

SOME RESULTS ON THE INDEPENDENCE POLYNOMIAL OF UNICYCLIC GRAPHS

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Abstract

Let G be a simple graph on n vertices. An independent set in a graph is a set of pairwise non-adjacent vertices. The independence polynomial of G is the polynomial $I(G, x) = \sum_{k=0}^n s(G, k)x^k$, where $s(G, k)$ is the number of independent sets of G with size k and $s(G, 0) = 1$. A unicyclic graph is a graph containing exactly one cycle. Let C_n be the cycle on n vertices. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs G on n vertices (except two of them), $I(G, t) > I(C_n, t)$ for sufficiently large t . Finally for every $n \geq 3$ we find all connected graphs H such that $I(H, x) = I(C_n, x)$.

Keywords: independence polynomial, independent set, unicyclic graphs.

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1. INTRODUCTION

Throughout this paper we will consider only simple graphs, the graphs with no loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph. The *order* of G denotes the number of vertices of G . Let e be an edge of G . By $e = uv$ we mean that e is an edge between vertices u and v . For every vertex $v \in V(G)$, the *closed neighborhood* of v denoted by $N[v]$ is defined as $\{u \in V(G) \mid uv \in E(G)\} \cup \{v\}$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of G_1 and G_2 denoted by $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The

graph rG denotes the disjoint union of r copies of G . For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v . A *pendant* vertex is a vertex of degree one. For a vertex $v \in V(G)$, $G \setminus v$ denotes the graph obtained from G by removing v . A *unicyclic graph* is a graph containing exactly one cycle. We denote the complete graph of order n , the complete bipartite graph with part sizes m, n , the cycle of order n , and the path of order n , by K_n , $K_{m,n}$, C_n , and P_n , respectively. Also $K_{1,n}$ is called a *star*.

A set $S \subseteq V(G)$ is an *independent set* if there is no edge between the vertices of S . If S is an independent set with $|S| = k$, then S is called a *k-independent set*. By $s(G, k)$ we mean the number of k -independent sets of G . The *independence number* of G , $\alpha(G)$, is the maximum cardinality of an independent set of G . The *independence polynomial* of G , $I(G, x)$, is defined as $I(G, x) = \sum_{k=0}^{\alpha(G)} s(G, k)x^k$, where $s(G, k)$ is the number of independent sets of G of size k and $s(G, 0) = 1$. This polynomial was introduced by Gutman and Harary in [10]. For example for every $n \geq 1$, $\alpha(K_n) = 1$ and $s(K_n, 1) = n$. Thus $I(K_n, x) = 1 + nx$. The independence polynomial has very nice properties, see [5, 6, 13] for more details. There are many polynomials associated with graphs. For example chromatic polynomial, clique polynomial, domination polynomial, edge cover polynomial and matching polynomial, see [1]–[16]. One of the most important problems related to graph polynomials is the following:

Problem. Which graphs are uniquely determined by their graph polynomials?

In many papers, researchers study the problem defined above for graph polynomials. For example in [3] the authors show that the complete graphs, the cycles and some complete bipartite graphs are determined by their edge cover polynomials. In [2] it is proved that the cycles are determined by their domination polynomials. In this paper we study the independence polynomial of unicyclic graphs. We show that among all connected unicyclic graphs G on n vertices except the cycle C_n and the graph D_n (see Figure 3), $I(G, t) > I(C_n, t)$ for sufficiently large t . We show that for every $n \geq 4$ there is only one connected graph H such that $H \not\cong C_n$ and $I(H, x) = I(C_n, x)$.

2. THE INDEPENDENCE POLYNOMIALS OF UNICYCLIC GRAPHS

In this section we study the independence polynomials of unicyclic graphs. We need the following basic properties of independence polynomials.

Theorem 1 [10, 11]. *Let G be a graph with connected components G_1, \dots, G_t . Then $I(G, x) = \prod_{i=1}^t I(G_i, x)$.*

Theorem 2 [10, 11]. *Let G be a graph and v be a vertex of G . Then*

$$I(G, x) = I(G \setminus v, x) + xI(G \setminus N[v], x).$$

Remark 3. We remark that by independence polynomials one can find the number of vertices and the number of edges of graphs. More precisely, if G is a graph with n vertices and m edges, then $n = s(G, 1)$ and $m = \binom{n}{2} - s(G, 2)$.

