is colored with 0, its neighbors are colored with 2, 3, 4, 5, and $x_{uv}^{3_2}$ is colored with 1. Hence f_2' is an inh-coloring with span 5.

Let $k \equiv 0$ (mod 3). We give an $L(2,1)$-coloring f_3 to $P_2(4P_k)$ as follows: $f_3(u) = f_3(v) = 0$; $f_3\left(x_{uv}^{1_1}\right) = 2$, $f_3\left(x_{uv}^{1_2}\right) = 5$, $f_3\left(x_{uv}^{1_3}\right) = 3$, $f_3\left(x_{uv}^{1_4}\right) = 1$, $f_3\left(x_{uv}^{1_5}\right) = 4$, $f_3\left(x_{uv}^{1_j}\right) = 0, 2$ or 4 according as $j \equiv 0, 1$ or 2 (mod 3) for $6 \le j \le k-1$; $f_3\left(x_{uv}^{2_1}\right) = 4$, $f_3\left(x_{uv}^{2_2}\right) = 1$, $f_3\left(x_{uv}^{2_3}\right) = 3$, $f_3\left(x_{uv}^{2_4}\right) = 5$, $f_3\left(x_{uv}^{2_5}\right) = 2$, $f_3\left(x_{uv}^{2_j}\right) = 0, 4$ or 2 according as $j \equiv 0, 1$ or 2 (mod 3) for $6 \le j \le k-1$; $f_3\left(x_{uv}^{3_1}\right) = 3$, $f_3\left(x_{uv}^{3_2}\right) = 1$, $f_3\left(x_{uv}^{3_3}\right) = 4$, $f_3\left(x_{uv}^{3_4}\right) = 2$, $f_3\left(x_{uv}^{3_5}\right) = 5$, $f_3\left(x_{uv}^{3_j}\right) = 0, 2$ or 5 according as $j \equiv 0, 1$ or 2 (mod 3) for $6 \le j \le k-1$; $f_3\left(x_{uv}^{4_1}\right) = 5$, $f_3\left(x_{uv}^{4_2}\right) = 2$, $f_3\left(x_{uv}^{4_3}\right) = 4$, $f_3\left(x_{uv}^{4_4}\right) = 1$, $f_3\left(x_{uv}^{4_5}\right) = 3$, $f_3\left(x_{uv}^{4_j}\right) = 0, 5$ or 3 according as $j \equiv 0, 1$ or 2 (mod 3) for $6 \le j \le k-1$. It can be checked that f_3 is an $L(2,1)$-coloring. We reduce f_3 until we arrive at an irreducible coloring f_3'. In f_3', u is colored with 0, its neighbors are colored with 2, 3, 4, 5, and $x_{uv}^{3_2}$ is colored with 1. Hence f_3' is an inh-coloring with span 5. Since $\lambda(P_2(4P_k)) = 5$ [12] for $k \ge 4$, we conclude that $\lambda_{inh}(P_2(4P_k)) = 5$ for $k \ge 4$. ■

In the theorem below we find the exact value of inh-span of $P_2(rP_k)$ for $r = 3$ and $k \ge 4$.

Theorem 7. *The value of $\lambda_{inh}(P_2(3P_k))$ is 5 for $4 \le k \le 6$, and 4 for $k \ge 7$.*

Proof. Let $P_2 = uv$. From the proof of Theorem 6 we see that for $4 \le k \le 6$, $P_2(3P_k)$ can be given an $L(2,1)$-coloring g with span 5 such that $g(u) = 0$, $g\left(x_{uv}^{3_2}\right) = 1$ and the neighbors of u are colored with 2, 3 and 4. We reduce g until we arrive at an irreducible coloring g'. Then $g'(u) = 0$, $g'\left(x_{uv}^{3_2}\right) = 1$, and g' assigns colors 2, 3 and 4 to neighbors of u. Since span $g' \le 5$, g' is an inh-coloring. Then for $4 \le k \le 6$, $\lambda_{inh}(P_2(3P_k)) = 5$ because $\lambda(P_2(3P_k)) = 5$ [12] for the same values of k.

For $k \ge 7$, Lü and Sun [12] have given an $L(2,1)$-coloring f to $P_2(3P_k)$ with span 4 such that $f(u) = 0$, $f\left(x_{uv}^{2_1}\right) = 3$, and the other neighbors of u are colored with 2 and 4. We reduce f until we arrive at an irreducible coloring f'. Then $f'(u) = 0$, $f'\left(x_{uv}^{2_1}\right) = 3$, and f' assigns colors 2 and 4 to the other neighbors of u. Since $\text{span}(f') = 4$, $f'\left(x_{uv}^{2_2}\right) = 1$. Thus f' is an inh-coloring with span 4. Since $\lambda(P_2(3P_k)) = 4$ for $k \geq 7$ [12], we conclude that $\lambda_{inh}(P_2(3P_k)) = 4$ for the same values of k. ■

Theorem 8. *For $r \geq 5$ and $k \geq 4$, $\lambda_{inh}(P_2(rP_k)) = r + 1$.*

Proof. Let the nodes of $P_2(rP_k)$ be u and v. Lü and Sun [12] have given an $L(2,1)$-coloring f to $P_2(rP_k)$ in which $f(u) = f(v) = 0$, and for $1 \leq i \leq r$, $f\left(x_{uv}^{i_1}\right) = i + 1$ and $f\left(x_{uv}^{i_{k-1}}\right) = i \pmod r + 2$. We recolor the vertices $x_{uv}^{2_j}$ for $2 \leq j \leq k-2$, and get the coloring g as below: for $k \equiv 1 \pmod 3$, $g\left(x_{uv}^{2_2}\right) = 1$, $g\left(x_{uv}^{2_3}\right) = 4$, $g\left(x_{uv}^{2_j}\right) = 0, 2$, or 4 according as $j \equiv 1, 2$ or $0 \pmod 3$, $4 \leq j \leq k-2$; for $k \equiv 2 \pmod 3$, $g\left(x_{uv}^{2_2}\right) = 5$, $g\left(x_{uv}^{2_3}\right) = 1$, $g\left(x_{uv}^{2_4}\right) = 4$, $g\left(x_{uv}^{2_j}\right) = 0, 2$, or 4 according as $j \equiv 2, 0$ or $1 \pmod 3$, $5 \leq j \leq k-2$; for $k \equiv 0 \pmod 3$, $g\left(x_{uv}^{2_2}\right) = 1$, $g\left(x_{uv}^{2_3}\right) = 5$, $g\left(x_{uv}^{2_4}\right) = 2$, $g\left(x_{uv}^{2_5}\right) = 4$, $g\left(x_{uv}^{2_j}\right) = 0, 2$, or 4 according as $j \equiv 0$, 1 or $2 \pmod 3$, $6 \leq j \leq k-2$. We reduce g until we arrive at an irreducible coloring, say g'. In the coloring g', u is colored with 0, its neighbors are colored with $2, 3, \ldots, r+1$ and either $x_{uv}^{2_2}$ or $x_{uv}^{2_3}$ is colored with 1. Hence g' is an inh-coloring with span $r+1$. Since $\lambda(P_2(rP_k)) = r + 1$ [12] we get the result. ■

3. Inh-Colorability of Graphs $G(rP_h)$ with $\Delta(G) = 2$

In our next few results we need the following greedy algorithm.

Algorithm 9 (Greedy coloring)**.** Let G be a graph whose few vertices might have been colored before. Then

1. Order the vertices of the given graph as $u_1, u_2, \ldots, u_n$ such that all colored vertices (if any) appear at the beginning of the list.
2. Let u_i be the first uncolored vertex that appears in the list.
3. Color u_i with the smallest possible color k such that no lower indexed neighbor of u_i in the list is colored with $k-1$, k or $k+1$ and no lower indexed vertex at distance two from u_i is colored with k.
4. If all the vertices of the graph have received color then stop; otherwise set $i = i + 1$ and go to 3.

The theorem below is obviously true.

Theorem 10. *Algorithm 9 gives an $L(2,1)$-coloring of G if and only if the pre-colored vertices of G satisfy constraints of $L(2,1)$-coloring in the graph G.*

Now we consider the case $\Delta(G) = 2$. We note that simple connected graphs with $\Delta(G) = 2$ are paths P_m and cycles C_m only, $m \geq 3$. In Theorems 11 and 13 we show respectively that the inh-span of $P_m(rP_2)$ and $P_m(rP_3)$ coincide with their span which was computed by Lü and Sun [12].

Theorem 11. *Let $r \geq 2$ and $m \geq 3$. Then*

$$\lambda_{inh}(P_m(rP_2)) = \begin{cases} 2r+1 & \text{for } 3 \leq m \leq 4, \\ 2r+2 & \text{for } m \geq 5. \end{cases}$$

Proof. Lü and Sun [12] proved that $\lambda(P_3(rP_2)) = 2r + 1$. Let $P_3 = u_1u_2u_3$. We give an $L(2,1)$-coloring f_1 to $P_3(rP_2)$ as below: $f_1(u_1) = 1$, $f_1(u_2) = 0$, $f_1(u_3) = r + 3$, $f_1\left(x^{i_1}_{u_1u_2}\right) = r + 1 + i$ and $f_1\left(x^{i_1}_{u_2u_3}\right) = i + 1$ for $1 \leq i \leq r$. We check that f_1 is an inh-coloring. Thus $\lambda_{inh}(P_3(rP_2)) = 2r + 1$.

Lü and Sun [12] proved that $\lambda(P_4(rP_2)) = 2r + 1$. Let $P_4 = u_1u_2u_3u_4$. We give an $L(2,1)$-coloring f_2 to $P_4(rP_2)$ as below: $f_2(u_1) = 1$, $f_2(u_2) = 0$, $f_2(u_3) = 2r+1$, $f_2(u_4) = 3$; $f_2(x^{i_1}_{u_1u_2}) = r+1+i$, $f_2\left(x^{i_1}_{u_2u_3}\right) = i+1$ for $1 \leq i \leq r$; $f_2\left(x^{1_1}_{u_3u_4}\right) = 0$, $f_2(x^{2_1}_{u_3u_4}) = 1$, and $f_2\left(x^{i_1}_{u_3u_4}\right) = r + i - 1$ for $3 \leq i \leq r$. We check that f_2 is an inh-coloring and thus $\lambda_{inh}(P_4(rP_2)) = 2r + 1$.

