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BOUNDING THE LOCATING-TOTAL DOMINATION NUMBER OF A TREE IN TERMS OF ITS ANNIHILATION NUMBER

Wenjie Ning

College of Science
China University of Petroleum (East China)
Qingdao 266580, China

e-mail: ningwenjie-0501@163.com

Mei Lu

Department of Mathematical Sciences
Tsinghua University, Beijing 100084, China

e-mail: mlu@math.tsinghua.edu.cn

and

Kun Wang

School of Mathematical Sciences
Anhui University, Hefei 230601, China

e-mail: wangkun26@163.com

Abstract

Suppose $G = (V, E)$ is a graph with no isolated vertex. A subset S of V is called a locating-total dominating set of G if every vertex in V is adjacent to a vertex in S, and for every pair of distinct vertices u and v in $V - S$, we have $N(u) \cap S \neq N(v) \cap S$. The locating-total domination number of G, denoted by $\gamma_t^L(G)$, is the minimum cardinality of a locating-total dominating set of G. The annihilation number of G, denoted by $a(G)$, is the largest integer k such that the sum of the first k terms of the nondecreasing degree sequence of G is at most the number of edges in G. In this paper, we show that for any tree of order $n \geq 2$, $\gamma_t^L(T) \leq a(T) + 1$ and we characterize the trees achieving this bound.

Keywords: total domination, locating-total domination, annihilation number, tree.

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1. Introduction

Given a graph $G = (V(G), E(G))$, we usually use n for the number of vertices and m for the number of edges. For a vertex v in G, the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ (or $N(v)$) is called the *neighborhood* of v. The *degree* of v in G, denoted by $d_G(v)$ or $d(v)$, is equal to $|N(v)|$. A vertex of degree one is a *leaf* and a vertex adjacent to a leaf is a *support vertex*. We will use $l(G)$ to denote the number of leaves of G. For arbitrary two vertices u and v in G, the *distance* between u and v, denoted by $d(u, v)$, is the number of edges in a shortest path joining u and v. If there is no such path, then we define $d(u, v) = \infty$. The *diameter* of G is the maximum distance among all pairs of vertices of G, denoted by $diam(G)$. For a subset $S \subseteq V(G)$, we use $G - S$ to denote the graph obtained from G by deleting the vertices in S and all edges incident with vertices in S. If $S = \{v\}$, we simply write $G - v$ rather than $G - \{v\}$. We define

$$\Sigma(S, G) = \sum_{v \in S} d_G(v).$$

Suppose $G = (V, E)$ is a graph with no isolated vertex. A subset S of V is called a *total dominating set* (TDS) of G if every vertex in V is adjacent to a vertex in S. A total dominating set S is called a *locating-total dominating set* (LTDS) if for every pair of distinct vertices u and v in $V - S$, we have $N(u) \cap S \neq N(v) \cap S$. The *locating-total domination number* of G, denoted by $\gamma_t^L(G)$, is the minimum cardinality of a locating-total dominating set of G. An LTDS of cardinality $\gamma_t^L(G)$ is called a $\gamma_t^L(G)$-*set*. The concept of a locating-total dominating set in a graph was first introduced in [9], since this time many results have been obtained on this parameter (see, for instance, [1–4, 10]).

The annihilation number was first introduced in Pepper's dissertation [13]. Originally it was defined in terms of a reduction process on the degree sequence akin to the Havel-Hakimi process (see, for example, [8, 14]). In [13], Pepper showed the following equivalent way to define the annihilation number. Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be the nondecreasing degree sequence of a graph G having n vertices and m edges. Then the *annihilation number* of G, denoted by $a(G)$, is the largest integer k such that $\sum_{i=1}^{k} d_i \leq m$ or, equivalently, the largest integer k such that

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i.$$

The relation between annihilation number and some graph parameters have been studied by several authors (see for example [5–7, 11–14]).

For a graph $G = (V(G), E(G))$ with m edges, an *a-set* of G is a subset S of $V(G)$ such that $|S| = a(G)$ and $\Sigma(S, G) \leq m$, where $a(G)$ is the annihilation number of G. An a_{min}-*set* of G is an a-set S of G with $\Sigma(S, G)$ minimum. Thus, if S is an a_{min}-set of G, then S is a set of vertices (not necessarily unique) corresponding to the first $a(G)$ vertices in the nondecreasing degree sequence of G.