Lemma 4. *Let T be a tree of order n . Then there exists a positive real number r_n such that for all $x \geq r_n$ we have*

$$I(T, x) > \begin{cases} x^{\lceil \frac{n}{2} \rceil}, & \text{if } n \text{ is odd;} \\ 2x^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Since T is a tree, T is bipartite. Assume that X and Y are partite sets of $V(T)$. Hence $\alpha(T) \geq |X|, |Y|$. This shows that $\alpha(T) \geq \lceil \frac{n}{2} \rceil$. First assume that n is odd. Thus for all $x \geq 1$, $x^{\alpha(T)} \geq x^{\lceil \frac{n}{2} \rceil}$. This shows that for all $x \geq 1$, $I(T, x) > x^{\lceil \frac{n}{2} \rceil}$. Now assume that n is even. If $\alpha(T) = \frac{n}{2}$, then $|X| = |Y| = \frac{n}{2}$. Thus $s(T, \frac{n}{2}) \geq 2$. Hence for all $x \geq 1$, $s(T, \alpha(T))x^{\alpha(T)} \geq 2x^{\frac{n}{2}}$ and so $I(T, x) > 2x^{\frac{n}{2}}$. Otherwise suppose that $\alpha(T) > \frac{n}{2}$. Thus $I(T, x) - 2x^{\frac{n}{2}}$ is a polynomial with positive leading coefficient. Therefore for sufficiently large x , $I(T, x) - 2x^{\frac{n}{2}} > 0$. This completes the proof. ■

Let G be a graph of order n with vertex set $\{v_1, \dots, v_n\}$. Let H_1, \dots, H_n be some disjoint graphs. Assume that $u_1 \in V(H_1), \dots, u_n \in V(H_n)$. By $G(H_1, \dots, H_n; u_1, \dots, u_n)$ we mean the graph that is obtained by identifying the vertices u_i and v_i for $i = 1, \dots, n$. Note that the order of $G(H_1, \dots, H_n; u_1, \dots, u_n)$ is $|V(H_1)| + \dots + |V(H_n)|$, see Figure 1. In particular, suppose that H_1, \dots, H_n are some stars, say $H_1 = K_{1, m_1}, \dots, H_n = K_{1, m_n}$, where m_1, \dots, m_n are some non-negative integers (by $K_{1,0}$ we mean the single vertex K_1). In addition let u_i be the vertex of K_{1, m_i} with degree m_i . Then we use $G(m_1, \dots, m_n)$ instead of $G(K_{1, m_1}, \dots, K_{1, m_n}; u_1, \dots, u_n)$. Note that the order of $G(m_1, \dots, m_n)$ is $m_1 + \dots + m_n + n$ and $G(0, \dots, 0) \cong G$. See Figure 2.

Lemma 5. *Let $k \geq 3$ be an integer. Let $V(C_k) = \{v_1, \dots, v_k\}$ and $E(C_k) = \{v_1v_2, \dots, v_{k-1}v_k, v_kv_1\}$. Let $G = C_k(n_1, \dots, n_k)$ and $n = n_1 + \dots + n_k + k$, where n_1, \dots, n_k are some non-negative integers. If $G \not\cong C_n$ and $n \geq 5$, then for sufficiently large x we have $I(G, x) > I(C_n, x)$.*

Proof. First we note that if $n = 3$, then $G \cong C_3$. Also if $n = 4$, then $G \cong C_4$ or $G \cong C_3(1, 0, 0)$. Since $I(C_3(1, 0, 0), x) = I(C_4, x) = 1 + 4x + 2x^2$, $I(G, x) = I(C_4, x)$. We note that $C_k(n_1, \dots, n_k) \cong C_n$ if and only if $n = k$. Now assume that $n \geq 5$ and $G \not\cong C_n$ ($G \not\cong C_k$). We have one of the following cases.