Lü and Sun [12] proved that for $m \geq 5$, $\lambda(P_m(rP_2)) = 2r + 2$. Let $P_m = u_1u_2 \cdots u_m$. We give an $L(2,1)$-coloring f_3 to $P_m(rP_2)$ as below: $f_3(u_1) = 2r+1$, $f_3(u_k) = 0$ if k is even, $f_3(u_k) = 1$ if $k > 1$ and k is odd, $f_3\left(x^{1_1}_{u_1u_2}\right) = 2$, $f_3\left(x^{i_1}_{u_1u_2}\right) = 2i - 1$ for $2 \leq i \leq r$, and we color the remaining paths of length r as $0, (2i + 2), 1$ or $1, (2i + 1), 0$, where $i = 1, 2, \ldots, r$. We check that f_3 is an inh-coloring and thus for $m \geq 5$, $\lambda_{inh}(P_m(rP_2)) = 2r + 2$. ■

Theorem 12. *If $m \geq 3$, $r \geq 3$, and $h(e) \geq 2$ with equality for at least one e but not for all, then $P_m(rP_h)$ is inh-colorable and $\lambda_{inh}(P_m(rP_h)) \leq 3r + 3$.*

Proof. Let P_m be the path $u_1u_2 \cdots u_m$. Let $E_1 = \{uv : uv \in E(P_m), h(uv) > 2\}$ and $E_2 = E(P_m) - E_1$. Without loss of generality we assume that E_2 has an element other than $u_{m-1}u_m$. We first give a coloring f to the nodes $u_1, u_2, \ldots, u_m$ in $P_m(rP_h)$ with the colors 0 and 1 such that $L(2,1)$-coloring constraints are satisfied. We choose an arbitrary edge u_ku_{k+1} in E_1. If $f(u_k) = 0$, then we rename f as f', otherwise we define $f'(u_p) = 1 - f(u_p)$ for $1 \leq p \leq m$. We reduce the colors of the colored vertices until color of no vertex can be reduced further and get the coloring g. There is a vertex colored with 0, a vertex colored with 1, and the maximum color used till now is 1. We color the vertex $x^{1_1}_{u_ku_{k+1}}$ greedily. Then $g\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$. We color the vertices $x^{i_1}_{u_ku_{k+1}}$, $2 \leq i \leq r$, greedily in any order. Let $S_1 = \{x^{i_1}_{u_pu_{p+1}} : p \neq k, u_pu_{p+1} \in E_1, 1 \leq i \leq r\}$. We color the vertices in S_1 greedily in any order. The maximum color used till now is at most $r + 2$. Let $S_2 = \left\{x^{i_1}_{u_pu_{p+1}} : u_pu_{p+1} \in E_2, 1 \leq i \leq r\right\}$. Then we color the vertices in S_2 greedily in any order. No hole is created so far and the maximum color used is at least

$2r+1$ and at the most $3r+2$. Let $E_3 = \{uv : uv \in E(G), h(uv) > 3\}$. For each edge $u_j u_{j+1}$ in E_3 we color the vertices $x^{1_2}_{u_j u_{j+1}}, x^{1_3}_{u_j u_{j+1}}, \ldots, x^{1_{h(u_j u_{j+1})-2}}_{u_j u_{j+1}}, x^{2_2}_{u_j u_{j+1}}, x^{2_3}_{u_j u_{j+1}}, \ldots, x^{2_{h(u_j u_{j+1})-2}}_{u_j u_{j+1}}, \ldots, x^{r_2}_{u_j u_{j+1}}, x^{r_3}_{u_j u_{j+1}}, \ldots, x^{r_{h(u_j u_{j+1})-2}}_{u_j u_{j+1}}$ greedily in the listed order. When such a vertex w is colored it has one colored neighbor and at most two colored vertices at distance two. Hence $g(w) \leq 5$. We color the remaining vertices greedily. When such a vertex w' is colored it has two colored neighbors and at most $2r$ colored vertices at distance two. Hence $g(w') \leq 2r+6 \leq 3r+3$. Since $5 < 2r+1$ and $r+2 < 2r+1$ no hole is created. Thus g is an inh-coloring of $P_m(rP_h)$ with span at most $3r+3$. ■

Theorem 13. *For $m \geq 3$ and $r \geq 2$, $\lambda_{inh}(P_m(rP_3)) = 2r+1$.*

Proof. Let $P_m = u_1 u_2 \cdots u_m$. We consider two cases depending on values of r.

Case 1. In this case we take $r = 2$. We give the following $L(2,1)$-coloring f_1 to $P_m(rP_3)$: for $1 \leq k \leq m-1$, $f_1(u_k) = 0$; for $1 \leq k \leq m-2$, $f_1\left(x^{1_1}_{u_k u_{k+1}}\right) = 4$, $f_1\left(x^{1_2}_{u_k u_{k+1}}\right) = 2$, $f_1\left(x^{2_1}_{u_k u_{k+1}}\right) = 3$, $f_1(x^{2_2}_{u_k u_{k+1}}) = 5$; $f_1(u_m) = 5$, $f_1\left(x^{1_1}_{u_{m-1} u_m}\right) = 4$, $f_1\left(x^{1_2}_{u_{m-1} u_m}\right) = 2$, $f_1\left(x^{2_1}_{u_{m-1} u_m}\right) = 3$ and $f_1\left(x^{2_2}_{u_{m-1} u_m}\right) = 1$. It can be checked that f_1 is an $L(2,1)$-coloring of $P_m(rP_3)$. We reduce f_1 until we arrive at an irreducible coloring, say f'_1. Since $f_1\left(x^{2_2}_{u_{m-1} u_m}\right) = 1$, $f_1(u_{m-1}) = 0$, and $d\left(x^{2_2}_{u_{m-1} u_m}, u_{m-1}\right) = 2$, the color of $x^{2_2}_{u_{m-1} u_m}$ cannot be reduced and so $f'_1\left(x^{2_2}_{u_{m-1} u_m}\right) = 1$. Now $f'_1(u_2) = 0$ and its neighbors are colored with 2, 3, 4 and 5. Thus f'_1 is an inh-coloring of $P_m(2P_3)$ with span 5.

Case 2. In this case we take $r \geq 3$. Here Lü and Sun [12] have given the following $L(2,1)$-coloring f_2 to $P_m(rP_3)$: $f_2(u_k) = 0$ for $1 \leq k \leq m$ and $f_2\left(x^{i_1}_{u_k u_{k+1}}\right) = 2i$ and $f_2\left(x^{i_2}_{u_k u_{k+1}}\right) = 2i \pmod{2r} + 3$ for $1 \leq k \leq m-1$, $1 \leq i \leq r$. We note that $f_2\left(x^{(r-1)_2}_{u_{m-1} u_m}\right) = 2r+1$. We recolor the vertices $x^{(r-1)_2}_{u_{m-1} u_m}$ and u_m with colors 1 and $2r+1$, respectively, and get the coloring f'_2. No vertex adjacent to $x^{(r-1)_2}_{u_{m-1} u_m}$ has got the color 0 or 2. No vertex adjacent to u_m has received color $2r$ or $2r+1$ and no vertex at distance two from u_m has got the color $2r+1$. Thus f'_2 is an $L(2,1)$-coloring. We reduce f'_2 until we arrive at an irreducible coloring, say f''_2. Since $f'_2\left(x^{(r-1)_2}_{u_{m-1} u_m}\right) = 1$, $f'_2(u_{m-1}) = 0$ and $d\left(x^{(r-1)_2}_{u_{m-1} u_m}, u_{m-1}\right) = 2$, we get $f''_2\left(x^{(r-1)_2}_{u_{m-1} u_m}\right) = 1$. Since $f''_2(u_2) = 0$ and neighbors of u_2 are colored with $2, 3, \ldots, 2r+1$, f''_2 is an inh-coloring of $P_m(rP_3)$ with span $2r+1$.

Thus $\lambda_{inh}(P_m(rP_3)) \leq 2r+1$. Since $\lambda(P_m(rP_3)) = 2r+1$ [12], we get $\lambda_{inh}(P_m(rP_3)) = 2r+1$. ■

The theorem below says that inh-span of $C_3(rP_3)$ is exactly one more than its span [12].

Theorem 14. *For* $r \geq 2$, $\lambda_{inh}(C_3(rP_3)) = 2r + 2$.

Proof. Let $C_3 = u_1u_2u_3u_1$. We first prove that $\lambda_{inh}(C_3(rP_3)) \leq 2r+2$. For this we consider two cases depending on values of r.

Case 1. In this case we take $r = 2$. Lü and Sun [12] have given the following $L(2,1)$-coloring f_1 to $C_3(rP_3)$: $f_1(u_k) = 0$ for $1 \leq k \leq 3$; $f_1\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $f_1\left(x^{1_2}_{u_ku_{k+1}}\right) = 4$, $f_1\left(x^{2_1}_{u_ku_{k+1}}\right) = 3$, $f_1\left(x^{2_2}_{u_ku_{k+1}}\right) = 5$ for $k = 1, 2$; $f_1\left(x^{1_1}_{u_3u_1}\right) = 2$, $f_1\left(x^{1_2}_{u_3u_1}\right) = 4$, $f_1\left(x^{2_1}_{u_3u_1}\right) = 3$ and $f_1\left(x^{2_2}_{u_3u_1}\right) = 5$. We recolor the vertices u_2 and $x^{2_2}_{u_1u_2}$ with colors 6 and 1 respectively and get the coloring g_1. Since no vertex adjacent to u_2 has got the color 5 and no vertex adjacent to $x^{2_2}_{u_1u_2}$ has received color 0 or 2, g_1 is an $L(2,1)$-coloring. If g_1 is not an irreducible coloring, then we reduce it until we arrive at an irreducible coloring, say g'_1. Since $g_1\left(x^{2_2}_{u_1u_2}\right) = 1$, $g_1(u_1) = 0$ and $d\left(x^{2_2}_{u_1u_2}, u_1\right) = 2$, we get $g'_1\left(x^{2_2}_{u_1u_2}\right) = 1$. Since the vertex u_1 is colored with 0, its neighbors are colored with 2, 3, 4, 5 and the vertex u_2 is colored with 6, g'_1 is an inh-coloring with span 6. Hence $\lambda_{inh}(C_3(rP_3)) \leq 2r + 2$ for $r = 2$.

Case 2. In this case we take $r \geq 3$. Lü and Sun [12] have given the following $L(2,1)$-coloring f_2 to $C_3(rP_3)$: $f_2(u_k) = 0$ for $1 \leq k \leq 3$; and for $1 \leq i \leq r$, $k = 1, 2$, $f_2\left(x^{i_1}_{u_ku_{k+1}}\right) = 2i$, $f_2\left(x^{i_1}_{u_ku_{k+1}}\right) = 2i \pmod{2r} + 3$, $f_2\left(x^{i_1}_{u_3u_1}\right) = 2i$ and $f_2\left(x^{i_1}_{u_3u_1}\right) = 2i \pmod{2r} + 3$. We note that $f_2\left(x^{(r-1)_2}_{u_1u_2}\right) = 2r + 1$. We recolor the vertices u_2 and $x^{(r-1)_2}_{u_1u_2}$ with colors $2r + 2$ and 1, respectively, and get the coloring g_2. Since no vertex adjacent to u_2 is colored with $2r + 1$ and no vertex adjacent to $x^{(r-1)_2}_{u_1u_2}$ is colored with 0 or 2, g_2 is an $L(2,1)$-coloring. If g_2 is not an irreducible coloring, then we reduce it until we get an irreducible coloring, say g'_2. Since $g_2(x^{(r-1)_2}_{u_1u_2}) = 1, g_2(u_1) = 0$ and $d(x^{(r-1)_2}_{u_1u_2}, u_1) = 2$, we get $g'_2\left(x^{(r-1)_2}_{u_1u_2}\right) = 1$. The vertex u_1 is colored with 0 and its neighbors are colored with $2, 3, \ldots, 2r + 1$ and $g'_2(u_2) = 2r + 2$. Thus g'_2 is an inh-coloring with span $2r + 2$. Hence $\lambda_{inh}(C_3(rP_3)) \leq 2r + 2$ for $r \geq 3$.

Now we prove that $\lambda_{inh}(C_3(rP_3)) \geq 2r+2$. From Proposition 4, $\lambda_{inh}(C_3(rP_3)) \geq 2r + 1$. Suppose $\lambda_{inh}(C_3(rP_3)) = 2r + 1$ and g_3 is an inh-coloring of $C_3(rP_3)$. From Proposition 1 the vertices u_1, u_2, u_3 are colored with 0 or $2r + 1$. If all the vertices u_1, u_2 and u_3 are colored with 0, then no vertex in $C_3(rP_3)$ will be colored with 1. Hence at least one of u_1, u_2, u_3 is colored with $2r + 1$. Similarly, at least one of u_1, u_2, u_3 is colored with 0. If two nodes, say u_1, u_2, receive the color 0 then u_3 receives the color $2r + 1$. Now from Proposition 1, a neighbor of u_3 in $C_3(rP_3)$, say v, is colored with 0. Since v is at distance two from u_1 or u_2, this is a contradiction. We also get a contradiction if two nodes are colored with $2r+1$. Thus $\lambda_{inh}(C_3(rP_3)) \geq 2r+2$ and we get $\lambda_{inh}(C_3(rP_3)) = 2r+2$. ■

In the following theorem we show that $C_m(rP_2)$ is inh-colorable and give an upper bound to its inh-span.