In order to prove our theorem, we introduce a variation of the annihilation number of a graph defined in [6]. The *upper annihilation number* of a graph G, denoted by $a^*(G)$, is the largest integer k such that the first k terms of the nondecreasing degree sequence of G is at most $|E(G)| + 1$. That is, if $d_1 \leq d_2 \leq \cdots \leq d_n$ is the nondecreasing degree sequence of a graph G with m edges, then $a^*(G)$ is the largest integer k such that $\sum_{i=1}^{k} d_i \leq m + 1$. Similarly, we define an a^*_{min}-*set* of G to be a set S of vertices in G such that $|S| = a^*(G)$ and S corresponds to the first $a^*(G)$ vertices in the nondecreasing degree sequence of G. By the definitions of the annihilation number and the upper annihilation number, we have $a(G) \leq a^*(G) \leq a(G) + 1$.

A path of order n is P_n. A star of order n is denoted by S_n. A tree is called a *double star* $S(p, q)$, if it is obtained from S_{p+2} and S_{q+1} by identifying a leaf of S_{p+2} with the center of S_{q+1}, where $p, q \geq 1$.

In this paper, we establish an upper bound on the locating-total domination number of a tree in terms of its annihilation number. We show that for any tree of order $n \geq 2$, $\gamma_t^L(T) \leq a(T) + 1$ and we characterize the trees achieving this bound.

2. THE MAIN RESULT

In order to characterize the trees satisfying $\gamma_t^L(T) = a(T) + 1$, we first introduce a family Γ of labeled trees defined in [4].

For each tree $T \in \Gamma$, every vertex v in T has a label $sta(v) \in \{A, B, C\}$, called its *status*. Let Γ be the family of labeled trees $T = T_k$ that can be obtained as follows. Let T_0 be a path P_6 in which the two leaves have status C, the two support vertices have status A and the remaining two vertices have status B. If $k \geq 1$, then T_k can be obtained from T_{k-1} by one of the following operations.

- **Operation $\boldsymbol{\tau_1}$.** For any $y \in V(T_{k-1})$, if $sta(y) = C$ and $d_{T_{k-1}}(y) = 1$, then add a path $xwvz$ and the edge xy. Let $sta(x) = sta(w) = B$, $sta(v) = A$ and $sta(z) = C$.

- **Operation $\boldsymbol{\tau_2}$.** For any $y \in V(T_{k-1})$, if $sta(y) = B$, then add a path xwv and the edge xy. Let $sta(x) = B$, $sta(w) = A$ and $sta(v) = C$.

Chen and Sohn [4] established the following upper bound of $\gamma_t^L(T)$ of a tree in terms of its order and number of leaves. Moreover, they gave a characterization of the trees achieving this bound.

Theorem 1 [4]**.** *If T is a tree of order $n \geq 3$ with l leaves, then* $\gamma_t^L(T) \leq \frac{n+l}{2}$.

Theorem 2 [4]**.** *If T is a tree of order $n \geq 3$ with l leaves, then* $\gamma_t^L(T) = \frac{n+l}{2}$ *if and only if* $T \in \Gamma$.

For each tree $T \in \Gamma$, we have the following lemma.

Lemma 3. *Let $T \in \Gamma$. Then*

(1) $\gamma_t^L(T) = a(T) + 1 = a^*(T)$.

(2) *For any vertex $v \in V(T)$ with $d(v) = 2$, there are an a^*_{min}-set S containing v and an a^*_{min}-set S' not containing v.*

(3) *For every a^*_{min}-set A, it contains no vertices of degree larger than two.*

Proof. Suppose $T \in \Gamma$ is obtained from T_0 by applying k_1 τ_1 operations and k_2 τ_2 operations. Then $n(T) = 6 + 4k_1 + 3k_2$, $l(T) = 2 + k_2$ and by Theorem 2,

$$\gamma_t^L(T) = \frac{n(T) + l(T)}{2} = \frac{(6 + 4k_1 + 3k_2) + (2 + k_2)}{2} = 4 + 2k_1 + 2k_2.$$

Note that $V(T)$ consists of $2 + k_2$ leaves with status C, $4 + 4k_1 + k_2$ vertices of degree two and k_2 vertices with status B and degree larger than two. By simple calculation, we have $a(T) = 3 + 2k_1 + 2k_2$ and $a^*(T) = 4 + 2k_1 + 2k_2$. Thus, (1) holds.

