

AN $O(mn^2)$ ALGORITHM FOR COMPUTING THE STRONG GEODETIC NUMBER IN OUTERPLANAR GRAPHS

MAURO MEZZINI

*Department of Education
Roma Tre University*

e-mail: mauro.mezzini@uniroma3.it

Abstract

Let $G = (V(G), E(G))$ be a graph and S be a subset of vertices of G . Let us denote by $\gamma[u, v]$ a geodesic between u and v . Let $\Gamma(S) = \{\gamma[v_i, v_j] \mid v_i, v_j \in S\}$ be a set of exactly $|S|(|S| - 1)/2$ geodesics, one for each pair of distinct vertices in S . Let $V(\Gamma(S)) = \bigcup_{\gamma[x, y] \in \Gamma(S)} V(\gamma[x, y])$ be the set of all vertices contained in all the geodesics in $\Gamma(S)$. If $V(\Gamma(S)) = V(G)$ for some $\Gamma(S)$, then we say that S is a *strong geodetic set* of G . The cardinality of a minimum strong geodetic set of a graph is the *strong geodetic number* of G . It is known that it is NP-hard to determine the strong geodetic number of a general graph. In this paper we show that the strong geodetic number of an outerplanar graph can be computed in polynomial time.

Keywords: outerplanar graph, strong geodetic set, strong geodetic number, geodetic set, geodetic number, geodesic convexity.

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1. INTRODUCTION

Given two vertices u and v of a graph, a *geodesic* is a shortest path between u and v . The *interval* of a pair of vertices u and v of G , denoted by $I_G[u, v]$, is the set of all vertices that lie on some geodesic between u and v [14, 16, 19]. The interval of a set of vertices S , $I_G[S]$, is the union of the intervals between pairs of vertices of S , taken over all pairs of vertices in S .

A set S is a *geodetic set* of G if $I_G[S] = V(G)$ [4, 17, 20, 22]. The *geodetic number* of a graph is the cardinality of the minimum geodetic set S , and given a graph G , the problem of deciding if there exists a geodetic set of cardinality less than an integer $k > 1$ is the *geodetic number problem* (GNP). In [3] it has been proved that the GNP is NP-complete for general graphs.

In [7] the GNP is solved for Ptolemaic graphs. In [5] it has been proved that the GNP is NP-complete for chordal or chordal bipartite graphs and is polynomially solvable for cographs and split graphs. In [6] is proved that the GNP is NP-complete even for cobipartite graphs; furthermore, a block decomposition approach to solve the GNP is given and it is used to prove that the GNP is polynomially solvable in cactus graphs. Bounds on the geodetic number are given in [5] for triangle-free graphs and for unit interval graphs.

In order to model a problem on social networks, a variant to the GNP has been introduced in [2] (see also [9, 11–13] and [8] where the state of the art on the strong geodetic number is summarized). Given a set S of supervisors of a social network, we suppose that these supervisors can communicate each other by using a single, fixed shortest path through the network. The maximum number of possible geodesics of communication among all the supervisors is therefore not more than $|S|(|S| - 1)/2$. Denote this set of geodesics as $\Gamma(S)$ and let $V(\Gamma(S))$ be the set of all the vertices contained in all selected fixed geodesics between each distinct pair of supervisors. If $V(\Gamma(S)) = V(G)$ for some $\Gamma(S)$, then we say that S is a *strong geodetic set*. The *strong geodetic number problem* on a graph G is the problem of finding the minimum number of supervisors S such that S is a strong geodetic set of G .

This problem is stronger than the GNP and has been shown to be NP-complete in general graphs but solved for Apollonian graphs [2]. An edge version of the problem was also introduced in [15].

The problem of computing the geodetic number has been solved for the class of maximal outerplanar graphs [1]. Furthermore in [18] the solution to the GNP has been extended to general outerplanar graphs.

In this paper, based on the results of [18], we show the existence of a polynomial time algorithm for computing a minimum strong geodetic set of general outerplanar graphs.