Theorem 15. *For $r \geq 2$, we have* $\lambda_{inh}(C_m(rP_2)) \leq \begin{cases} 2r+3 & if\ m \text{ is even}, \\ 3r+2 & if\ m \text{ is odd}. \end{cases}$

Proof. Let $C_m = u_1u_2\cdots u_mu_1$. For even m, we give an $L(2,1)$-coloring f_1 to $C_m(rP_2)$ as below: $f_1(u_k) = 0$ for odd k; $f_1(u_k) = 1$ if $k \neq m$ and k is even; $f_1(u_m) = 2r+3$; $f_1\left(x^{i_1}_{u_ku_{k+1}}\right) = 2i+1$ for $1 \leq k \leq m-1$, $1 \leq i \leq r$ and k odd; $f_1\left(x^{i_1}_{u_ku_{k+1}}\right) = 2i+2$ for $1 \leq k \leq m-2$, $1 \leq i \leq r$ and k even; and $f_1\left(x^{i_1}_{u_mu_1}\right) = 2i$ for $1 \leq i \leq r$. We check that f_1 is an inh-coloring and thus $\lambda_{inh}(C_m(rP_2)) \leq 2r+3$ for m even. For odd m, we give an $L(2,1)$-coloring f_2 to $C_m(rP_2)$ as below: $f_2(u_k) = 0$ for odd k, $k \neq m$; $f_2(u_k) = 1$ if k is even; $f_2(u_m) = 2$; $f_2\left(x^{i_1}_{u_ku_{k+1}}\right) = 2i+1$ for odd k and $1 \leq i \leq r$; $f_2\left(x^{i_1}_{u_ku_{k+1}}\right) = 2i+2$ for even k and $1 \leq i \leq r$, and $f_2\left(x^{i_1}_{u_mu_1}\right) = 2r+2+i$ for $1 \leq i \leq r$. We check that f_2 is an inh-coloring and thus $\lambda_{inh}(C_m(rP_2)) \leq 3r+2$ for odd m. ■

Theorem 16. *If $m \geq 3$, $r \geq 3$, and $h(e) \geq 2$ with equality for at least one e but not for all, then $C_m(rP_h)$ is inh-colorable and $\lambda_{inh}(C_m(rP_h)) \leq 3r+3$.*

Proof. Let C_m be the cycle $u_1u_2\cdots u_mu_1$. Let $E_1 = \{uv : uv \in E(C_m), h(uv) > 2\}$ and $E_2 = E(C_m) - E_1$. For our convenience we call the edge u_mu_1 as u_mu_{m+1} too. We first give a coloring f to the nodes $u_1, u_2, \ldots, u_m$ in $C_m(rP_h)$ using the colors 0 and 1 only such that $L(2,1)$-coloring constraints are satisfied. This is possible since $h(e) > 2$ for at least one edge e of C_m. We choose an arbitrary edge u_ku_{k+1} in E_1. If $f(u_k) = 0$, then we rename f as f', otherwise we define $f'(u_p) = 1 - f(u_p)$ for $1 \leq p \leq m$. We reduce the colors of the colored vertices until color of no vertex can be reduced further and get the coloring g. There is a vertex colored with 0, a vertex colored with 1, and the maximum color used till now is 1. We color the vertex $x^{1_1}_{u_ku_{k+1}}$ greedily. Then $g\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$. We color the vertices $x^{i_1}_{u_ku_{k+1}}$, $2 \leq i \leq r$, greedily in any order. Let $S_1 = \left\{x^{i_1}_{u_pu_{p+1}} : p \in [1, k-1] \cup [k+1, m], u_pu_{p+1} \in E_1, 1 \leq i \leq r\right\}$. We color the vertices in S_1 greedily in any order. The maximum color used till now is at most $r+2$. Let $S_2 = \left\{x^{i_1}_{u_pu_{p+1}} : p \in [1, m], u_pu_{p+1} \in E_2, 1 \leq i \leq r\right\}$. Then we color the vertices in S_2 greedily in any order. No hole is created till now and the maximum color used is at least $2r+1$ and at the most $3r+2$. Let $E_3 = \{uv : uv \in E(G), h(uv) > 3\}$. For each edge u_ju_{j+1} in E_3 we color the vertices $x^{1_2}_{u_ju_{j+1}}, x^{1_3}_{u_ju_{j+1}}, \ldots, x^{1_{h(u_ju_{j+1})-2}}_{u_ju_{j+1}}, x^{2_2}_{u_ju_{j+1}}, x^{2_3}_{u_ju_{j+1}}, \ldots, x^{2_{h(u_ju_{j+1})-2}}_{u_ju_{j+1}}, \ldots, x^{r_2}_{u_ju_{j+1}},$

$x^{r_3}_{u_ju_{j+1}}, \ldots, x^{r_{h(u_ju_{j+1})-2}}_{u_ju_{j+1}}$ greedily in the listed order. When such a vertex w is colored it has one colored neighbor and at most two colored vertices at distance two. Hence $g(w) \leq 5$. We color the remaining vertices greedily. When such a vertex w' is colored it is adjacent to two colored vertices and there are at most $2r$ vertices at distance two from it. Hence $g(w') \leq 2r + 6 \leq 3r + 3$. Since $5 < 2r + 1$ and $r + 2 < 2r + 1$, no hole is created. Thus g is an inh-coloring of $C_m(rP_h)$ with span at most $3r + 3$. ■

The theorem below gives an upper bound to inh-span of $C_m(2P_3)$, $m \geq 4$, which is one more than the exact value of its span [12].

Theorem 17. *For $m \geq 4$, $\lambda_{inh}(C_m(2P_3)) \leq 6$.*

Proof. Let $C_m = u_1u_2 \cdots u_mu_1$. For $m \geq 4$, Lü and Sun [12] have given the following $L(2,1)$-coloring f to $C_m(2P_3)$: $f(u_k) = 0$ for $1 \leq k \leq m$; $f\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $f\left(x^{1_2}_{u_ku_{k+1}}\right) = 4$, $f\left(x^{2_1}_{u_ku_{k+1}}\right) = 3$, $f\left(x^{2_2}_{u_ku_{k+1}}\right) = 5$ for $1 \leq k \leq m-1$; and $f\left(x^{1_1}_{u_mu_1}\right) = 2$, $f\left(x^{1_2}_{u_mu_1}\right) = 4$, $f\left(x^{2_1}_{u_mu_1}\right) = 3$ and $f\left(x^{2_2}_{u_mu_1}\right) = 5$. We recolor the vertices u_2 and $x^{2_2}_{u_1u_2}$ with colors 6 and 1, respectively and get the coloring g. Since no vertex adjacent to u_2 has got the color 5 and no vertex adjacent to $x^{2_2}_{u_1u_2}$ has received the color 0 or 2, g is an $L(2,1)$-coloring. If g is not an irreducible coloring we reduce it until we arrive at an irreducible coloring, say g'. Since $g\left(x^{2_2}_{u_1u_2}\right) = 1$, $g(u_1) = 0$ and $d\left(x^{2_2}_{u_1u_2}, u_1\right) = 2$, we get $g'\left(x^{2_2}_{u_1u_2}\right) = 1$. The vertex u_1 is colored with 0 and its neighbors are colored with 2, 3, 4 and 5. Thus g' is an inh-coloring with span 6 and hence $\lambda_{inh}(C_m(2P_3)) \leq 6$ for $m \geq 4$. ■

In the next theorem we show that inh-span of $C_m(rP_3)$ is equal to its span [12] for $m \geq 4$ and $r \geq 3$.

Theorem 18. *For $m \geq 4$ and $r \geq 3$, $\lambda_{inh}(C_m(rP_3)) = 2r + 1$.*

Proof. Let $C_m = u_1u_2 \cdots u_mu_1$. We give an $L(2,1)$-coloring f to $C_m(rP_3)$ as follows: $f(u_1) = f(u_2) = 2r+1$ and $f(u_k) = 0$ for $3 \leq k \leq m$; $f\left(x^{1_1}_{u_1u_2}\right) = 0$ and $f\left(x^{i_1}_{u_1u_2}\right) = i$ for $2 \leq i \leq r$; $f\left(x^{i_2}_{u_1u_2}\right) = r + i$ for $1 \leq i \leq r-1$ and $f\left(x^{r_2}_{u_1u_2}\right) = 0$; $f\left(x^{i_1}_{u_2u_3}\right) = i$ and $f\left(x^{i_2}_{u_2u_3}\right) = r + i$ for $1 \leq i \leq r$; $f\left(x^{1_1}_{u_3u_4}\right) = 2r + 1$ and $f\left(x^{i_1}_{u_3u_4}\right) = i$ for $2 \leq i \leq r$; $f\left(x^{i_2}_{u_3u_4}\right) = r+i+1$ for $1 \leq i \leq r$; $f\left(x^{i_1}_{u_ku_{k+1}}\right) = i+1$ and $f\left(x^{i_2}_{u_ku_{k+1}}\right) = r + i + 1$ for $1 \leq i \leq r$ and $4 \leq k \leq m-1$; $f\left(x^{i_1}_{u_mu_1}\right) = i + 1$ for $1 \leq i \leq r$; $f\left(x^{i_2}_{u_mu_1}\right) = r + i$ for $1 \leq i \leq r-1$; and $f\left(x^{r_2}_{u_mu_1}\right) = 1$.

Now we check that f is an $L(2,1)$-coloring. We note that either a node is colored with 0 and its neighbors are colored with $2, 3, \ldots, 2r + 1$ or a node is colored with $2r + 1$ and its neighbors are colored with $0, 1, \ldots, 2r - 1$. In the coloring f, if a node is colored with 0 (respectively $2r + 1$), no vertex at distance two from it is colored with 0 (respectively $2r+1$). Now $\left|f\left(x^{1_1}_{u_1u_2}\right) - f\left(x^{1_2}_{u_1u_2}\right)\right| = r+1$ and $\left|f\left(x^{i_1}_{u_1u_2}\right) - f\left(x^{i_2}_{u_1u_2}\right)\right| = r$ for $2 \leq i \leq r$, $\left|f\left(x^{i_1}_{u_2u_3}\right) - f\left(x^{i_2}_{u_2u_3}\right)\right| = r$ for $1 \leq i \leq r$, $\left|f\left(x^{1_1}_{u_3u_4}\right) - f\left(x^{1_2}_{u_3u_4}\right)\right| = r-1$ and $\left|f\left(x^{i_1}_{u_3u_4}\right) - f\left(x^{i_2}_{u_3u_4}\right)\right| = r+1$ for $2 \leq i \leq r$, $\left|f\left(x^{i_1}_{u_ku_{k+1}}\right) - f\left(x^{i_2}_{u_ku_{k+1}}\right)\right| = r$ for $1 \leq i \leq r$ and $4 \leq k \leq m-1$, $\left|f\left(x^{i_1}_{u_mu_1}\right) - f\left(x^{i_2}_{u_mu_1}\right)\right| = r-1$ for $1 \leq i \leq r-1$ and $\left|f\left(x^{r_1}_{u_mu_1}\right) - f\left(x^{r_2}_{u_mu_1}\right)\right| = r$. Thus $\left|f\left(x^{i_1}_{uv}\right) - f\left(x^{i_2}_{uv}\right)\right| \geq 2$ for every edge uv of C_m and $1 \leq i \leq r$. Hence f is an $L(2,1)$-coloring. In $C_m(rP_3)$ every vertex is either a maximum degree vertex or adjacent to a maximum degree vertex. We have span$(f) = 2r+1$ and maximum degree of $C_m(rP_3) = 2r$. Thus f is an irreducible coloring. Since $f\left(x^{1_1}_{u_2u_3}\right) = 1$, u_3 is colored with 0 and its neighbors are colored with $2, 3, \ldots, 2r+1$, and f is an inh-coloring with span $2r+1$, we get $\lambda_{inh}(C_m(rP_3)) \leq 2r+1$. Since $\lambda(C_m(rP_3)) = 2r+1$ [12], we get $\lambda_{inh}(C_m(rP_3)) = 2r+1$. ■

If G is any graph with $\Delta = 2$ and $h : E(G) \to \mathbb{N} - \{1, 2\}$ with $h(e) > 3$ for at least one e, then the next theorem gives span as well as inh-span of $G(rP_h)$, and shows that both the spans are equal.