By the definition of an a^*_{min}-set, for any a^*_{min}-set S, S consists of $2+k_2$ leaves and $2 + 2k_1 + k_2$ vertices of degree two. Note that T has exactly $4 + 4k_1 + k_2$ vertices of degree two and k_2 vertices of degree larger than two. Thus, (2) and (3) hold. ■

Now we present our main result.

Theorem 4. *For a tree T of order $n \geq 2$, the following hold.*

(1) $\gamma_t^L(T) \leq a^*(T)$.

(2) $\gamma_t^L(T) \leq a(T) + 1$.

(3) $\gamma_t^L(T) = a(T) + 1$ *if and only if* $T = P_2$ *or* $T \in \Gamma$.

Proof. We proceed by induction on the order n. If $n = 2$, then $T = P_2$ and $\gamma_t^L(T) = 2 = a^*(T) = a(T) + 1$. If $n = 3$, then $T = P_3 \notin \{P_2\} \cup \Gamma$ and $\gamma_t^L(T) = 2 = a^*(T) = a(T)$. This establishes the base cases. Next we assume that every tree T' of order $3 \leq n' < n$ satisfies properties (1)–(3) in the statement of the theorem. Let T be a tree of order n.

If $diam(T) = 2$, then T is a star. Obviously, $T \notin \{P_2\} \cup \Gamma$ and $\gamma_t^L(T) = n - 1 = a^*(T) = a(T)$. If $diam(T) = 3$, then T is a double star, i.e., $T \cong S_{p,q}$.

Note that $T \notin \{P_2\} \cup \Gamma$, $\gamma_t^L(T) = n-2 = a(T)$, $a^*(T) = n-1$ if $\min\{p,q\} = 1$ and $a^*(T) = n-2$ if $\min\{p,q\} \geq 2$. Hence we may assume $diam(T) \geq 4$.

Let $P = x_0 x_1 \cdots x_d$ be a path of length $d = diam(T)$ in T. We root T at x_d.

Claim 1. *We may assume that* $d(x_1) = 2$.

Proof. Suppose $d(x_1) \geq 3$. Then $T \notin \Gamma$. Let $Q = N(x_1) \setminus \{x_2\}$. Then Q is the set of all leaves adjacent to x_1. Let $T' = T - Q \cup \{x_1\}$ and S be an a^*_{min}-set of T'. Then $|E(T)| = |E(T')| + |Q| + 1$ and $\Sigma(S,T') \leq |E(T')| + 1$. Letting $S_1 = S \cup Q$, we have

$$\begin{aligned}\Sigma(S_1,T) &= \Sigma(S \cup Q, T) = \Sigma(S,T) + |Q| \\ &\leq \Sigma(S,T') + 1 + |Q| \\ &\leq |E(T')| + 2 + |Q| = |E(T)| + 1.\end{aligned}$$

Then $a^*(T) \geq a^*(T') + |Q|$. Note that every LTDS of T' can extend to an LTDS of T by combining it with $(Q \setminus \{x_0\}) \cup \{x_1\}$. Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + |Q|$. By the inductive hypothesis, we have

$$\begin{aligned}\gamma_t^L(T) &\leq \gamma_t^L(T') + |Q| \leq a^*(T') + |Q| \leq a^*(T) \\ &\leq a(T) + 1.\end{aligned}$$

Thus (1) and (2) hold. Next we will show that $\gamma_t^L(T) \leq a(T)$.