The paper is organized as follows. In Section 2 we give some definitions and preliminaries. In Section 3 we give an algorithm for finding a minimum strong geodetic set in a outerplanar graph, prove its correctness and determine its complexity.

2. DEFINITIONS AND PRELIMINARIES

In what follows G will be a finite, connected, undirected and simple graph. Sets $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G , respectively.

We will use standard definition for edges, paths and cycles. Given two vertices a, b of a path P we will denote with $P[a, b]$ the *subpath* of P between a and b .

Let u and v be two vertices of G . The *distance*, $d_G(u, v)$, between u and v in G is the length of a geodesic between u and v in G .

A subset X of $V(G)$ is *geodetic* in G if $I_G[X] = V(G)$ and is a *minimum geodetic set* (MGS) of G if it is geodetic and of minimum cardinality. The cardinality of an MGS in a graph G is denoted as $gn(G)$ and is called the *geodetic number* of G .

Let us denote by $\gamma[u, v]$ a geodesic between u and v . Given a subset S of vertices of G , let $\binom{S}{2} = \{\{u, v\} \mid u \in S \text{ and } v \in S \text{ and } u \neq v\}$ be the set of all two-elements subsets of S . By $\Gamma(S) = \{\gamma[v_i, v_j] \mid \{v_i, v_j\} \in \binom{S}{2}\}$ we denote a set of exactly $|S|(|S| - 1)/2$ geodesics, one for each pair of distinct vertices in S . Let $V(\Gamma(S)) = \bigcup_{\gamma \in \Gamma(S)} V(\gamma)$ be the set of vertices of G contained in all the geodesics in $\Gamma(S)$. If $V(\Gamma(S)) = V(G)$ for some $\Gamma(S)$, then we say that S is a *strong geodetic set* of G . A strong geodetic set of a graph with minimum cardinality is called a *minimum strong geodetic set* (MSGS), and its cardinality is the *strong geodetic number* of G , denoted as $sgn(G)$.

Since a strong geodetic set is a geodetic set we have the following.

Fact 1. *In a graph G we have that $gn(G) \leq sgn(G)$.*

Therefore we have the following conclusion.

Corollary 1. *If S is a minimum geodetic set and there is a set $\Gamma(S)$ such that $V(\Gamma(S)) = V(G)$, then S is a minimum strong geodetic set.*

A *chord* of a cycle C (or path) is an edge uv of G such that $\{u, v\} \subseteq V(C)$ and $uv \notin E(C)$. A cycle (path) is *chordless* or *induced* if no edge of the graph is a chord of the cycle (path).

A vertex v of G is a *cutpoint* of G if v is a separator of G . A *biconnected component* of a graph G is a maximal subgraph of G having no cutpoints.

An *edge subdivision* is an operation that substitutes in a graph G an edge uv with the two edges uw and wv , where $w \notin V(G)$. Two graphs are *homeomorphic* if both can be obtained from the same graph by a sequence of subdivisions of edges [10].

A graph is *planar* if it can be embedded in the plane in such a way that its edges intersect only at their endpoints. Such a drawing subdivides the plane into regions called *faces* and the unbounded region is called the *outer face*. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the outer face [10].

Lemma 2 [10]. *A graph is outerplanar if and only if has no subgraph homeomorphic to K_4 or homeomorphic to $K_{2,3}$, except the graph obtained by removing one edge from K_4 .*

Lemma 3 [23]. *A biconnected outerplanar graph has the unique Hamiltonian cycle.*

In the following, we always assume that a biconnected outerplanar graph G is not a single edge and is not a chordless cycle because in these cases the problem of computing an MSGS of G is trivial.

If G is a biconnected outerplanar graph, we denote by C_H the Hamiltonian cycle of G .