Theorem 19. *For any graph G with $\Delta = 2$, $r \geq 2$, and $h(e) \geq 3$ with strict inequality for at least one e, $\lambda_{inh}(G(rP_h)) = \lambda(G(rP_h)) = 2r+1$.*

Proof. Here G is either a path $P_m = u_1u_2\cdots u_m$ or cycle $C_m = u_1u_2\cdots u_mu_1$, $m \geq 3$. For our convenience we call the edge u_mu_1 as u_mu_{m+1} too. We give an $L(2,1)$-coloring to $G(rP_h)$ in three cases depending on values of r. In all these cases, u_ku_{k+1} is an arbitrary edge of G.

Case 1. In this case we take $r = 2$. We give an $L(2,1)$-coloring g_1 to $G(rP_h)$ as follows: $g_1(u) = 0$ for all nodes u of $G(rP_h)$; if $h(u_ku_{k+1}) = 3$, then $g_1\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $g_1\left(x^{1_2}_{u_ku_{k+1}}\right) = 4$, $g_1\left(x^{2_1}_{u_ku_{k+1}}\right) = 5$, and $g_1\left(x^{2_2}_{u_ku_{k+1}}\right) = 3$; if $h(u_ku_{k+1}) = 6$, then $g_1\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $g_1\left(x^{1_2}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{1_3}_{u_ku_{k+1}}\right) = 3$, $g_1\left(x^{1_4}_{u_ku_{k+1}}\right) = 1$, $g_1\left(x^{1_5}_{u_ku_{k+1}}\right) = 4$, $g_1\left(x^{2_1}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{2_2}_{u_ku_{k+1}}\right) = 3$, $g_1\left(x^{2_3}_{u_ku_{k+1}}\right) = 0$, $g_1\left(x^{2_4}_{u_ku_{k+1}}\right) = 5$, and $g_1\left(x^{2_5}_{u_ku_{k+1}}\right) = 3$; for $h(u_ku_{k+1}) \geq 4$, $h(u_ku_{k+1}) \neq 6$: if $h(u_ku_{k+1}) \equiv 0 \pmod 4$ then $g_1\left(x^{1_j}_{u_ku_{k+1}}\right) = 0, 2, 5$ or 3 according as $j \equiv 0, 1, 2$ or $3 \pmod 4$ and $g_1(x^{2_j}_{u_ku_{k+1}}) = 0, 5, 1$ or 4 according as $j \equiv 0, 1, 2$ or $3 \pmod 4$; if $h(u_ku_{k+1}) \equiv 1 \pmod 4$ then $g_1\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $g_1\left(x^{1_2}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{1_3}_{u_ku_{k+1}}\right) = 1$, $g_1\left(x^{1_4}_{u_ku_{k+1}}\right) = 3$ and for $j \geq 5$, $g_1\left(x^{1_j}_{u_ku_{k+1}}\right) = 0, 2, 5$ or 3 according as $j \equiv 1, 2, 3$ or $0 \pmod 4$, $g_1\left(x^{2_1}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{2_2}_{u_ku_{k+1}}\right) = 3$, $g_1\left(x^{2_3}_{u_ku_{k+1}}\right) = 1$, $g_1\left(x^{2_4}_{u_ku_{k+1}}\right) = 4$ and for $j \geq 5$, $g_1\left(x^{2_j}_{u_ku_{k+1}}\right) = 0, 5, 1$ or 4 according as $j \equiv 1, 2, 3$ or $0 \pmod 4$; if $h(u_ku_{k+1}) \equiv 2 \pmod 4$, then $g_1\left(x^{1_1}_{u_ku_{k+1}}\right) =$

2, $g_1\left(x^{1_2}_{u_ku_{k+1}}\right) = 4$, $g_1\left(x^{1_3}_{u_ku_{k+1}}\right) = 0$, $g_1\left(x^{1_4}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{1_5}_{u_ku_{k+1}}\right) = 3$ and for $j \geq 6$, $g_1\left(x^{1_j}_{u_ku_{k+1}}\right) = 0, 2, 5$ or 3 according as $j \equiv 2, 3, 0$ or 1 (mod 4), $g_1\left(x^{2_1}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{2_2}_{u_ku_{k+1}}\right) = 3$, $g_1\left(x^{2_3}_{u_ku_{k+1}}\right) = 0$, $g_1(x^{2_4}_{u_ku_{k+1}}) = 2$, $g_1\left(x^{2_5}_{u_ku_{k+1}}\right) = 4$ and for $j \geq 6$, $g_1\left(x^{2_j}_{u_ku_{k+1}}\right) = 0, 5, 1$ or 4 according as $j \equiv 2, 3, 0$ or 1 (mod 4); if $h(u_ku_{k+1}) \equiv 3$ (mod 4), then $g_1\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $g_1\left(x^{1_2}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{1_3}_{u_ku_{k+1}}\right) = 3$, $g_1\left(x^{1_4}_{u_ku_{k+1}}\right) = 0$, $g_1\left(x^{1_5}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{1_6}_{u_ku_{k+1}}\right) = 3$ and for $j \geq 7$, $g_1\left(x^{1_j}_{u_ku_{k+1}}\right) = 0, 2, 5$ or 3 according as $j \equiv 3, 0, 1$ or 2 (mod 4), $g_1\left(x^{2_1}_{u_ku_{k+1}}\right) = 5$, $g_1\left(x^{2_2}_{u_ku_{k+1}}\right) = 1$, $g_1\left(x^{2_3}_{u_ku_{k+1}}\right) = 4$, $g_1\left(x^{2_4}_{u_ku_{k+1}}\right) = 0$, $g_1\left(x^{2_5}_{u_ku_{k+1}}\right) = 2$, $g_1\left(x^{2_6}_{u_ku_{k+1}}\right) = 4$ and for $j \geq 7$, $g_1\left(x^{2_j}_{u_ku_{k+1}}\right) = 0, 5, 1$ or 4 according as $j \equiv 3, 0, 1$ or 2 (mod 4).

For every edge uv in G the $L(2,1)$-coloring constraints are satisfied within the paths P^i_h for $1 \leq i \leq 2$ and colors assigned to neighbors of a node are different. Hence g_1 is an $L(2,1)$-coloring with span 5. Now we reduce g_1 until we arrive at an irreducible coloring, say g'_1. We prove that g'_1 is a no-hole coloring. From the way g_1 is defined there is at least one vertex w colored with 1 and lying at distance two from a vertex colored with 0 in g_1. Hence $g'_1(w) = 1$. A vertex in $G(rP_h)$ with degree 4 is colored with 0 and its neighbors are colored with 2, 3, 4, 5. Thus g'_1 is an inh-coloring with span 5.

Case 2. In this case we take $r = 3$. Now we give an $L(2,1)$-coloring g_2 to $G(rP_h)$ as follows: $g_2(u) = 0$ for all nodes u of $G(rP_h)$; if $h(u_ku_{k+1}) = 3$ then $g_2\left(x^{i_1}_{u_ku_{k+1}}\right) = i+1, g_2\left(x^{i_2}_{u_ku_{k+1}}\right) = i+4$ for $1 \leq i \leq 3$; if $h(u_ku_{k+1}) \equiv 1$ (mod 3), then $g_2\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $g_2\left(x^{1_2}_{u_ku_{k+1}}\right) = 7$, $g_2\left(x^{1_3}_{u_ku_{k+1}}\right) = 5$, $g_2\left(x^{2_1}_{u_ku_{k+1}}\right) = 3$, $g_2\left(x^{2_2}_{u_ku_{k+1}}\right) = 1$, $g_2\left(x^{2_3}_{u_ku_{k+1}}\right) = 6$, $g_2\left(x^{3_1}_{u_ku_{k+1}}\right) = 4$, $g_2\left(x^{3_2}_{u_ku_{k+1}}\right) = 1$, $g_2\left(x^{3_3}_{u_ku_{k+1}}\right) = 7$, for $1 \leq i \leq 3$ and $j \geq 4$, $g_2\left(x^{i_j}_{u_ku_{k+1}}\right) = 0$, $i+1$ or $i+4$ according as $j \equiv 1, 2$ or 0 (mod 3); if $h(u_ku_{k+1}) \equiv 2$ (mod 3) then $g_2\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $g_2\left(x^{1_2}_{u_ku_{k+1}}\right) = 7$, $g_2\left(x^{1_3}_{u_ku_{k+1}}\right) = 1$, $g_2\left(x^{1_4}_{u_ku_{k+1}}\right) = 5$, $g_2\left(x^{2_1}_{u_ku_{k+1}}\right) = 3$, $g_2\left(x^{2_2}_{u_ku_{k+1}}\right) = 1$, $g_2\left(x^{2_3}_{u_ku_{k+1}}\right) = 4$, $g_2\left(x^{2_4}_{u_ku_{k+1}}\right) = 6$, $g_2\left(x^{3_1}_{u_ku_{k+1}}\right) = 4$, $g_2\left(x^{3_2}_{u_ku_{k+1}}\right) = 1$, $g_2\left(x^{3_3}_{u_ku_{k+1}}\right) = 3$, $g_2\left(x^{3_4}_{u_ku_{k+1}}\right) = 7$, for $1 \leq i \leq 3$ and $j \geq 5$, $g_2(x^{i_j}_{u_ku_{k+1}}) = 0, i+1$ or $i+4$ according as $j \equiv 2, 0$ or 1 (mod 3); if $h(u_ku_{k+1}) \geq 6$ and $h(u_ku_{k+1}) \equiv 0$ (mod 3), then $g_2\left(x^{1_1}_{u_ku_{k+1}}\right) = 2$, $g_2\left(x^{1_2}_{u_ku_{k+1}}\right) = 5$, $g_2\left(x^{1_3}_{u_ku_{k+1}}\right) = 0$, $g_2\left(x^{1_4}_{u_ku_{k+1}}\right) = 2$, $g_2\left(x^{1_5}_{u_ku_{k+1}}\right) = 5$, $g_2\left(x^{2_1}_{u_ku_{k+1}}\right) = 3$,

$g_2\left(x^{2_2}_{u_k u_{k+1}}\right) = 1$, $g_2\left(x^{2_3}_{u_k u_{k+1}}\right) = 4$, $g_2\left(x^{2_4}_{u_k u_{k+1}}\right) = 2$, $g_2\left(x^{2_5}_{u_k u_{k+1}}\right) = 6$, $g_2\left(x^{3_1}_{u_k u_{k+1}}\right) = 4$, $g_2\left(x^{3_2}_{u_k u_{k+1}}\right) = 1$, $g_2\left(x^{3_3}_{u_k u_{k+1}}\right) = 3$, $g_2\left(x^{3_4}_{u_k u_{k+1}}\right) = 5$, $g_2\left(x^{3_5}_{u_k u_{k+1}}\right) = 7$, and $g_2\left(x^{i_j}_{u_k u_{k+1}}\right) = 0, i+1$ or $i+4$ according as $j \equiv 0, 1$ or $2 \pmod 3$, where $1 \leq i \leq 3$ and $j \geq 6$.