Suppose $\gamma_t^L(T) = a(T) + 1$. Then equalities hold throughout the above inequalities, that is, $\gamma_t^L(T) = \gamma_t^L(T') + |Q|$, $\gamma_t^L(T') = a^*(T')$ and $a^*(T) = a^*(T') + |Q| = a(T) + 1$. Let A be an a_{min}-set of T' and $A_1 = A \cup Q$. Then

$$\begin{aligned}\Sigma(A_1,T) &= \Sigma(A \cup Q, T) = \Sigma(A,T) + |Q| \\ &\leq \Sigma(A,T') + 1 + |Q| \\ &\leq |E(T')| + 1 + |Q| = |E(T)|.\end{aligned}$$

Hence $a(T) \geq a(T') + |Q|$. If $\gamma_t^L(T') \leq a(T')$, then $\gamma_t^L(T) = \gamma_t^L(T') + |Q| \leq a(T') + |Q| \leq a(T)$, a contradiction to the assumption of $\gamma_t^L(T) = a(T) + 1$. Thus, $\gamma_t^L(T') = a(T') + 1$. Since $d \geq 4$, we have $n(T') \geq 3$. By the inductive hypothesis, we have $T' \in \Gamma$. Thus, $\gamma_t^L(T') = (n(T') + l(T'))/2$ by Theorem 2.

If $d_{T'}(x_2) \geq 2$, then there is an a^*_{min}-set B not containing x_2 by Lemma 3 (2) and (3). Let $B_1 = B \cup Q$. Then

$$\begin{aligned}\Sigma(B_1,T) &= \Sigma(B \cup Q, T) = \Sigma(B,T) + |Q| \\ &= \Sigma(B,T') + |Q| \\ &\leq |E(T')| + 1 + |Q| = |E(T)|,\end{aligned}$$

and so $a(T) \geq a^*(T') + |Q|$, a contradiction to $a^*(T') + |Q| = a(T) + 1$. Thus, $d_{T'}(x_2) = 1$. Now, we have

$$\begin{aligned}\gamma_t^L(T) &= \gamma_t^L(T') + |Q| = (n(T') + l(T'))/2 + |Q| \\ &= ((n(T) - |Q| - 1) + (l(T) - |Q| + 1))/2 + |Q| \\ &= (n(T) + l(T))/2.\end{aligned}$$

By Theorem 2, $T \in \Gamma$, a contradiction. □

By Claim 1, $d(x_1) = 2$. Let $Y = \{y_1, \ldots, y_l\}$ be the children of x_2, where $y_1 = x_1$. By Claim 1, we may assume $1 \leq d(y_i) \leq 2$ for all $y_i \in Y \setminus \{y_1\}$.

Claim 2. *We may assume that $d_T(y) = 1$ for any $y \in Y \setminus \{y_1\}$.*

Proof. Suppose there is a vertex, say $y_2 \in Y \setminus \{y_1\}$, such that $d_T(y_2) = 2$. Then $T \notin \Gamma$. Let z_2 be the leaf adjacent to y_2. Let $T' = T - \{x_0, x_1\}$ and S be an a^*_{min}-set of T'. Then $|E(T)| = |E(T')| + 2$ and $\Sigma(S, T') \leq |E(T')| + 1$. Since $d \geq 4$, $d_T(x_2) \geq 3$ and $d_{T'}(x_2) \geq 2$.

If $x_2 \in S$, by letting $S_2 = (S \cup \{x_0, x_1\}) \setminus \{x_2\}$, we have

$$\begin{aligned}\Sigma(S_2, T) &= \Sigma(S \setminus \{x_2\}, T) + 3 = \Sigma(S \setminus \{x_2\}, T') + 3 \\ &= \Sigma(S, T') - d_{T'}(x_2) + 3 \leq \Sigma(S, T') - 2 + 3 \\ &\leq |E(T')| + 1 + 1 = |E(T)|.\end{aligned}$$

If $x_2 \notin S$, by letting $S_2 = S \cup \{x_0\}$, we have

$$\begin{aligned}\Sigma(S_2, T) &= \Sigma(S, T) + 1 = \Sigma(S, T') + 1 \\ &\leq |E(T')| + 1 + 1 = |E(T)|.\end{aligned}$$

In both cases, we have $\Sigma(S_2, T) \leq |E(T)|$ which implies $a(T) \geq a^*(T') + 1$.

Let D be a $\gamma_t^L(T')$-set of T' that contains a minimum number of leaves. Then $\{x_2, y_2\} \subseteq D$, and so $D \cup \{x_1\}$ is an LTDS of T. Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 1$. By the inductive hypothesis, we have

$$\gamma_t^L(T) \leq \gamma_t^L(T') + 1 \leq a^*(T') + 1 \leq a(T) \leq a^*(T)$$

and we are done. □

Claim 3. *We may assume that $Y = \{x_1\}$, i.e., $l = 1$.*

Proof. Suppose $l \geq 2$. Then $T \notin \Gamma$ and every vertex in $Y \setminus \{y_1\}$ is a leaf in T by Claim 2.