Let G be a biconnected outerplanar graph and C_H its Hamiltonian cycle. A *leaf* of G is a chordless cycle C such that $E(C)$ contains only one chord of C_H . If C is a leaf and x_1x_2 is the only chord of C_H in $E(C)$, then we call x_1x_2 the *chord* of C ; furthermore we denote by $ex(C)$ the set of those vertices in $V(C)$ such that $v \in ex(C)$ if either $d_C(v, x_1) = \max\{d_C(x_1, u) \mid u \in V(C)\}$ or $d_C(v, x_2) = \max\{d_C(x_2, u) \mid u \in V(C)\}$. Note that if C is odd, then $|ex(C)| = 1$ and if C is even, then $|ex(C)| = 2$. For example, with reference to the graph of Figure 1, we have that $C_0 = (0, 1, 12, 13)$ is a leaf with chord $\{1, 12\}$ and $ex(C_0) = \{0, 13\}$.

Let $C_H = (v_0, \dots, v_{|V(G)|-1})$ be the Hamiltonian cycle of G . Given an edge uv of C_H we say that $u \prec v$ if we encounter u before v in a clockwise traverse of C_H starting from v_0 . We define $Arc(u, v)$ to be the set of internal vertices of the path in C_H between u and v that starts at u and traverses around C_H clockwise, while $Arc[u, v]$ denotes $Arc(u, v) \cup \{u, v\}$. Note that if $uv \in E(C_H)$, then either $Arc(u, v) = \emptyset$ or $Arc(v, u) = \emptyset$. For example in the graph of Figure 1, $Arc(2, 8) = \{3, 4, 5, 6, 7\}$.

For convenience we borrow the notation of [1, 18] and denote by $L(u, v) = Arc(u, v) \setminus I_G[u, v]$. For example, in the graph of Figure 1, we have that $L(0, 8) = \{2, 3, 4, 5, 6\}$.

The following two lemmas (see [18]) are needed for the proof of the correctness of the algorithm.

Lemma 4 [18]. *Let G be a biconnected outerplanar graph and let $u, v \in V(G)$. If $z \in I_G[u, v] \cap Arc(u, v)$, then for every $x \in Arc[u, z]$ and for every $y \in Arc[z, v]$ we have that $z \in I_G[x, y]$.*

By the above lemma we have the following.

Corollary 5. *Let G be a biconnected outerplanar graph and let $u, v \in V(G)$. If $L(u, v) = \emptyset$, then for every $z \in Arc[u, v]$, for every $x \in Arc[u, z]$ and every $y \in Arc[z, v]$, we have that $L(x, y) = \emptyset$.*

Lemma 6 [18]. *Let G be a biconnected outerplanar graph. Let X be a geodetic set of G . Let C be a leaf of G and xy be the chord of C . Then $|X \cap (V(C) \setminus \{x, y\})| \geq 1$.*

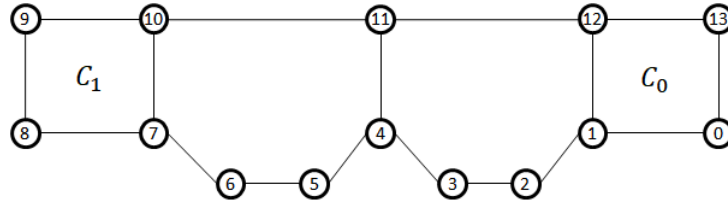


Figure 1. A biconnected outerplanar graph having two leaves C_0 and C_1 .

3. THE ALGORITHM

Based on the work in [18] we will show in this section an algorithm for finding an MSGS in polynomial time. We limit our discussion to biconnected outerplanar graphs, being the generalizations to non-biconnected outerplanar graphs very similar and easy to make by using results from [18].

First we report here a simplified version of the algorithm presented in [18] for computing in polynomial time an MGS of a biconnected outerplanar graph (see Algorithm 1). Then we show how to compute an MSGS of G based on the output of the algorithm.