(i) For some $i \in \{1, \dots, k\}$, $n_i \geq 2$. Without losing the generality assume that $n_1 \geq 2$. Note that $\alpha(G) \geq n_1 + \alpha(P_{k-1}(n_2, \dots, n_k))$, where $V(P_{k-1}) = \{v_2, \dots, v_k\}$ and $E(P_{k-1}) = \{v_2v_3, \dots, v_{k-1}v_k\}$. Since $P_{k-1}(n_2, \dots, n_k)$ is a tree

of order $n - n_1 - 1$ (by the proof of Lemma 4), $\alpha(P_{k-1}(n_2, \dots, n_k)) \geq \lceil \frac{n-n_1-1}{2} \rceil$. Hence

$$\alpha(G) \geq n_1 + \left\lceil \frac{n - n_1 - 1}{2} \right\rceil = \left\lceil \frac{n + n_1 - 1}{2} \right\rceil \geq \left\lceil \frac{n + 1}{2} \right\rceil > \lfloor \frac{n}{2} \rfloor = \alpha(C_n).$$

Thus $\alpha(G) > \alpha(C_n)$. Since the coefficients of independence polynomials are positive, for sufficiently large x we have $I(G, x) > I(C_n, x)$.

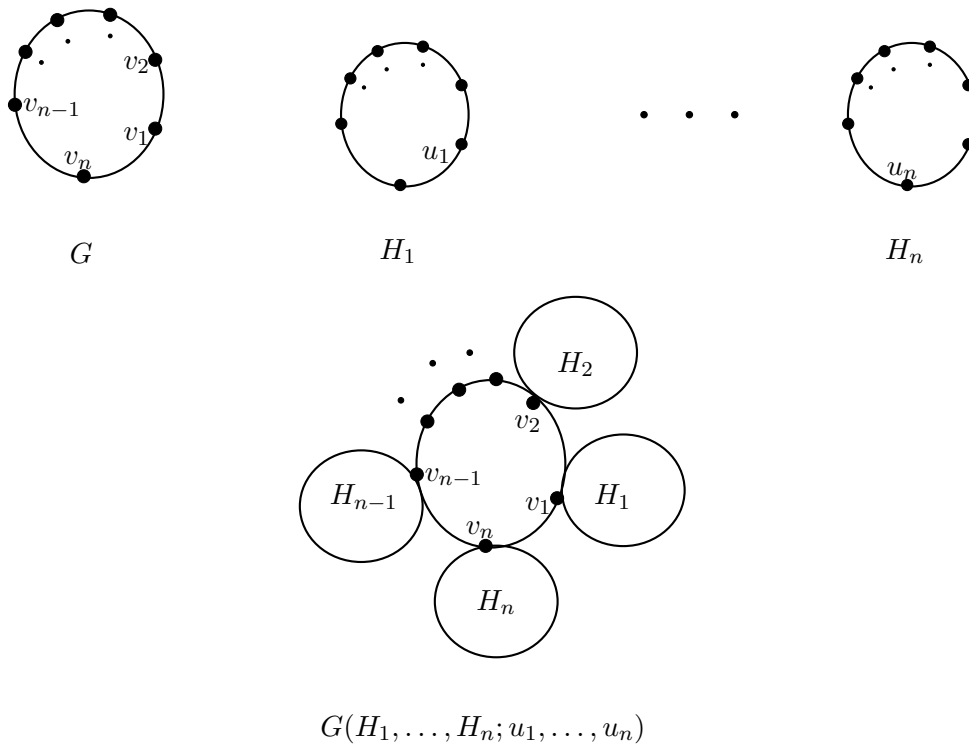
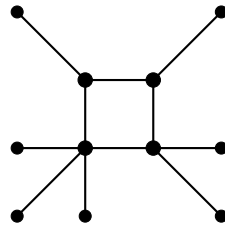


Figure 1. The graph $G(H_1, \dots, H_n; u_1, \dots, u_n)$.

