Discussiones Mathematicae
Graph Theory 42 (2022) 485–500
https://doi.org/10.7151/dmgt.2289

ARBITRARILY PARTITIONABLE $\{2K_2, C_4\}$-FREE GRAPHS

Fengxia Liu, Baoyindureng Wu

and

Jixiang Meng

College of Mathematics and System Sciences
Xinjiang University
Urumqi, Xinjiang, 830046, P.R.China

e-mail: xjulfx@163.com
baoywu@163.com
mjx@xju.edu.cn

Abstract

A graph $G = (V, E)$ of order n is said to be arbitrarily partitionable if for each sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of positive integers with $\lambda_1 + \cdots + \lambda_p = n$, there exists a partition $(V_1, V_2, \ldots, V_p)$ of the vertex set V such that V_i induces a connected subgraph of order λ_i in G for each $i \in \{1, 2, \ldots, p\}$. In this paper, we show that a threshold graph is arbitrarily partitionable if and only if it admits a perfect matching or a near perfect matching. We also give a necessary and sufficient condition for a $\{2K_2, C_4\}$-free graph being arbitrarily partitionable, as an extension for a result of Broersma, Kratsch and Woeginger [*Fully decomposable split graphs*, European J. Combin. 34 (2013) 567–575] on split graphs.

Keywords: arbitrarily partitionable graphs, arbitrarily vertex decomposable, threshold graphs, $\{2K_2, C_4\}$-free graphs.

2010 Mathematics Subject Classification: 05C70.

1. Introduction

Let $G = (V, E)$ be a simple, undirected graph of order n. A set M of edges of G is called a *matching* of G if any pair of two elements of G have no common end vertex. Furthermore, M is called a *perfect matching* (respectively, a *near perfect matching* if every vertex of G (all but one vertex) is incident with an edge of M.

The matching number of G, denoted by $\alpha'(G)$, is the cardinality of a maximum matching of G. A graph G is called *traceable* if G has a Hamilton path. A subset $S \subseteq V$ is an *independent set* of G if no pair of vertices in S are adjacent in G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G.

A sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ of positive integers is called a *partition* of n if $\lambda_1 + \cdots + \lambda_p = n$. The graph G is called λ-*decomposable* (or λ is *realizable*) if there exists a partition $(V_1, V_2, \dots, V_p)$ of the vertex set V such that $|V_i| = \lambda_i$ and $G[V_i]$ is connected for each $i \in \{1, \dots, k\}$. In this case, we call such a partition of G a λ-*decomposition* of G, and $G[V_i]$ (or V_i) a λ_i-*component*. Furthermore, G is called *arbitrarily partitionable* (AP, for short) if G is λ-*decomposable* for every partition λ of n. Note that if G is traceable, then it is AP; if G is AP, then it admits a perfect matching or near perfect matching, and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$.

The notion of AP graphs was first introduced by Barth, Baudon and Puech [1], and independently, by Horňák and Woźniak [20]. It is also called arbitrarily vertex decomposable [20] or fully decomposable [12] or decomposable [1]. Similarly, a graph G is called k-*partitionable* if G is λ-decomposable for each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n with length k.

A classical theorem of Győri [17] and Lovász [26] is stated as follows.

Theorem 1 (Győri [17] and Lovász [26]). *Every k-connected graph is k-partitionable.*

Structure of AP graphs and minimal AP graphs are investigated in [7, 9]. The problem of deciding whether a given admissible sequence is realizable in a given graph G is NP-complete [2]. Moreover, it is true even if we restrict the problem to the class of trees of degree at most 3 [2]. More results for the algorithmic aspects of AP graphs can be found in [2, 12, 10]. However, it still remains to be an open problem for deciding whether a tree is AP is NP-complete. Barth, Baudon and Puech [1] showed that this problem is polynomial in number of vertices for the class of tripodes. Horňák and Woźniak [20] showed that the maximum degree of a AP tree is at most 6. Later in [2], this bound was dropen to 4. Cichacz, Görlich, Marczyk and Przybyło [15] gave a complete characterization of AP caterpillars with four leaves. They also exhibited two infinite families of AP trees with maximum degree three or four. Ravaux [29] focused on trees with a large diameter. There are also some results on AP star-like trees [21], unicyclic AP graphs [24] and the shape of AP trees [3].