Algorithm 1 Computing an MGS of a biconnected outerplanar graph

Input: A biconnected outerplanar graph G and a vertex v_0 of an MGS of G

Output: An MGS of G

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1: procedure SIMPLEMGS( $G$ )
2:   Let  $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$  such that  $v_i \prec v_{i+1}$ ,  $i = 0, \dots, n - 2$ 
3:    $X \leftarrow \{v_0\}$ 
4:    $p = v_0$ 
5:   for  $i = 1, \dots, n - 1$  do
6:     if  $L(p, v_i) \neq \emptyset$  then
7:        $X \leftarrow X \cup \{v_{i-1}\}$ 
8:        $p \leftarrow v_{i-1}$ 
9:       if  $I_G[X] = V(G)$  then exit and return  $X$ 
10:    end if
11:    if  $I_G[X \cup \{v_i\}] = V(G)$  then
12:       $X \leftarrow X \cup \{v_i\}$ 
13:      exit and return  $X$ 
14:    end if
15:  end for
16: end procedure

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Theorem 7 [18]. *The procedure SIMPLEMGS (Algorithm 1), given a biconnected outerplanar graph G and a vertex v_0 belonging to an MGS of G , produces as output an MGS of G .*

Proof. The first 14 lines in the procedure MGS reported in [18] consist in an algorithm for finding a vertex v_0 which belongs to an MGS. After, beginning from line 15, there is a greedy algorithm which, starting with v_0 , builds an MGS X . The procedure SIMPLEMGS consists in exactly the lines of procedure MGS reported in [18] from lines 15 to 29. So, the proof of correctness of procedure SIMPLEMGS is exactly the same as reported in Theorem 22 of [18], if we assume that v_0 is a vertex in an MGS of G . ■

Furthermore in [18] it has been proved the following.

Lemma 8 ([18], Lemma 12). *Let G be a biconnected outerplanar graph and C be a leaf of G . There exists an MGS X of G such that if C is odd, then X contains $ex(C)$, and if C is even, then $X \cap V(C) \subseteq ex(C)$.*

Based on Lemma 8, we can build an “oracle” which, at the cost of running the Algorithm 1 at most two times, finds a vertex v that is in an MGS of G which is a requirement of input of Algorithm 1. The oracle searches this vertex in an arbitrary leaf of G . If C is an odd leaf, then, by Lemma 8, there exists an MGS X such that $ex(C) \subset X$. In this case the oracle will output the vertex in $ex(C)$. If C is even, then $ex(C) = \{u_1, u_2\}$. By Lemma 8, there exists an MGS X such that either u_1 or u_2 is in X . We run the algorithm starting at u_i obtaining as a output a set X_i , $i = 1, 2$. By Theorem 7, at least one of X_1 or X_2 should be an MGS of G and we can recognize it by taking the set with minimum cardinality.

For example consider the graph G of Figure 1. If we start the algorithm from vertex $u_0 = 0$, we obtain that $L(0, 7) \neq \emptyset$. Thus we insert the vertex $u_1 = 6$ in X . Then we find that $L(6, 8) = \emptyset$ and that $I_G[X \cup \{8\}] = V(G)$, and then the algorithm stops here outputting the set $X = \{0, 6, 8\}$. As another example we may start the algorithm at vertex $u_0 = 13$. In this case we find that $L(13, 4) \neq \emptyset$ and thus we insert in X the vertex $u_1 = 3$. Then, since $L(3, 8) = \emptyset$ and since $I_G[X \cup \{8\}] = V(G)$, the algorithm stops here outputting the set $X = \{13, 3, 8\}$. In both case $|X| = 3$, thus both the vertices 0 and 13 are contained in an MGS of G .

Theorem 9. *Let X be the output of the procedure SIMPLEMGS (Algorithm 1) with input a biconnected outerplanar graph G and a vertex v_0 in an MGS of G . If $|X| = 2$, then we have that $sgn(G) = 3$, otherwise $sgn(G) = |X|$.*

Proof. Let $X = \{x_0, \dots, x_{k-1}\}$ be the output of the algorithm where the vertices are ordered by the time of insertion into X by the algorithm (from now on all indexes are taken modulo k if not otherwise specified). We prove that the path

P_i induced by $Arc[x_{i-1}, x_i]$ is a geodesic of G , for $i = 1, \dots, k - 1$. Suppose, by contradiction, that there is an i such that P_i is not a geodesic. When we insert x_i in X , we have that $L(x_{i-1}, x_i) = \emptyset$. This means that $Arc[x_{i-1}, x_i] \subseteq I_G[x_{i-1}, x_i]$. Since P_i is not a geodesic, every geodesic between x_{i-1} and x_i does not contain all the vertices in $Arc[x_{i-1}, x_i]$.