(ii) For $i = 1, \dots, k$, $n_i \in \{0, 1\}$ and n is odd. Since $G \not\cong C_n$, for some i , $n_i = 1$. Without losing the generality let $n_1 = 1$. Since n is odd, similar to part (i), $\alpha(G) \geq 1 + \lceil \frac{n-2}{2} \rceil = \lceil \frac{n}{2} \rceil > \lfloor \frac{n}{2} \rfloor = \alpha(C_n)$. Thus the result follows.

(iii) For $i = 1, \dots, k$, $n_i \in \{0, 1\}$ and n is even. Since $G \not\cong C_n$, for some t , $n_t = 1$. First suppose that there is only one i such that $n_i = 1$. Without losing the generality assume that $n_1 = \dots = n_{k-1} = 0$ and $n_k = 1$. Hence $k = n - 1$. In other words, $G \cong C_{n-1}(0, \dots, 0, 1)$. Hence $\alpha(G) = \frac{n}{2}$. Let $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{v_1v_2, \dots, v_{n-2}v_{n-1}, v_{n-1}v_1, v_{n-1}v_n\}$. Since $n \geq 6$, $\{v_1, v_3, \dots, v_{n-3}, v_n\}$, $\{v_1, v_3, \dots, v_{n-5}, v_{n-2}, v_n\}$ and $\{v_2, v_4, \dots, v_{n-2}, v_n\}$ are three independent sets of G with cardinality $\frac{n}{2}$. Hence $s(G, \frac{n}{2}) \geq 3$. On the



$C_4(1, 1, 2, 3)$

Figure 2. The graph $C_4(1, 1, 2, 3) = G(H_1, H_2, H_3, H_4; u_1, u_2, u_3, u_4)$, where $G = C_4$ and u_1 is the vertex of $H_1 = K_{1,1}$ of degree one, u_2 is the vertex of $H_2 = K_{1,1}$ of degree one, u_3 is the vertex of $H_3 = K_{1,2}$ of degree two and u_4 is the vertex of $H_4 = K_{1,3}$ of degree three.

other hand, since n is even, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. By the fact that $\alpha(G) = \alpha(C_n) = \frac{n}{2}$ and $s(G, \frac{n}{2}) > s(C_n, \frac{n}{2})$, for sufficiently large x we obtain $I(G, x) > I(C_n, x)$. Now assume that there are some $i \neq j$ such that $n_i = 1$ and $n_j = 1$. This shows that G has at least two vertices of degree one (G has two pendant vertices). Let u and v be two pendant vertices of G . Applying Theorem 2 for vertex u we obtain $I(G, x) = I(G \setminus u, x) + xI(T_1, x)$, where T_1 is a tree of order $n - 2$. Using Theorem 2 for v and $G \setminus u$ we have

$$I(G, x) = I(G \setminus \{u, v\}, x) + xI(T_2, x) + xI(T_1, x),$$

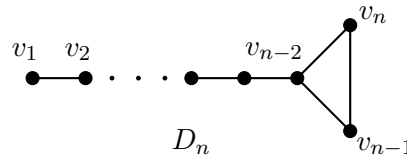
where T_2 is a tree of order $n - 3$. Hence for $x \geq 0$, $I(G, x) > xI(T_2, x) + xI(T_1, x)$. Using Lemma 4 for trees T_1 and T_2 we obtain that for sufficiently large x ,

$$I(G, x) > xx^{\lceil \frac{n-3}{2} \rceil} + 2xx^{\frac{n-2}{2}} = 3x^{\frac{n}{2}}.$$

On the other hand, $\alpha(C_n) = \frac{n}{2}$ and $s(C_n, \frac{n}{2}) = 2$. Hence for sufficiently large x , $3x^{\frac{n}{2}} > I(C_n, x)$. Thus for sufficiently large x , $I(G, x) > 3x^{\frac{n}{2}} > I(C_n, x)$. The proof is complete. ■

3. GRAPHS WHOSE INDEPENDENCE POLYNOMIALS COINCIDE WITH INDEPENDENCE POLYNOMIALS OF CYCLES

In this section we study the graphs G such that $I(G, x) = I(C_n, x)$, where $n \geq 3$. We show that there is only one connected graph $G \not\cong C_n$ satisfying $I(G, x) = I(C_n, x)$. Let $n \geq 4$ be an integer. By D_n we mean the graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\} \cup \{v_{n-2}v_n\}$, see Figure 3. In addition by D_3 we mean the cycle C_3 . The next result shows that the independence polynomials of C_n and D_n are the same.