For every edge uv in G the $L(2,1)$-coloring constraints are satisfied within the paths P^i_h for $1 \leq i \leq 3$ and colors assigned to neighbors of a node are different. Hence g_2 is an $L(2,1)$-coloring with span 7. Now we reduce g_2 until we arrive at an irreducible coloring, say g'_2. We prove that g'_2 is a no-hole coloring. From the way g_2 is defined there is at least one vertex w' colored with 1 and lying at distance two from a vertex colored with 0 in g_2. Hence $g'_2(w') = 1$. A vertex in $G(rP_h)$ with degree 6 is colored with 0 and its neighbors are colored with 2, 3, 4, 5, 6, 7. Thus g'_2 is an inh-coloring with span 7.

Case 3. In this case we take $r \geq 4$. We give an $L(2,1)$-coloring g_2 to $G(rP_h)$ as follows: $g_2(u) = 0$ for all nodes u of $G(rP_h)$; $g_3\left(x^{i_1}_{u_k u_{k+1}}\right) = i+1$ and $g_3\left(x^{i_{h(u_k u_{k+1})-1}}_{u_k u_{k+1}}\right) = r+i+1$ for $1 \leq i \leq r$. $L(2,1)$-coloring constraints are satisfied for the colored vertices so far. We take an edge $u'v'$ of G such that $h(u'v') > 3$ and assign $g_3\left(x^{2_2}_{u'v'}\right) = 1$. Since no vertex adjacent to $x^{2_2}_{u'v'}$ is colored with color 0 or 2, $L(2,1)$-coloring constraints are satisfied for the colored vertices. The maximum color used till now is $2r+1$. We color the remaining vertices greedily. If z is such a vertex then it has two neighbors and there are two vertices at distance two apart from it. Hence $g_3(z) \leq 8$. Thus $\operatorname{span}(g_3) = 2r+1$ because $r \geq 4$. Now we reduce g_3 until we arrive at an irreducible coloring, say g'_3. A vertex in $G(rP_h)$ with degree $2r$ is colored with 0 and its neighbors are colored with $2, 3, \ldots, 2r+1$. Since $g_3\left(x^{2_2}_{u'v'}\right) = 1, g_3(u') = 0$ and $d\left(x^{2_2}_{u'v'}, u'\right) = 2$, we get $g'_3\left(x^{2_2}_{u'v'}\right) = 1$. Hence g'_3 is an inh-coloring with span $2r+1$.

Combining all these cases we conclude that $G(rP_h)$ is inh-colorable and $\lambda_{inh}(G(rP_h)) \leq 2r+1$. Thus from Proposition 4 we get $\lambda_{inh}(G(rP_h)) = \lambda(G(rP_h)) = 2r+1$. ■

4. Inh-Colorability of Graphs $G(rP_h)$ with $\Delta(G) \geq 3$

In this section we first consider the case $\Delta(G) = 3$. In Theorem 20 below we find the exact value of span of $G(rP_3)$, $r \geq 2$, which were not computed by Lü and Sun [12]. Moreover, this value of $\lambda(G(rP_3))$ agrees with $\lambda\left(G_{(3)}\right)$ for $r = 1$, computed by Chang *et al.* [1], for some graphs G.

Theorem 20. *If G is a graph with $\Delta(G) = 3$, then for $r \geq 2$, $\lambda(G(rP_3)) = 3r+1$.*

Proof. We first consider the graph $G_{(3)}$. Let $S = V\left(G_{(3)}\right) - V(G)$. Since every vertex in S is at distance two apart from at most two other vertices in S, we can give a coloring f to vertices in S using colors 0, 1 and 2 only such that vertices at distance two in $G_{(3)}$ have different colors.

Now we give an $L(2,1)$-coloring g to $G(rP_3)$. We assign $g(u) = 0$ for all $u \in V(G)$. For every edge uv of G we assign colors to the vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_2}$, $1 \le i \le r$, as below: if $f\left(x_{uv}^{1}\right) = f\left(x_{uv}^{2}\right) = 0$ then $g\left(x_{uv}^{i_1}\right) = 3i-1$ for $1 \le i \le r$, $g\left(x_{uv}^{1_2}\right) = 3r-1$ and $g\left(x_{uv}^{i_2}\right) = 3i-4$ for $2 \le i \le r$; if $f\left(x_{uv}^{1}\right) = f\left(x_{uv}^{2}\right) = 1$ then $g(x_{uv}^{i_1}) = 3i$ for $1 \le i \le r$, $g(x_{uv}^{1_2}) = 3r$ and $g(x_{uv}^{i_2}) = 3i-3$ for $2 \le i \le r$; if $f\left(x_{uv}^{1}\right) = f\left(x_{uv}^{2}\right) = 2$, then $g\left(x_{uv}^{i_1}\right) = 3i+1$ for $1 \le i \le r$, $g\left(x_{uv}^{1_2}\right) = 3r+1$ and $g\left(x_{uv}^{i_2}\right) = 3i-2$ for $2 \le i \le r$; if $f\left(x_{uv}^{1}\right) = 0$ and $f\left(x_{uv}^{2}\right) = 1$, then $g\left(x_{uv}^{i_1}\right) = 3i-1$ for $1 \le i \le r$, $g\left(x_{uv}^{1_2}\right) = 3r$ and $g\left(x_{uv}^{i_2}\right) = 3i-3$ for $2 \le i \le r$; if $f\left(x_{uv}^{1}\right) = 1$ and $f\left(x_{uv}^{2}\right) = 2$, then $g\left(x_{uv}^{i_1}\right) = 3i$ for $1 \le i \le r$, $g\left(x_{uv}^{1_2}\right) = 3r+1$ and $g\left(x_{uv}^{i_2}\right) = 3i-2$ for $2 \le i \le r$; if $f\left(x_{uv}^{1}\right) = 0$ and $f\left(x_{uv}^{2}\right) = 2$, then $g\left(x_{uv}^{i_1}\right) = 3i-1$ for $1 \le i \le r$ and $g\left(x_{uv}^{i_2}\right) = 3i+1$ for $1 \le i \le r$.

For any edge uv of G and for $1 \le i \le r$, $|g(x_{uv}^{i_1}) - g(x_{uv}^{i_2})| \ge 2$. Colors of the vertices of $G(rP_3)$ adjacent to a node are distinct. Colors of the nodes are 0 and colors of the other vertices are greater than or equal to 2. Hence g is an $L(2,1)$-coloring with span $3r+1$. Thus $\lambda(G(rP_3)) \le 3r+1$. Now from Proposition 4 we get $\lambda(G(rP_3)) = 3r+1$. ■

The theorem below gives an upper bound to inh-span of $G(rP_3)$, $r \ge 2$. We note that this bound agrees with the upper bound of $\lambda_{inh}\left(G_{(3)}\right)$ given by Mandal and Panigrahi [13] for $r = 1$.

Theorem 21. *If G is a graph with $\Delta(G) = 3$, then for $r \ge 2$, $G(rP_3)$ is inh-colorable and $\lambda_{inh}(G(rP_3)) \le 3r+2$.*

Proof. We consider the same $L(2,1)$-coloring g of $G(rP_3)$ as given in the proof of Theorem 20. Note that g has a hole only at 1. Also note that colors of neighbors of a vertex colored with 2 lies in the set $\{0, 4, 3r-1, 3r\}$. Let u be a vertex in G with $deg(u) = 3$ if G is a regular graph and $deg(u) \ne 3$ otherwise. From the way g is defined, we get that u is adjacent to a vertex in $G(rP_3)$ that is colored with 2, 3 or $3r+1$. We consider three cases depending on colors of neighbors of u.

Case 1. Here u is adjacent to a vertex colored with $3r+1$. Let $g\left(x_{uv_1}^{i_1}\right) = 3r+1$. Then $g\left(x_{uv_1}^{i_2}\right) \ne 2$. We give an another $L(2,1)$-coloring g_1 to $G(rP_3)$ as follows: $g_1(y) = g(y)$ if $y \ne u, x_{uv_1}^{i_1}$ and $g_1\left(x_{uv_1}^{i_1}\right) = 1$. Since $x_{uv_1}^{i_1}$ is adjacent to u and $x_{uv_1}^{i_2}$ only, and the vertex $x_{uv_1}^{i_1}$ receives the color 1, the $L(2,1)$-coloring constraints are satisfied so far. We color the vertex u with the least available color such that $L(2,1)$ coloring constraints are satisfied. Since u is not adjacent to any vertex colored with $3r+1$ in g_1, $g_1(u) \le 3r+2$. Now we reduce g_1 until we arrive at an irreducible coloring g_1'. Then $\text{span}(g_1') \le 3r+2$. Since

$g_1\left(x_{uv_1}^{i_1}\right) = 1$, $g_1(v_1) = 0$ and $d\left(x_{uv_1}^{i_1}, v_1\right) = 2$, color of $x_{uv_1}^{i_1}$ cannot be reduced, and so $g_1'\left(x_{uv_1}^{i_1}\right) = 1$. There is a vertex of degree $3r$ in $G(rP_3)$ colored with 0 and its neighbors are colored with $2, 3, \ldots, 3r+1$. Hence g_1' is an inh-coloring.

Case 2. Here u is not adjacent to any vertex colored with $3r+1$ and adjacent to a vertex colored with 2. Let $g\left(x_{uv_2}^{i_1}\right) = 2$. Then $g\left(x_{uv_2}^{i_2}\right) \neq 2$. We give an another $L(2,1)$-coloring g_2 to $G\left(rP_3\right)$ as follows: $g_2(y) = g(y)$ if $y \neq u, x_{uv_2}^{i_1}$ and $g_2\left(x_{uv_2}^{i_1}\right) = 1$. Since $x_{uv_2}^{i_1}$ is adjacent to u and $x_{uv_2}^{i_2}$ only, and the vertex $x_{uv_2}^{i_1}$ is colored with 1, the $L(2,1)$-coloring constraints are satisfied so far. Then we color the vertex u with the least available color such that $L(2,1)$-coloring constraints are satisfied. Since u is not adjacent to any vertex colored with $3r+1$, $g_2(u) \leq 3r+2$. Now we reduce g_2 until we arrive at an irreducible coloring g_2'. Then $\text{span}(g_2') \leq 3r+2$. Since $g_2\left(x_{uv_2}^{i_1}\right) = 1$, $g_2(v_2) = 0$ and $d\left(x_{uv_2}^{i_1}, v_2\right) = 2$, color of $x_{uv_2}^{i_1}$ cannot be reduced, and so $g_2'\left(x_{uv_2}^{i_1}\right) = 1$. There is a vertex of degree $3r$ in $G(rP_3)$ colored with 0 and its neighbors are colored with $2, 3, \ldots, 3r+1$. Hence g_2' is an inh-coloring.

Case 3. In this case, u is not adjacent to any vertex colored with $3r+1$ or 2. Then u is adjacent to a vertex colored with 3. Let $g\left(x_{uv_3}^{i_1}\right) = 3$. Then $g\left(x_{uv_3}^{i_2}\right) \neq 2$. We give an another $L(2,1)$-coloring g_3 to $G\left(rP_3\right)$ as follows: $g_3(y) = g(y)$ if $y \neq u, x_{uv_3}^{i_1}$ and $g_3\left(x_{uv_3}^{i_1}\right) = 1$. Since $x_{uv_3}^{i_1}$ is adjacent to u and $x_{uv_3}^{i_2}$ only, and the vertex $x_{uv_3}^{i_1}$ is colored with 1, the $L(2,1)$-coloring constraints are satisfied so far. Then we color the vertex u with the least available color such that $L(2,1)$-coloring constraints are satisfied. Since u is not adjacent to any vertex colored with $3r+1$, $g_3(u) \leq 3r+2$. Now we reduce g_3 until we arrive at an irreducible coloring g_3'. Then $\text{span}(g_3') \leq 3r+2$. Since $g_3\left(x_{uv_3}^{i_1}\right) = 1$, $g_3(v_3) = 0$ and $d\left(x_{uv_3}^{i_1}, v_3\right) = 2$, color of $x_{uv_3}^{i_1}$ cannot be reduced, and so $g_3'\left(x_{uv_3}^{i_1}\right) = 1$. There is a vertex of degree $3r$ in $G\left(rP_3\right)$ colored with 0 and its neighbors are colored with $2, 3, \ldots, 3r+1$. Hence g_3' is an inh-coloring.