Let γ be a geodesic between x_{i-1} and x_i such that $V(\gamma) \cap Arc[x_{i-1}, x_i]$ has maximum cardinality. By hypothesis, there exist two vertices x_a and x_b in $V(\gamma) \cap Arc[x_{i-1}, x_i]$ such that $Arc(x_a, x_b) \neq \emptyset$, $Arc[x_a, x_b] \subseteq Arc[x_{i-1}, x_i]$ and the path P_{ab} induced by $Arc[x_a, x_b]$ is not a geodesic. Furthermore, by the choice of γ , we find vertices x_a and x_b such that $V(\gamma) \cap Arc[x_a, x_b] = \{x_a, x_b\}$. Now we show that $\gamma_1 = \gamma[x_a, x_b]$ is the only geodesic in G between x_a and x_b . First of all note that $V(\gamma_1) \neq \{x_a, x_b\}$, for otherwise $x_a x_b$ would be a chord of $Arc[x_{i-1}, x_i]$. But then the subgraph of G induced by $Arc[x_{i-1}, x_i]$ would contain at least one leaf C of G with chord $x_c x_d$ such that $Arc(x_c, x_d) \subseteq Arc(x_{i-1}, x_i)$. By Lemma 6, at least one vertex of $Arc(x_c, x_d)$ should be in X contradicting the fact that X is a geodetic set.

Now suppose, by contradiction, that there exists a geodesic γ_2 between x_a and x_b , distinct from γ_1 . By the choice of γ , we have that $V(\gamma_2) \cap Arc[x_a, x_b] = \{x_a, x_b\}$ and, by the discussion above, we have that $V(\gamma_2) \neq \{x_a, x_b\}$. Furthermore, since γ_1 is a geodesic distinct from γ_2 , there exists a vertex $w \in \gamma_2$ such that $w \notin \gamma_1$. But then the subgraph of G induced by $V(P_{ab}) \cup V(\gamma_1) \cup V(\gamma_2)$ would be homeomorphic to $K_{2,3}$ contradicting the hypothesis that G is outerplanar (Lemma 2). Since γ_1 is the only geodesic in G between x_a and x_b , we have that $I_G[x_a, x_b] = V(\gamma_1)$ and, therefore, that $L(x_a, x_b) \neq \emptyset$. But since $L(x_{i-1}, x_i) = \emptyset$, this contradicts Corollary 5. So we proved that $Arc[x_{i-1}, x_i]$ is a geodesic for $i = 0, \dots, k - 1$. Now if $|X| = 2$, we have that $\Gamma(X)$ can contain only one geodesic and we could not cover with a single geodesic all the vertices in G . Thus, if v is any vertex in $Arc(x_0, x_1)$, it is easy to see that $X \cup \{v\}$ is a strong geodetic set of G . If $|X| > 2$, since P_i is a geodesic, it is easy to see that $\bigcup_{i=0}^{k-1} V(P_i) = V(G)$, thus X is an SGS of G . The theorem then follows by Fact 1 and Corollary 1. ■

In the example above, we have already seen that $X = \{0, 6, 8\}$ is an MGS of the graph in Figure 1. It is immediate to check that the paths induced by $Arc[0, 6]$, $Arc[6, 8]$ and $Arc[8, 0]$ are geodesics. So X is also an MSGS of G .

As for the complexity of the algorithm we have the following.

Theorem 10. *Let G be a biconnected outerplanar graph, $n = |V(G)|$ and $m = |G|$. The Algorithm 1 has $O(mn^2)$ time complexity.*

Proof. Finding the Hamiltonian cycle of a biconnected outerplanar graph requires linear time [21]. By [22], it requires at most $O(nm)$ for finding the interval

of a set X . The theorem follows by the fact that the algorithm requires to find $L(p, v_i)$ at most $O(n)$ times, and requires to find the interval of a set X at most $O(n)$ times. ■

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