Figure 3. The graph D_n .

Lemma 6. Let $n \geq 4$ be an integer. Then

- (i) $I(C_n, x) = I(C_{n-1}, x) + xI(C_{n-2}, x)$, where C_2 is the path P_2 .
- (ii) $I(C_n, x) = I(D_n, x)$.

Proof. It is easy to check the result for $n = 4$. Thus let $n \geq 5$. Using Theorem 2 for one of the vertices of C_n we obtain that

$$(1) \quad I(C_n, x) = I(P_{n-1}, x) + xI(P_{n-3}, x).$$

On the other hand, by Theorem 2 for one of the pendant vertices of P_t we have

$$(2) \quad I(P_t, x) = I(P_{t-1}, x) + xI(P_{t-2}, x), \text{ for } t \geq 2, \text{ where } I(P_0, x) = 1.$$

Using equations (1) and (2) one can see that

$$I(C_n, x) = I(P_{n-2}, x) + xI(P_{n-4}, x) + x(I(P_{n-3}, x) + xI(P_{n-5}, x)).$$

So by equation (1) the first part is proved. Now we prove the second part. Using Theorem 2 for the vertex v_n of D_n (see Figure 3) we obtain that $I(D_n, x) = I(P_{n-1}, x) + xI(P_{n-3}, x)$. Hence by equation (1), $I(D_n, x) = I(C_n, x)$. The proof is complete. ■

We recall that a unicyclic graph is a graph with exactly one cycle. The next result shows that among all connected unicyclic graphs the cycles have the smallest independence polynomials.

Theorem 7. Let G be a connected unicyclic graph of order n . Assume that $G \not\cong C_n$ and $G \not\cong D_n$. Then for sufficiently large x we have $I(G, x) > I(C_n, x)$.

Proof. Assume that H is a connected unicyclic graph of order n . Thus $n \geq 3$. If $n = 3$, then $H \cong C_3$. If $n = 4$, then $H \cong C_4$ or $H \cong D_4$. If $n = 5$, then $H \cong C_5$ or $H \cong D_5$ or $H \cong C_4(1, 0, 0, 0)$ or $H \cong C_3(2, 0, 0)$ or $H \cong C_3(1, 1, 0)$. So by the fact that G is unicyclic and $G \not\cong C_n$ and $G \not\cong D_n$ we obtain that $n \geq 5$. We use induction on n to prove the result. If $n = 5$, then $G \cong C_4(1, 0, 0, 0)$ or $G \cong C_3(2, 0, 0)$ or $G \cong C_3(1, 1, 0)$. One can see that $I(C_4(1, 0, 0, 0), x) = 1 + 5x + 5x^2 + x^3$, $I(C_3(2, 0, 0), x) = 1 + 5x + 5x^2 + 2x^3$ and

$I(C_3(1, 1, 0), x) = 1 + 5x + 5x^2 + x^3$. On the other hand $I(C_5, x) = 1 + 5x + 5x^2$. Thus the result holds for $n = 5$.

Now assume that $n \geq 6$. Suppose that the length of the unique cycle of G is k . Assume that v_1, \dots, v_k are the vertices of this cycle. Since G is unicyclic there are some trees T_1, \dots, T_k such that $G = C_k(T_1, \dots, T_k; v_1, \dots, v_k)$. If each tree T_1, \dots, T_k is a star, then by Lemma 5 the result follows. Now without losing the generality assume that T_1 is not a star. Let u_1 be a pendant vertex of T_1 which has the maximum distance from v_1 among all pendant vertices of T_1 . We consider the three following cases for $G \setminus u_1$.