Combining all these cases we get that $G(rP_3)$ is inh-colorable and $\lambda_{inh}(G(rP_3)) \leq 3r+2$. ■

In Theorem 22 below we find span as well as inh-span of $G(rP_h)$, where $r \geq 2$ and $h(e) \geq 3$ with strict inequality for at least one e. Moreover, here we settle the case $h(e) = 4$, for all e, which was left by Lü and Sun [12].

Theorem 22. *If G is a graph with $\Delta(G) = 3$, $r \geq 2$ and $h(e) \geq 3$ with strict inequality for at least one e, then $\lambda_{inh}(G(rP_h)) = \lambda(G(rP_h)) = 3r+1$.*

Proof. We choose an edge $u'v'$ in G such that $h(u'v') > 3$. We first consider the graph $G_{(3)}$. Let $S = V\left(G_{(3)}\right) - V(G)$. Since every vertex in S is at distance two from at most two other vertices in S, we can give a coloring f to S using the colors 0, 1 and 2 only such that vertices at distance two get different colors and $f\left(x_{u'v'}^1\right) = f\left(x_{u'v'}^2\right) = 1$. Then we give an $L(2,1)$-coloring g to $G\left(rP_3\right)$ following the same method of coloring in the proof of Theorem 20. Now we consider two cases depending on values of r.

Case 1. In this case we take $r = 2$. We give a coloring g_1 to $G(rP_h)$ as below. For any edge uv of G and for $i = 1, 2$, $g_1\left(x_{uv}^{i_1}\right) = g\left(x_{uv}^{i_1}\right)$ and $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = g\left(x_{uv}^{i_2}\right)$. To color the remaining vertices we have the following subcases depending on values of $g_1\left(x_{uv}^{i_1}\right)$ and $g_1\left(x_{uv}^{i_{h(uv)-1}}\right)$ for $i = 1, 2$ and an arbitrary edge uv in G.

Subcase 1. $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 5$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$, then $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 5 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_2}\right) = 7$, $g_1\left(x_{uv}^{i_3}\right) = 5$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 5 according as $j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_2}\right) = 7$, $g_1\left(x_{uv}^{i_3}\right) = 3$, $g_1\left(x_{uv}^{i_4}\right) = 5$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 5 according as $j \equiv 2, 0$ or $1 \pmod 3$.

Subcase 2. $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 6$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$ then $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 6 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_2}\right) = 4$, $g_1\left(x_{uv}^{i_3}\right) = 6$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 6 according as $j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_2}\right) = 4$, $g_1\left(x_{uv}^{i_3}\right) = 1$, $g_1\left(x_{uv}^{i_4}\right) = 6$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 6 according as $j \equiv 2, 0$ or $1 \pmod 3$.

Subcase 3. $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 5$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$ then $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 5 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 5$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 5 according as $j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 7$, $g_1\left(x_{uv}^{i_4}\right) = 5$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 5 according as $j \equiv 2, 0$ or $1 \pmod 3$.

Subcase 4. $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 4$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$ then $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 4 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$, then $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_2}\right) = 6$, $g_1\left(x_{uv}^{i_3}\right) = 4$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 4 according as

$j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 2$, $g_1\left(x_{uv}^{i_2}\right) = 6$, $g_1\left(x_{uv}^{i_3}\right) = 1$, $g_1\left(x_{uv}^{i_4}\right) = 4$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 2$ or 4 according as $j \equiv 2, 0$ or $1 \pmod 3$.

Subcase 5. $g_1\left(x_{uv}^{i_1}\right) = 5$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 7$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$ then $g_1\left(x_{uv}^{i_j}\right) = 0, 5$ or 7 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 5$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 7$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 5$ or 7 according as $j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$, $g_1\left(x_{uv}^{i_1}\right) = 5$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 3$, $g_1\left(x_{uv}^{i_4}\right) = 7$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 5$ or 7 according as $j \equiv 2, 0$ or $1 \pmod 3$.

Subcase 6. $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 6$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$ and $h(uv) \geq 6$ then $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 4$, $g_1\left(x_{uv}^{i_4}\right) = 2$, $g_1\left(x_{uv}^{i_5}\right) = 6$, and for $j \geq 6$, $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 6 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$, then $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 6$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 6 according as $j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$, then $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 4$, $g_1\left(x_{uv}^{i_4}\right) = 6$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 6 according as $j \equiv 2, 0$ or $1 \pmod 3$.

Subcase 7. $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 7$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$ then $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 7 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 7$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 7 according as $j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 3$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 4$, $g_1\left(x_{uv}^{i_4}\right) = 7$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 3$ or 7 according as $j \equiv 2, 0$ or $1 \pmod 3$.

Subcase 8. $g_1\left(x_{uv}^{i_1}\right) = 4$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 6$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0 \pmod 3$ then $g_1\left(x_{uv}^{i_j}\right) = 0, 4$ or 6 according as $j \equiv 0, 1$ or $2 \pmod 3$. If $h(uv) \equiv 1 \pmod 3$, then $g_1\left(x_{uv}^{i_1}\right) = 4$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 6$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 4$ or 6 according as $j \equiv 1, 2$ or $0 \pmod 3$. If $h(uv) \equiv 2 \pmod 3$ then $g_1\left(x_{uv}^{i_1}\right) = 4$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 3$, $g_1\left(x_{uv}^{i_4}\right) = 6$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 4$ or 6 according as

$j \equiv 2, 0$ or 1 (mod 3).

Subcase 9. $g_1\left(x_{uv}^{i_1}\right) = 4$, $g_1\left(x_{uv}^{i_{h(uv)-1}}\right) = 7$. Then g_1 assigns colors to the remaining vertices as follows. If $h(uv) \equiv 0$ (mod 3) then $g_1\left(x_{uv}^{i_j}\right) = 0, 4$ or 7 according as $j \equiv 0, 1$ or 2 (mod 3). If $h(uv) \equiv 1$ (mod 3) then $g_1\left(x_{uv}^{i_1}\right) = 4$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 7$, and for $j \geq 4$, $g_1\left(x_{uv}^{i_j}\right) = 0, 4$ or 7 according as $j \equiv 1, 2$ or 0 (mod 3). If $h(uv) \equiv 2$ (mod 3) then $g_1\left(x_{uv}^{i_1}\right) = 4$, $g_1\left(x_{uv}^{i_2}\right) = 1$, $g_1\left(x_{uv}^{i_3}\right) = 3$, $g_1\left(x_{uv}^{i_4}\right) = 7$, and for $j \geq 5$, $g_1\left(x_{uv}^{i_j}\right) = 0, 4$ or 7 according as $j \equiv 2, 0$ or 1 (mod 3).

Note that the colors of the vertices $x_{uv}^{i_1}$ and $x_{uv}^{i_{h(uv)-1}}$ remain unchanged. We reduce g_1 until we arrive at an irreducible coloring g_1'. Now we prove that g_1' is a no hole coloring. Since $f\left(x_{u'v'}^1\right) = f\left(x_{u'v'}^2\right) = 1$ we get $g_1\left(x_{u'v'}^{1_1}\right) = 3$ and $g_1\left(x_{u'v'}^{1_{h(uv)-1}}\right) = 6$. Thus $g_1\left(x_{u'v'}^{1_2}\right) = 1$. Since $g_1(u') = 0$ and $d\left(x_{u'v'}^{1_2}, u'\right) = 2$, color of the vertex $x_{u'v'}^{1_2}$ cannot be reduced and so $g_1'\left(x_{u'v'}^{1_2}\right) = 1$. A maximum degree vertex is colored with 0 and its neighbors are colored with 2, 3, 4, 5, 6 and 7. Hence g_1' is an inh-coloring with span 7.

Case 2. In this case we take $r \geq 3$. We give a coloring g_2 to $G(rP_h)$ as below. For any edge uv of G, $g_2\left(x_{uv}^{i_1}\right) = g\left(x_{uv}^{i_1}\right)$ and $g_2\left(x_{uv}^{i_{h(uv)-1}}\right) = g(x_{uv}^{i_2})$, $1 \leq i \leq r$. Since $g_2\left(x_{u'v'}^{1_1}\right) = 3$ and $g_2\left(x_{u'v'}^{1_{h(uv)-1}}\right) = 3r$ we take $g_2\left(x_{u'v'}^{1_2}\right) = 1$. Then we color the remaining vertices greedily in any order. If w is such a vertex, then it has two neighbors and there are two vertices at distance two from it. Hence $g_2(w) \leq 8$. Since $r \geq 3$, we get $3r + 1 > 8$. Thus $\text{span}(g_2) = 3r + 1$. We reduce g_2 until we arrive at an irreducible coloring g_2'. Now we prove that g_2' is a no hole coloring. Since $g_2\left(x_{u'v'}^{1_2}\right) = 1$, $g_2(u') = 0$ and $d\left(x_{u'v'}^{1_2}, u'\right) = 2$, color of the vertex $x_{u'v'}^{1_2}$ cannot be reduced and so $g_2'\left(x_{u'v'}^{1_2}\right) = 1$. A maximum degree vertex is colored with 0 and its neighbors are colored with $2, 3, \ldots, 3r + 1$. Hence g_2' is an inh-coloring with span $3r + 1$.

From these two cases we conclude that $G(rP_h)$ is inh-colorable and $\lambda(G(rP_h)) \leq \lambda_{inh}(G(rP_h)) \leq 3r + 1$. Thus from Proposition 4 we get $\lambda_{inh}(G(rP_h)) = \lambda(G(rP_h)) = 3r + 1$. ∎

We state the following lemma by Mandal and Panigrahi [13] which will be used in our next few results.

Lemma 23 [13]**.** *Let f be an irreducible coloring of a graph G. Then no two consecutive numbers can be holes in f. Further, if l is a hole in f then every vertex colored with $l + 1$ is adjacent to a vertex colored with $l - 1$.*

Now we consider graphs G with $\Delta(G) \geq 3$. The theorem below gives upper bound to $\lambda_{inh}(G(rP_2))$ which agrees with the upper bound of $\lambda_{inh}\left(G_{(2)}\right)$ given by Mandal and Panigrahi [13], for $r = 1$. If G is either a tree or a non-bipartite graph then the bound agrees with the upper bound of $\lambda(G(rP_2))$ given by Lü and Sun [12].

Theorem 24. *Let G be a graph with $\Delta(G) \geq 3$. Then for $r \geq 2$, $G(rP_2)$ is inh-colorable and*

$$\lambda_{inh}(G(rP_2)) \leq \begin{cases} \chi(G) + r\chi'(G) + 3 & \textit{if } G \textit{ is a bipartite graph other than a tree,} \\ \chi(G) + r\chi'(G) & \textit{otherwise,} \end{cases}$$

where $\chi(G)$ and $\chi'(G)$ are respectively the chromatic number and edge chromatic number of G.

Proof. Let G be a bipartite graph other than a tree. Now let f'_1 be an edge coloring of G starting with color 1 and ending with $\chi'(G)$. Mandal and Panigrahi [13] have given an inh-coloring f_1 to $G_{(2)}$. We describe the coloring f_1 below.