Marczyk [27] showed that if G is connected, $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, and $d_G(x) + d_G(y) \geq n - 2$ for all nonadjacent vertices $x, y \in V(G)$, then G is AP. Later, he [28] further showed that if G is a connected graph on n vertices with independence number at most $\lceil \frac{n}{2} \rceil$ and such that the degree sum of any pair of nonadjacent vertices is at least $n - 3$, then G is AP or is isomorphic to one of two exceptional graphs. Horňák, Marczyk, Schiermeyer and Woźniak [18] showed that if for a connected graph G of order n, the degree sum of any pair of nonadjacent vertices is at least $n-5$, then G is AP. Dense arbitrarily partitionable graphs have been studied in [23].

Various variations of AP graphs, such as on-line arbitrarily partitionable graphs [19, 22, 25], recursively arbitrarily partitionable graphs [4, 8] and AP+k graphs [5, 6] are also investigated.

A graph G is called a *split graph* if its vertex set can be partitioned into two sets I and C, where I is an independent set of G, and C is a clique of G, that is, a set of mutually adjacent vertices in G. For an integer $n \geq 2$, a partition λ of n is called 2-3-*primitive* if it has one of the following forms.

- $\lambda = (1, 3, 3, \ldots, 3)$ consists of threes and a single one;
- $\lambda = (2, \ldots, 2, 3, 3, \ldots, 3)$ only consists of twos and threes.

Broersma, Kratsch and Woeginger [12] characterized AP split graphs as follows.

Theorem 2 (Broersma, Kratsch, Woeginger [12])**.** *A split graph on n vertices is AP if and only if it is λ-decomposable for each* 2-3-*primitive partition λ of n.*

For $n \geq 2$, the *canonical* 2-3-*primitive* partition λ of n is defined as follows.

- If $n = 2k$ is even, then the canonical 2-primitive partition of n consists of k twos. If $n = 2k+1$ is odd, then the canonical 2-primitive partition of n consists of $k-1$ twos and a single three.
- If $n = 3k$, then the canonical 3-primitive partition of n consists of k threes. If $n = 3k+1$, then the canonical 3-primitive partition of n consists of k threes and a single one. If $n = 3k+2$, then the canonical 3-primitive partition of n consists of k threes and a single two.

The canonical 2-3-primitive partitions are a crucial subfamily of the 2-3-primitive partitions.

Theorem 3 (Broersma, Kratsch, Woeginer [12])**.** *A split graph on n vertices is AP if and only if it is λ-decomposable for the canonical* 2-3-*primitive partition λ of n.*

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if it contains no induced subgraph isomorphic to a member $F \in \mathcal{F}$. Földes and Hammer[16] proved that a graph is a split graph if and only if it is $\{2K_2, C_4, C_5\}$-free. Hence, split graphs are a subclass of $\{2K_2, C_4\}$-free graphs.

In Section 2, we show that a connected threshold graph is AP if and only if it admits a perfect matching or a near perfect matching (a matching omitting exactly one vertex). In Section 3, we extend the result of Theorem 3 to $\{2K_2, C_4\}$-free graphs, by showing that a $\{2K_2, C_4\}$-free graph is AP if and only if it is λ-decomposable for the canonical 2-3-primitive partition λ of n.

2. Threshold Graphs

Threshold graphs were first introduced and studied by Chvátal and Hammer [14]. Let $a_1, a_2, \ldots, a_n$ be distinct real numbers, and define a simple graph G with vertex set $\{a_1, a_2, \ldots, a_n\}$, in which two vertices a_i and a_j are adjacent if and only if $a_i + a_j > 0$. Without loss of generality, let $a_1 \leq \cdots \leq a_n$. Note that G is connected if and only if $a_1 + a_n > 0$. It is clear that threshold graphs are split graphs. Chvátal and Hammer [14] showed that a graph G is a threshold graph if and only if it is $\{2K_2, P_4, C_4\}$-free.

Theorem 4. *A connected threshold graph G of order n is AP if and only if $\alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. To prove the necessity, we consider the admissible sequence $\lambda = \left(2^k 1^{n-2k}\right)$, where $k = \left\lfloor \frac{n}{2} \right\rfloor$. Since G is AP, there is λ-decomposition $(V_1, \ldots, V_k)$ of G. Since $G[V_i]$ is connected for each i, we have $\alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

Next we show its sufficiency. Let $V(G) = \{a_1, a_2, \ldots, a_n\}$, and let $V^+(G) = \{a_i |\ a_i \geq 0\}$ and $V^-(G) = \{a_i |\ a_i < 0\}$. By the definition of the threshold graph, $V^+(G)$ is a clique and $V^-(G)$ is an independent set of G. Since $\alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$, $|V^+(G)| \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition of n. We show that G is λ-decomposable. We proceed with induction on n. If $n \leq 2$, then $G \cong K_n$, the result holds trivially. Next let us consider the case when $n \geq 3$. Without loss of generality, let $\lambda_1 \leq \cdots \leq \lambda_l$ and $a_1 < a_2 < \cdots < a_n$. By the definition of threshold graph, $N(a_i) \setminus \{a_j\} \subseteq N(a_j) \setminus \{a_i\}$ for each i, j with $i < j$. Combining this fact with the assumption $\alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$, it follows that there exists a maximum matching M of G with

$$M = \begin{cases} \left\{a_i a_{n+1-i} \,|\, 1 \leq i \leq \frac{n}{2}\right\}, & \text{if } n \text{ is even,} \\ \left\{a_i a_{n+2-i} \,|\, 2 \leq i \leq \frac{n+1}{2}\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