(i) Assume that $G \setminus u_1$ is the cycle C_{n-1} . Hence $G = C_{n-1}(1, 0, \dots, 0)$ and $T_1 = P_2$, a contradiction (since T_1 is not a star). Thus this case does not happen.

(ii) Assume that $G \setminus u_1$ is the graph D_{n-1} . Hence $G \cong D_n$ or $G \cong H$, where H is obtained by identifying the pendant vertex of D_{n-2} with the non-pendant vertex of P_3 . Thus it suffices to check the result for H . Let z be a pendant vertex of H . Thus $H \setminus z \cong D_{n-1}$ and $H \setminus N[z] \cong D_{n-3} + K_1$. Hence by Theorems 1 and 2, $I(H, x) = I(H \setminus z, x) + xI(H \setminus N[z], x) = I(D_{n-1}, x) + x(1 + x)I(D_{n-3}, x)$. So by the second part of Lemma 6 we obtain

$$(3) \quad I(H, x) = I(C_{n-1}, x) + x(1 + x)I(C_{n-3}, x).$$

On the other hand, by the first part of Lemma 6 for $n \geq 7$, $I(C_{n-3}, x) = I(C_{n-4}, x) + xI(C_{n-5}, x)$. This shows that for $x > 0$, $I(C_{n-3}, x) > I(C_{n-4}, x)$ (this inequality also holds for $n = 6$, where C_2 is the path P_2). Hence for $x > 0$, $xI(C_{n-3}, x) > xI(C_{n-4}, x)$. Thus for every $x > 0$ we have

$$(1 + x)I(C_{n-3}, x) = I(C_{n-3}, x) + xI(C_{n-3}, x) > I(C_{n-3}, x) + xI(C_{n-4}, x).$$

Therefore by the first part of Lemma 6 we obtain that

$$(4) \quad \text{for } x > 0, (1 + x)I(C_{n-3}, x) > I(C_{n-2}, x).$$

The equations (3) and (4) show that for $x > 0$, $I(H, x) > I(C_{n-1}, x) + xI(C_{n-2}, x)$. Hence by the first part of Lemma 6 for every $x > 0$, $I(H, x) > I(C_n, x)$.

(iii) Suppose that $G \setminus u_1 \not\cong C_{n-1}$ and $G \setminus u_1 \not\cong D_{n-1}$. Since $G \setminus u_1$ is a connected unicyclic graph of order $n - 1$, by the induction hypothesis for sufficiently large x , $I(G \setminus u_1, x) > I(C_{n-1}, x)$. As we defined above, u_1 is a pendant vertex of T_1 which has the maximum distance from v_1 among all pendant vertices of T_1 . Assume that w_1 is the neighbor of u_1 . Since T_1 is not a star, $d(u_1, v_1) \geq 2$. We note that $w_1 \neq v_1$. Let $\text{deg}(w_1) = t + 1$. Thus $t \geq 1$. By the definition of u_1 , exactly t neighbors of w_1 have degree one. Hence $G \setminus N[u_1]$ is the union of a unicyclic graph of order $n - t - 1$, say L , with exactly $t - 1$ isolated vertices. In other words, $G \setminus N[u_1] = L + (t - 1)K_1$. Hence by Theorem 1, $I(G \setminus N[u_1], x) = I(L, x)(1 + x)^{t-1}$. On the other hand, by the induction hypothesis for sufficiently large x , $I(L, x) \geq$

$I(C_{n-t-1}, x)$ (if $L \neq C_{n-t-1}$ and $L \neq D_{n-t-1}$, $I(L, x) > I(C_{n-t-1}, x)$ for large x). Since $n \geq t + 4$, similar to the previous part one can see that for $x > 0$, $(1+x)I(C_{n-t-1}, x) > I(C_{n-t}, x)$. Hence for $x > 0$, $(1+x)^2 I(C_{n-t-1}, x) > (1+x)I(C_{n-t}, x)$. Similarly for $x > 0$, $(1+x)I(C_{n-t}, x) > I(C_{n-t+1})$. By applying this method $t - 1$ times, we obtain that if $t \geq 2$, then

$$(5) \quad \text{for } x > 0, (1+x)^{t-1} I(C_{n-t-1}, x) > I(C_{n-2}, x).$$

Hence for $t \geq 1$ we conclude that

$$(6) \quad \text{for } x > 0, (1+x)^{t-1} I(C_{n-t-1}, x) \geq I(C_{n-2}, x).$$

The equation (6) shows that for sufficiently large x ,

$$I(G \setminus N[u_1], x) = I(L, x)(1+x)^{t-1} \geq (1+x)^{t-1} I(C_{n-t-1}, x) \geq I(C_{n-2}, x).$$