Let y be a vertex in G of degree at least 3 and y_1, y_2, y_3 be its neighbors with degree of y_1 greater than or equal to 2. Let y_{11} be a neighbor of y_1 different from y. We give an $L(2,1)$-coloring c_1 to $G_{(2)}$ as below. $c_1(y) = 1$, $c_1(y_i) = 0$ and $c_1\left(x^1_{yy_i}\right) = i + 2$ for $i = 1, 2, 3$, $c_1\left(x^1_{y_1y_{11}}\right) = 2$ and $c_1(y_{11}) = 4$. We color all the uncolored vertices in $V(G)$ with the colors 0 and 1 so that $L(2,1)$-coloring constraints are satisfied in $G_{(2)}$ and any vertex in $V(G)$ colored with 1 is at distance 2 in $G_{(2)}$ from a vertex colored with 0. We color the remaining uncolored vertices of $G_{(2)}$ with the colors $6, 7, \ldots, \chi'(G) + 5$ such that $L(2,1)$-coloring constraints are satisfied. We reduce c_1 until we arrive at an irreducible coloring f_1. Mandal and Panigrahi [13] have proved that f_1 is an inh-coloring of $G_{(2)}$ with span less than or equal to $\chi(G) + \chi'(G) + 3$ and greater than 4 such that color of each node is less than or equal to 4.

Let $\mathrm{span}(f_1) = \lambda_1$. Let $S_1 = V(G) \cup \left\{x^{1_1}_{uv} : uv \in E(G)\right\}$ and $S'_1 = V(G(rP_2)) - S_1$. We give an $L(2,1)$-coloring g_1 to $G(rP_2)$ as below: $g_1(u) = f_1(u)$ for all $u \in V(G)$, $g_1\left(x^{1_1}_{uv}\right) = f_1(x^1_{uv})$ for all edges uv of G, and we assign $g_1\left(x^{i_1}_{uv}\right) = \chi(G) + \chi'(G) + 3 + (i-2)\chi'(G) + f'_1(uv)$ for $2 \leq i \leq r$. Then all the vertices adjacent to a node have different colors. Since colors of nodes are less than 5 and colors of vertices in S'_1 are greater than 5, g_1 is an $L(2,1)$-coloring. We reduce g_1 until we arrive at an irreducible coloring g'_1. In this process color of vertices in S'_1 are only reduced. We prove that g'_1 is a no-hole coloring. Let l be a hole in g'_1. Then $l \geq \lambda_1 + 1 \geq 6$. From Lemma 23, a vertex colored with $l+1$ is adjacent to a vertex colored with $l-1$. A vertex colored with $l+1$ lies in S'_1 and it is adjacent to vertices in $V(G)$ only. Hence $l - 1 \leq 4$. This is a contradiction. Hence g'_1 is an inh-coloring with $\mathrm{span}(g'_1) \leq \chi(G) + r\chi'(G) + 3$ and $\lambda_{inh}(G(rP_2)) \leq \chi(G) + r\chi'(G) + 3$.

Next let G be a tree. We take a leaf s of G. Let t be the vertex adjacent to s. We give an $L(2,1)$-coloring g_2 to $G(rP_2)$ as below. We color the vertex t with color 0. We color the vertices $x_{ts}^{1_1}, x_{ts}^{2_1}, \ldots, x_{ts}^{r_1}$ and s with colors $2, 3, \ldots, r+1$ and $r+3$ respectively. We order the uncolored nodes of $G(rP_2)$ in increasing order of their distances from t and color them greedily. We note that colors 0 and 1 are only used by these nodes. We order the remaining vertices of of $G(rP_2)$ in increasing order of their distance from t and color them greedily. When such a vertex w is colored it is adjacent to two vertices colored with 0 and 1 and there are at most $r\Delta - 1$ colored vertices at distance two from it. Thus $g_2(w) \le r\Delta + 2$ and so $\text{span}(g_2) \le r\Delta + 2$. Now g_2 is an irreducible coloring and the only possible hole is $r+2$. But a neighbor of t is colored with $r+2$. Thus g_2 is an inh-coloring. Since $r\Delta + 2 = \chi(G) + r\chi'(G)$, we get $\lambda_{inh}(G(rP_2)) \le \chi(G) + r\chi'(G)$.

Finally, let G be not a bipartite graph. Now let f_3' be an edge coloring of G starting with color 1 and ending with $\chi'(G)$. Mandal and Panigrahi [13] have given an inh-coloring f_3 to $G_{(2)}$. We describe the coloring f_3 below.

Let c be a proper coloring of G which uses $\chi(G)$ colors starting from 1 such that color of no vertex can be reduced. There is at least one vertex z colored with 1 and adjacent to at least one vertex of every other color class. Let $z_1, z_2, \ldots, z_{\chi(G)-1}$ be vertices adjacent to z and colored with $2, 3, \ldots, \chi(G)$, respectively. From c we construct an $L(2,1)$-coloring c_3 of $G_{(2)}$ as below. $c_3(u) = c(u) - 1$ if $u \in V(G)$ and $c_3\left(x_{zz_1}^1\right) = 3$. If $\chi(G) = 3$ we do not assign color to $x_{zz_2}^1$ now. If $\chi(G) > 3$ then $c_3\left(x_{zz_2}^1\right) = 4$. If $\chi(G) = 4$ we do not assign color to $x_{zz_3}^1$ now. If $\chi(G) > 4$ then for $3 \le i \le \chi(G) - 2$, $c_3(x_{zz_i}^1) = i+2$ and $c_3\left(x_{zz_{\chi(G)-1}}^1\right) = 2$. For any uncolored vertex z' in $G_{(2)}$ we define $c_3(z') = \chi(G) + f'(e_{z'})$, where $e_{z'}$ is the edge of G that is subdivided by z. If c_3 is not irreducible we reduce c_3 until we arrive at an irreducible coloring f_3. Mandal and Panigrahi [13] have proved that f_3 is an inh-coloring of $G_{(2)}$ with span less than or equal to $\chi(G) + \chi'(G)$ and greater than $\chi(G) - 1$ such that color of each node is less than or equal to $\chi(G) - 1$.

Let $\text{span}(f_3) = \lambda_3$. Let $S_3 = V(G) \cup \left\{x_{uv}^{1_1} : uv \in E(G)\right\}$ and $S_3' = V(G(rP_2)) - S_3$. We give an $L(2,1)$-coloring g_3 to $G(rP_2)$ as below: $g_3(u) = f_3(u)$ for all $u \in V(G)$, $g_3\left(x_{uv}^{1_1}\right) = f_3\left(x_{uv}^1\right)$ for all edges uv of G, $g_3(x_{uv}^{i_1}) = \chi(G) + (i-1)\chi'(G) + f_3'(uv)$ for $2 \le i \le r$. Then g_3 is an $L(2,1)$-coloring because all the vertices adjacent to a node have different colors, all nodes have colors less than $\chi(G)$, and colors of vertices in S_3' are greater than $\chi(G)$. We reduce g_3 until we arrive at an irreducible coloring g_3'. In this process color of vertices in S_3' are only reduced. We prove that g_3' is a no-hole coloring. If l is a hole in g_3', then $l \ge \lambda_3 + 1 \ge \chi(G) + 1$. From Lemma 23, a vertex colored with $l+1$ is adjacent to a vertex colored with $l-1$. A vertex colored with $l+1$ lies in S_3' and is adjacent to vertices in $V(G)$ only. Hence $l - 1 \le \chi(G) - 1$. This is a contradiction. Hence g_3' is an inh-coloring with $\text{span}(g_3') \le \chi(G) + r\chi'(G)$ and $\lambda_{inh}(G(rP_2)) \le \chi(G) + r\chi'(G)$. ■

Theorem 25. *Let G be a graph with $\Delta(G) \ge 3$, $h(e) \ge 2$, and $h(e) = 2$ for at least one e but not for all. Then for $r \ge 5$, $G(rP_h)$ is inh-colorable and $\lambda_{inh}(G(rP_h)) \le 2r\Delta - r + 5$.*

Proof. Let uv be an edge in G with $h(uv) > 2$ and $deg(u) \ge deg(v)$. We give a coloring f to vertices in $G(rP_h)$ as below: $f(u) = f(v) = 0$, and uncolored nodes of $G(rP_h)$ are colored greedily using Algorithm 9 in any order. Let c be the maximum color used by f till now. When a node is colored it has no colored neighbor and there are most Δ colored vertices at distance two from it. Also there exist at least two nodes distance two apart from each other. Hence $1 \le c \le \Delta$. We color the vertex $x_{uv}^{1_1}$ greedily and get $f\left(x_{uv}^{1_1}\right) = 2$. If $c = 1$ then the maximum color used till now is 2 and no hole is created. If $c > 1$ then there is a vertex y_1 in $V(G)$ colored with $c - 1$. Since the coloring is obtained greedily, y_1 is at distance two from at least $c - 1$ vertices in $V(G)$, say, $z_1, z_2, \ldots, z_{c-1}$ colored with $0, 1, \ldots, c - 2$, respectively. We color the vertices $x_{y_1z_1}^{1_1}, x_{y_1z_2}^{1_1}, \ldots, x_{y_1z_{c-1}}^{1_1}$ greedily in the order they are listed. Then the colors $c-2, c-1, c, f\left(x_{y_1z_1}^{1_1}\right), f\left(x_{y_1z_2}^{1_1}\right), \ldots, f\left(x_{y_1z_{c-2}}^{1_1}\right)$, and $f\left(x_{y_1z_{c-1}}^{1_1}\right)$ are all distinct and one of them is $c + 1$. Therefore, $c + 1$ is not a hole. Hence the maximum color used till now is at least $c + 1$ and no hole is created. Let $V_1 = \left\{x_{u_1v_1}^{i_1} : h(u_1v_1) = 2, 1 \le i \le r\right\}$. We color the vertices in V_1 greedily in any order. No hole is created till now. We color the vertices $x_{uv}^{i_1}$, $2 \le i \le r$, greedily in any order. We choose a maximum degree vertex w of G, where $w \ne v$. This is possible since $deg(u) \ge deg(v)$. We color the uncolored vertices adjacent to w greedily in any order. No hole is created till now and the maximum color used so far is at least $r\Delta + 1$, since w is a maximum degree vertex. Let $V_2 = \left\{x_{uv}^{1_1}\right\} \cup \left\{x_{wv_2}^{1_1} : h(wv_2) > 2\right\} \cup \left\{x_{u_3v_3}^{1_1} : h(u_3v_3) > 2, x_{u_3v_3}^{1_{h(u_3v_3)-1}} \notin V_2\right\}$ and $V_3 = \left\{x_{u_3v_3}^{i_1} : x_{u_3v_3}^{1_1} \in V_2, 1 \le i \le r\right\}$. We color the uncolored vertices in V_3 greedily in any order. Let c_1 be the maximum color used by vertices in V_3. No hole is created till now. Let $V_4 = \left\{x_{u_3v_3}^{i_{h(u_3v_3)-1}} : f\left(x_{u_3v_3}^{i_1}\right) = c_1, x_{u_3v_3}^{i_1} \in V_3\right\}$ and $V_5 = \left\{x_{u_3v_3}^{i_{h(u_3v_3)-1}} : x_{u_3v_3}^{i_1} \in V_3\right\}$. We color the vertices in V_4 greedily in any order. When a vertex $x_{u_3v_3}^{i_{h(u_3v_3)-1}}$ in V_4 is colored it can use any color other than $f(v_3), f(v_3) \pm 1, f(u_3), f\left(x_{u_3v_3}^{i_1}\right), f\left(x_{u_3v_3}^{i_1}\right) \pm 1$ and the colors of at most $r\Delta - r$ colored neighbors of v_3. Hence $f\left(x_{u_3v_3}^{i_{h(u_3v_3)-1}}\right) \le r\Delta - r + 7 \le r\Delta + 2$. No hole is created since $c_1 \ge r\Delta + 1$. We color the remaining vertices in V_5 greedily in any order. No hole is created so far. Finally, we color all the remaining vertices greedily in any order. Let $x_{u_4v_4}^{i_j}$ be such a vertex. Number of vertices adjacent to $x_{u_4v_4}^{i_j}$ is 2 and number of vertices at distance two from $x^{i_j}_{u_4v_4}$ is also 2. Thus color of $x^{i_j}_{u_4v_4}$ is at most 8. Hence f is an inh-coloring of $G(rP_h)$ and $G(rP_h)$ is inh-colorable.