Let $T' = T - \{x_0\} \cup (Y \setminus \{y_1\})$ and S be an a^*_{min}-set of T'. Then $|E(T)| = |E(T')| + l$ and $\Sigma(S, T') \leq |E(T')| + 1$. Since $d_{T'}(x_1) = 1$ and $d_{T'}(x_2) = 2$, we can choose S so that $\{x_1, x_2\} \subseteq S$. Let $S_3 = (S \setminus \{x_2\}) \cup \{x_0\} \cup (Y \setminus \{y_1\})$. Then

$$\begin{aligned}\Sigma(S_3, T) &= \Sigma(S \setminus \{x_2\}, T) + l = \Sigma(S \setminus \{x_2\}, T') + 1 + l \\ &= \Sigma(S, T') - d_{T'}(x_2) + 1 + l = \Sigma(S, T') + l - 1 \\ &\leq |E(T')| + l = |E(T)|,\end{aligned}$$

and so $a(T) \geq a^*(T') + l - 1$. Let D be a $\gamma_t^L(T')$-set of T' that contains a minimum number of leaves. Then $x_2 \in D$. Thus, $D \cup (Y \setminus \{y_l\})$ is an LTDS of T and $\gamma_t^L(T) \leq \gamma_t^L(T') + l - 1$. By the inductive hypothesis, we have

$$\gamma_t^L(T) \leq \gamma_t^L(T') + l - 1 \leq a^*(T') + l - 1 \leq a(T) \leq a^*(T)$$

and we are done. □

By Claim 3, we have $Y = \{x_1\}$. Since $d \geq 4$, $d_T(x_3) \geq 2$. We will finish the proof by considering the following two cases.

Case 1. $d_T(x_3) \geq 3$. Let $T' = T - \{x_0, x_1, x_2\}$ and S be an a^*_{min}-set of T'. Then $|E(T)| = |E(T')| + 3$, $d_{T'}(x_3) \geq 2$ and $\Sigma(S, T') \leq |E(T')| + 1$.

If $x_3 \notin S$, by letting $S_4 = S \cup \{x_0, x_1\}$, we have

$$\begin{aligned}\Sigma(S_4, T) &= \Sigma(S, T) + 3 = \Sigma(S, T') + 3 \\ &\leq |E(T')| + 1 + 3 = |E(T)| + 1.\end{aligned}$$

If $x_3 \in S$, by letting $S_4 = (S \setminus \{x_3\}) \cup \{x_0, x_1, x_2\}$, we have

$$\begin{aligned}\Sigma(S_4, T) &= \Sigma(S \setminus \{x_3\}, T') + 5 = \Sigma(S, T') - d_{T'}(x_3) + 5 \\ &\leq \Sigma(S, T') - 2 + 5 \leq |E(T')| + 1 + 3 \\ &= |E(T)| + 1.\end{aligned}$$

In both cases, we have $\Sigma(S_4, T) \leq |E(T)| + 1$ which implies $a^*(T) \geq a^*(T') + 2$. Note that every LTDS of T' can extend to an LTDS of T by combining it with $\{x_1, x_2\}$. Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 2$. By the inductive hypothesis, we have

$$\gamma_t^L(T) \leq \gamma_t^L(T') + 2 \leq a^*(T') + 2 \leq a^*(T) \leq a(T) + 1.$$

It remains to show that T satisfies property (3). By Lemma 3, if $T \in \Gamma$, then $\gamma_t^L(T) = a(T) + 1$, as desired. Suppose now $\gamma_t^L(T) = a(T) + 1$. Then equalities hold throughout the above inequalities, that is, $\gamma_t^L(T) = \gamma_t^L(T') + 2$, $\gamma_t^L(T') =$

$a^*(T')$ and $a^*(T) = a^*(T') + 2 = a(T) + 1$. Since $d_T(x_3) \geq 3$, x_3 is not a leaf of T'. Thus, by Theorem 2, we have

$$\begin{aligned}\gamma_t^L(T) &= \gamma_t^L(T') + 2 = (n(T') + l(T'))/2 + 2 \\ &= (n(T) - 3 + l(T) - 1)/2 + 2 = (n(T) + l(T))/2,\end{aligned}$$

and then $T \in \Gamma$.