Since for large x , $I(G \setminus u_1, x) > I(C_{n-1}, x)$, by Theorem 2, the equation (5) and the first part of Lemma 6, we find that for large x ,

$$I(G, x) = I(G \setminus u_1, x) + xI(G \setminus N[u_1], x) > I(C_{n-1}, x) + xI(C_{n-2}, x) = I(C_n, x).$$

The proof is complete. ■

Now we are in a position to prove the main result of this section.

Theorem 8. *Let $n \geq 3$ be an integer. Assume that G is a connected graph such that $I(G, x) = I(C_n, x)$. Then $G \cong C_n$ or $G \cong D_n$.*

Proof. Since $I(G, x) = I(C_n, x)$ and C_n has n vertices and n edges, by Remark 3 we find that G has exactly n vertices and n edges. Since the number of vertices and the number of edges of G are the same and G is connected, G is unicyclic. If $G \not\cong C_n$ or $G \not\cong D_n$, then by Theorem 7 for large x we have $I(G, x) > I(C_n, x)$, a contradiction. This completes the proof. ■

Let $n \geq 3$ be an integer. One might ask whether there is a disconnected graph G satisfying $I(G, x) = I(C_n, x)$. We check this question for $n \leq 9$.

Remark 9. Let $3 \leq n \leq 9$ and G be a graph of order n . Assume that $I(G, x) = I(C_n, x)$. We find that if $n \in \{3, 4, 5, 7, 8\}$, then $G \cong C_n$ or $G \cong D_n$ (see Theorem 8). We obtain that $I(G, x) = I(C_6, x)$ if and only if $G \in \{C_6, D_6, K_2 + K_4 \setminus e\}$, where e is an edge of K_4 . We find that $I(G, x) = I(C_9, x)$ if and only if $G \in \{C_9, D_9, H_1, H_2, H_3\}$, where H_1, H_2 and H_3 have been shown in Figure 4. In fact $I(C_6, x) = 1 + 6x + 9x^2 + 2x^3 = (1 + 4x + x^2)(1 + 2x) = I(K_4 \setminus e, x)I(K_2, x)$ and $I(C_9, x) = 1 + 9x + 27x^2 + 30x^3 + 9x^4 = (1 + 6x + 9x^2 + 3x^3)(1 + 3x)$. These examples show that the structure of all non-connected graphs G with $I(G, x) = I(C_m, x)$ is not clear, where $m \geq 10$.

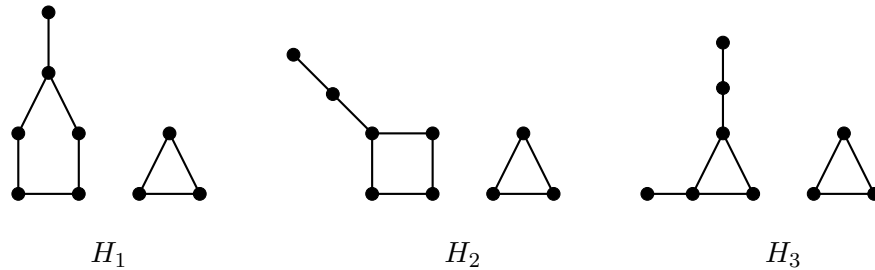


Figure 4. All non-connected graphs G such that $I(G, x) = I(C_9, x)$.

We finish the paper by the following problem.

Problem. Let $n \geq 10$ be an integer. Find all non-connected graphs G such that $I(G, x) = I(C_n, x)$.

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