We prove that span$(f) \leq 2r\Delta - r + 5$. Color of a node is less than $\Delta + 1$. Let $x^{i_1}_{u_5v_5}$ be a vertex such that $h(u_5v_5) = 2$. Then u_5 and v_5 have at most $r\Delta$ neighbors each, of which r neighbors are common. Thus u_5 and v_5 have at most $2r\Delta - r$ neighbors in total. Hence number of vertices adjacent to $x^{i_1}_{u_5v_5}$ is 2 and number of vertices at distance two from it is at most $2r\Delta - r - 1$. Thus color of $x^{i_1}_{u_5v_5}$ is at most $2r\Delta - r + 5$. Let $x^{i_1}_{u_6v_6}$ be a vertex such that $h(u_6v_6) > 2$. Number of vertices adjacent to $x^{i_1}_{u_6v_6}$ is 2 and number of vertices at distance two from it is at most $r\Delta$. Thus color of $x^{i_1}_{y_6v_6}$ is at most $r\Delta + 6$. Let $x^{i_j}_{u_7v_7}$ be a vertex not adjacent to u_7 or v_7. Number of vertices adjacent to $x^{i_j}_{u_7v_7}$ is 2 and number of vertices at distance two from $x^{i_j}_{u_7v_7}$ is 2. Thus color of $x^{i_j}_{u_7v_7}$ is at most 8. Therefore span$(f) \leq 2r\Delta - r + 5$ and hence the result follows. ■

Finally, we consider graphs having maximum degree at least 4. In the theorem below we obtain an upper bound of $\lambda_{inh}(G(rP_3))$, $r \geq 2$, which agrees with the upper bound of $\lambda_{inh}\left(G_{(3)}\right)$ given by Mandal and Panigrahi [13], for $r = 1$. Moreover, we find the exact value of $\lambda_{inh}(G(rP_3))$ if $\Delta(G)$ is at least four times the minimum degree of G.

Theorem 26. *If G is a graph with $\Delta \geq 4$, then for $r \geq 2$, $G(rP_3)$ is inh-colorable and $\lambda_{inh}(G(rP_3)) \leq r\Delta + 2$. Further, if $\Delta \geq 4\delta$ then $\lambda_{inh}(G(rP_3)) = r\Delta + 1$, where δ is the minimum degree in G.*

Proof. Let G_1 be the subgraph of $G(rP_3)$ induced on the vertex set $V(G) \cup \{x^{1_j}_{uv} : j \in \{1, 2\}, uv \in E(G)\}$. Then G_1 is isomorphic to $G_{(3)}$. According to Proposition 3 we get a λ-perfect labeling f of G_1. We note that 1 is the only hole in f because nodes are colored with 0, vertices adjacent to a maximum degree vertex are colored with $2, 3, \ldots, \Delta + 1$ and every vertex is either a node or adjacent to a node. Now we use f to construct an $L(2,1)$-coloring g of $G(rP_3)$ with span $r\Delta + 1$ as below: $g(w) = f(w)$ if $w \in V(G_1)$, for all edges uv of G, $2 \leq i \leq r$ and $1 \leq j \leq 2$, $g\left(x^{i_j}_{uv}\right) = g\left(x^{1_j}_{uv}\right) + (i-1)\Delta$. We check that g is an $L(2,1)$-coloring with span $r\Delta + 1$ and having a hole at 1. A maximum degree vertex in $G(rP_3)$ is colored with 0 and its neighbors are colored with $2, 3, \ldots, r\Delta + 1$. Thus 1 is the only hole in g. Let u' be a minimum degree vertex of G. Let y be a vertex having the maximum color among the neighbors of u' in $G(rP_3)$. Then $y = x^{r_1}_{u'v'}$ for some neighbor v' of u' in G. Thus $(r-1)\Delta + 2 \leq g\left(x^{r_1}_{u'v'}\right) \leq r\Delta + 1$ and $(r-1)\Delta + 2 \leq g(x^{r_2}_{u'v'}) \leq r\Delta + 1$. We give another $L(2,1)$-coloring g' to $G(rP_3)$ where $g'(z) = g(z)$ if $z \neq u', x^{r_1}_{u'v'}$ and $g'\left(x^{r_1}_{u'v'}\right) = 1$. Since no vertex adjacent to $x^{r_1}_{u'v'}$ is colored with 0 or 2, $L(2,1)$-coloring constraints are satisfied. Next, u' is colored with the least available color such that $L(2,1)$-coloring constraints are satisfied by g'. Since no vertex adjacent to u' is colored with $r\Delta+1$, $g'(u') \leq r\Delta+2$

and thus span$(g') \leq r\Delta+2$. Again, the number of vertices adjacent to u' is $r\delta$ and the number of vertices at distance two from u' is also $r\delta$. Hence the number of colors not available for u' is at most $4r\delta$ and so $g'(u') \leq 4r\delta$. Therefore, if $\Delta \geq 4\delta$ then $g'(u') \leq r\Delta + 1$ and so span$(g') = r\Delta + 1$. Now we reduce g' until we arrive at an irreducible coloring g''. Since $g'\left(x^{r_1}_{u'v'}\right) = 1$, $g'(v') = 0$ and $d\left(x^{r_1}_{u'v'}, v'\right) = 2$, color of $x^{r_1}_{u'v'}$ cannot be reduced and so $g''\left(x^{r_1}_{u'v'}\right) = 1$. There is a maximum degree vertex in $G(rP_3)$ colored with 0 and its neighbors are colored with $2, 3, \ldots, r\Delta+1$ by g''. Thus g'' is an inh-coloring of $G(rP_3)$ and $\lambda_{inh}(G(rP_3)) \leq r\Delta+2$. Further, by Proposition 4 if $\Delta \geq 4\delta$ then $\lambda_{inh}(G(rP_3)) = r\Delta + 1$. ■

Lü and Sun [12] have found the exact value of $\lambda(G(rP_3))$, $r \geq 2$, when $\Delta(G)$ is even. In the corollary below we find the same when $\Delta(G)$ is odd.

Corollary 27. *If G is a graph with $\Delta(G) \geq 5$, $\Delta(G)$ odd, then for $r \geq 2$, $\lambda(G(rP_3)) = r\Delta + 1$.*

Proof. In the proof of Theorem 26 we have given an $L(2,1)$-coloring g to $G(rP_3)$ with span $r\Delta + 1$. Thus from Proposition 4 we get $\lambda(G(rP_3)) = r\Delta + 1$. ■

In Theorem 28 below we find the exact value of span and inh-span of $G(rP_h)$ (with some restrictions on h) which coincide with span and inh-span of $G_{(h)}$, for $r = 1$, given by Chang *et al.* [1] and Mandal and Panigrahi [13], respectively. In particular, for $\Delta(G)$ odd we get the exact value of $\lambda(G(rP_4))$ which was not found by Lü and Sun [12].

Theorem 28. *If G is a graph with $\Delta(G) \geq 4$, $r \geq 2$, and $h(e) \geq 3$ with strict inequality for at least one e, then $\lambda_{inh}(G(rP_h)) = \lambda(G(rP_h)) = r\Delta + 1$.*

Proof. Here we consider the same $L(2,1)$-coloring g of $G(rP_3)$ that appears in the proof of Theorem 26. Note that span of g is $r\Delta + 1$ and g assigns color 0 to all nodes. Let $u'v'$ be an edge in G such that $h(u'v') > 3$. Then we give an $L(2,1)$-coloring g' to $G(rP_h)$ as below. For any edge uv in G and for $1 \leq i \leq r$, $g'\left(x^{i_1}_{uv}\right) = g\left(x^{i_1}_{uv}\right)$, $g'\left(x^{i_{h(uv)-1}}_{uv}\right) = g\left(x^{i_2}_{uv}\right)$, and $g'\left(x^{2_2}_{u'v'}\right) = 1$ (this is possible since $g'\left(x^{2_1}_{u'v'}\right) > \Delta + 1$ and $g'\left(x^{2_{h(u'v')-1}}_{u'v'}\right) > \Delta + 1$). We color the remaining vertices greedily in any order applying Algorithm 9. When such a vertex w is colored it is adjacent to two vertices and there are two vertices at distance two from it. Hence $g'(w) \leq 8$. Since $r \geq 2$ and $\Delta \geq 4$, $g'(w) \leq r\Delta + 1$. Thus span$(g') = r\Delta + 1$. Now we reduce g' until we arrive at an irreducible coloring g''. We prove that g'' is a no-hole coloring. Since $g'\left(x^{2_2}_{u'v'}\right) = 1$, $g'(u') = 0$ and $d\left(x^{2_2}_{u'v'}, u'\right) = 1$, the color of $x^{2_2}_{u'v'}$ cannot be reduced and thus $g''\left(x^{2_2}_{u'v'}\right) = 1$. A maximum degree vertex is colored with 0 and its neighbors are colored with $2, 3, \ldots, r\Delta + 1$ by g''. Hence g'' is an inh-coloring with span $r\Delta + 1$.

Thus $\lambda(G(rP_h)) \leq \lambda_{inh}(G(rP_h)) \leq r\Delta + 1$. Now from Proposition 4 we get $\lambda_{inh}(G(rP_h)) = \lambda(G(rP_h)) = r\Delta + 1$. ■

5. Concluding Remarks

In this paper we show that for any graph G with $h(e) \geq 3$ and $r \geq 2$, $G(rP_h)$ is inh-colorable and for $\Delta(G) \geq 2$, $G(rP_2)$ is inh-colorable. We also prove that if G is a graph with $\Delta(G) \geq 2$, $h(e) \geq 2$ for all e in $E(G)$ and $h(e) = 2$ for at least one e but not for all, and $r \geq 2$, then $G(rP_h)$ is inh-colorable except possibly the following cases: $\Delta(G) = 2$, $r = 2$; and $\Delta(G) \geq 3$, $2 \leq r \leq 4$. We have found the exact value of $\lambda_{inh}(G(rP_h))$ in several cases and given upper bounds in the remaining. However, some of the upper bounds given in the paper may not be sharp. So the following problems remain open.

1. Is $G(2P_h)$ inh-colorable for any G with $\Delta = 2$, $h(e) \geq 2$ for all edges and equality for at least one but not for all?
2. Is $G(rP_h)$ inh-colorable for any graph G with $\Delta \geq 3$, $h(e) \geq 2$ for all edges and equality for at least one but not for all, and $2 \leq r \leq 4$?
3. Can the upper bound of $\lambda_{inh}(G(rP_h))$, when $\Delta(G) = 2$, $r \geq 2$ and $h(e) \geq 2$ with equality for at least one e but not for all (Theorems 12 and 16) be improved?
4. Can the upper bound of $\lambda_{inh}(G(rP_h))$, when $\Delta(G) \geq 3$, $r \geq 5$ and $h(e) \geq 2$ with equality for at least one e but not for all (Theorem 25) be improved?
5. Whether $\lambda_{inh}(G(rP_3)) = r\Delta + 1$, for every graph G with $\Delta \geq 3$ and $r \geq 2$ (Theorems 21 and 26)?
6. Is the upper bound 6 for $\lambda_{inh}(C_m(2P_3))$, $m \geq 4$ (Theorem 17) sharp?
7. Can the upper bound for $\lambda_{inh}(C_m(rP_2))$ given in Theorem 15 be improved?

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