Case 2. $d_T(x_3) = 2$. Let $T' = T - \{x_0, x_1, x_2, x_3\}$. Then $|E(T)| = |E(T')| + 4$. If $n(T') = 1$, then $T = P_5 \notin \{P_2\} \cup \Gamma$ and $\gamma_t^L(T) = 3 = a(T) = a^*(T)$. Thus, we may assume that $n(T') \geq 2$. Let S be an a^*_{min}-set of T'. Then $\Sigma(S, T') \leq |E(T')| + 1$.

If $x_4 \notin S$, by letting $S_5 = S \cup \{x_0, x_1\}$, we have $\Sigma(S_5, T) = \Sigma(S, T') + 3 \leq |E(T')| + 4 = |E(T)|$, and so $a(T) \geq a^*(T') + 2$. Note that every LTDS of T' can extend to an LTDS of T by combining it with $\{x_1, x_2\}$. Thus, $\gamma_t^L(T) \leq \gamma_t^L(T') + 2$. By the inductive hypothesis, we have

$$\gamma_t^L(T) \leq \gamma_t^L(T') + 2 \leq a^*(T') + 2 \leq a(T) \leq a^*(T).$$

Suppose now $x_4 \in S$. Let $S_6 = (S \setminus \{x_4\}) \cup \{x_0, x_1, x_2\}$. Then we have

$$\begin{aligned}\Sigma(S_6, T) &= \Sigma(S \setminus \{x_4\}, T') + 5 = \Sigma(S, T') - d_{T'}(x_4) + 5 \\ &\leq \Sigma(S, T') + 4 \leq |E(T')| + 5 \\ &= |E(T)| + 1\end{aligned}$$

which implies that $a^*(T) \geq a^*(T') + 2$. By the inductive hypothesis, we have

$$\gamma_t^L(T) \leq \gamma_t^L(T') + 2 \leq a^*(T') + 2 \leq a^*(T) \leq a(T) + 1.$$

It remains to show that T satisfies property (3). By Lemma 3, if $T \in \Gamma$, then $\gamma_t^L(T) = a(T) + 1$, as desired. Suppose now $\gamma_t^L(T) = a(T) + 1$.

Obviously, we have $\gamma_t^L(T) = \gamma_t^L(T') + 2$, $\gamma_t^L(T') = a^*(T')$ and $a^*(T) = a^*(T') + 2 = a(T) + 1$. If $d_{T'}(x_4) \geq 2$, then we have

$$\Sigma(S_6, T) = \Sigma(S, T') - d_{T'}(x_4) + 5 \leq \Sigma(S, T') + 3 \leq |E(T)|,$$

implying that $a(T) \geq a^*(T') + 2$, a contradiction to $a^*(T) = a^*(T') + 2 = a(T) + 1$. Thus, $d_{T'}(x_4) = 1$. If $T' = P_2$, then $T = P_6 \in \Gamma$. If $n(T') \geq 3$, then

$$\begin{aligned}\gamma_t^L(T) &= \gamma_t^L(T') + 2 = (n(T') + l(T'))/2 + 2 \\ &= (n(T) - 4 + l(T))/2 + 2 = (n(T) + l(T))/2,\end{aligned}$$

and then $T \in \Gamma$ by Theorem 2. ∎

3. COROLLARIES

Since $\gamma_t(T) \leq \gamma_t^L(T)$ for any tree T of order $n \geq 2$ and $\gamma_t(T_0) = \gamma_t^L(T_0)$ for any tree $T_0 \in \Gamma$ (see [4]), by Theorems 2 and 4, we easily obtain the following corollaries which are stated as main theorems in [5].

Corollary 5 [5]**.** *If T is a nontrivial tree, then $\gamma_t(T) \leq a(T)+1$, and this bound is sharp.*

Corollary 6 [5]**.** *Let T be a nontrivial tree of order n with n_1 vertices of degree 1. Then, $\gamma_t(T) = a(T)+1$ if and only if $\gamma_t(T) = (n+n_1)/2$.*

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