Case 1. $\lambda_1 = 1$. Clearly, $G - a_1$ is also a connected threshold graph of order $n-1$ with $\alpha'(G - a_1) = \left\lfloor \frac{n-1}{2} \right\rfloor$. By induction hypothesis, $\lambda' = (\lambda_2, \ldots, \lambda_l)$ is realizable for $G - a_1$. Hence, $\lambda = (\lambda_1, \ldots, \lambda_l)$ is realizable for G.

Case 2. $\lambda_1 \geq 2$. Then $\lambda_l \geq 2$. Since G is connected, $a_n a_1 \in E(G)$, i.e., $a_n + a_1 > 0$. It follows that $G[\{a_n, a_1, \ldots, a_{l-1}\}]$ is connected. Let $V_l = \{a_n, a_1, \ldots, a_{l-1}\}$. Note that $\alpha'(G - V_l) \geq \left\lfloor \frac{n-l}{2} \right\rfloor$. By the induction hypothesis, $(\lambda_1, \lambda_2, \ldots, \lambda_{l-1})$ is realizable for $G - V_l$. Thus, λ is realizable in G. ■

3. $\{2K_2, C_4\}$-Free Graphs

Blázsik, Hujter, Pluhár and Tuza [11] gave a structural characterization of $\{2K_2, C_4\}$-free graphs.

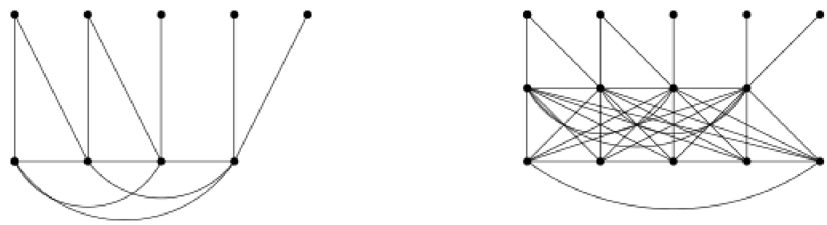
Theorem 5 (Blázsik, Hujter, Pluhár and Tuza [11])**.** *A graph $G = (V, E)$ is $\{2K_2, C_4\}$-free if and only if there is a partition $V_1 \cup V_2 \cup V_3 = V$ with the following properties.*

(i) *V_1 is an independent set in G.*

(ii) *V_2 is the vertex set of a complete subgraph in G.*

(iii) *$V_3 = \emptyset$ or $|V_3| = 5$, and in the latter case V_3 induces a 5-cycle in G.*

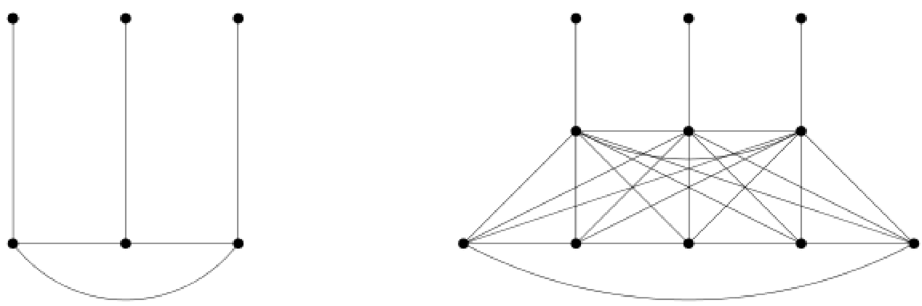
(iv) *If $V_3 \neq \emptyset$, then for all $v_i \in V_i$, $i = 1, 2, 3$, $v_1v_3 \notin E$ and $v_2v_3 \in E$ hold.*



H_1 G_1

Figure 1. A AP split graph H_1 and a $\{2K_2, C_4\}$-free graph G_1 which is not AP.

The graph H_1 in Figure 1 is a split graph. It can be checked that $(2, 2, 2, 3)$ and $(3, 3, 3)$ are realizable in H_1, by Theorem 3, H_1 is AP. Since the admissible sequence $(2, \ldots, 2)$ is not realizable for G_1 in Figure 1, G_1 is not AP.



H_2 G_2

Figure 2. A split graph H_2 that is not AP and a AP $\{2K_2, C_4\}$-free graph G_2.

On the other hand, the graph H_2 in Figure 2 is a $\{2K_2, C_4\}$-free graph, which is not AP, because $(3, 3)$ is not realizable in H_2. However, it is easy to check that G_2 is AP.

Theorem 6. *A $2K_2$-free graph G on n vertices is AP if and only if every 2-3-primitive partition λ of n is realizable in G.*

Proof. The necessity is obvious. Next we prove the sufficiency. Let $\lambda = (\lambda, \ldots, \lambda_m)$ be an admissible sequence for G. It is well known that any integer $l \geq 2$ can be expressed as $l = 2a + 3b$, where a and b are two nonnegative integers.

(1) Replace each $\lambda_i \geq 4$ in λ with a_i twos and b_i threes, and denote the resultant partition as λ'.

(2) Let λ_0 denote the number of ones in the vector λ. If $\lambda_0 \geq 2$, then replace the ones in vector λ with a_0 twos and b_0 threes, where $\lambda_0 = 2a_0 + 3b_0$. If $\lambda_0 = 1$ and there is a two in α, then replace the one and a two by a three, otherwise leave the one as it is.

The resultant new partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ of n has the form $(1, 3, \ldots, 3)$ or $(2, \ldots, 2, 3, \ldots, 3)$, and hence is a 2-3-primitive. By the assumption, let $(V'_1, \ldots, V'_m)$ be a realization of λ'. Since G is $2K_2$-free, for any $\lambda'_i \geq 2$ and $\lambda'_j \geq 2$, the union of the λ'_i-component and the λ'_j-component is connected. Therefore, the a_i 2-components and the b_i 3-components are combined into a λ_i-component.

This proves that λ is realizable in G. Thus, G is AP. ■

Let G be a $\{2K_2, C_4\}$-free graph. In view of Theorem 5, we denote G by (I, C, C_5, E), in which C_5 also denotes $V(C_5)$ in sequel. Assume that T is a connected subgraph of G with $|V(T)| = 3$. We say that T is of type-T_{ijk} if $|V(T) \cap I| = i$, $|V(T) \cap C| = j$ and $|V(T) \cap C_5| = k$. For the special case when $i = 1$, $j = 2$ and $k = 0$, we denote T_{ijk} by $\overline{T_{120}}$ if $T \cong K_3$, otherwise by T_{120}. The types of all connected subgraphs of G with order 3 belong to

$$\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}, T_{210}\}.$$

By Theorem 5, one can see that $T_{120} \cong T_{210} \cong T_{111} \cong T_{003} \cong P_3$, $T_{030} \cong T_{021} \cong K_3$. Moreover, we may assume that $T_{012} \cong K_3$, since by Theorem 5, any two vertices $v_2 \in C$ and $v_3 \in V(C_5)$ are adjacent in G.

We use $2^r 3^s$ to denote an admissible partition of n into r (possibly $r = 0$) twos and s (possibly $s = 0$) threes. A partition of $n = 3k + 1$ into k threes and 1 one is denoted by $3^k 1$. We say that G is $(3, 3)$-*reducible* if and only if $2^r 3^s$ is realizable for some $r \geq 0$ and $s \geq 4$ in G, then $2^{r+3} 3^{s-2}$ is also realizable in G. Similarly, we say that G is $(1, 3)$-*reducible* if $3^k 1$ is realizable for some $k \geq 3$ in G, then $2^2 3^{k-1}$ is also realizable in G.

Lemma 7. *Let $G = (I, C, C_5, E)$ be a $\{2K_2, C_4\}$-free graph of order n. If a canonical 2-3-primitive partition λ of n is realizable in G, then G is $(3, 3)$-reducible.*

Proof. Suppose $\lambda = 2^r 3^s$ is realizable in G for $r \geq 0$ and $s \geq 4$ and Λ be a realization of λ. Assume first that there exist two 3-components in Λ, say T_1 and T_2, of type other than T_{210}, i.e., of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$. One can see from Figure 3, that $G[T_1 \cup T_2]$ has a perfect matching for all possible cases, except possible the only case when $T_1 \cong T_{111} \cong T_2$. For this case, we may assume that the two vertices of $T_1 \cap C_5$ and $T_2 \cap C_5$ are adjacent in G, because each vertex in C_5 is adjacent to every vertex of C. Thus, by transposing such two 3-components into three 2-components in Λ, we obtain a realization Λ' of $2^{r+3}3^{s-2}$ in G.

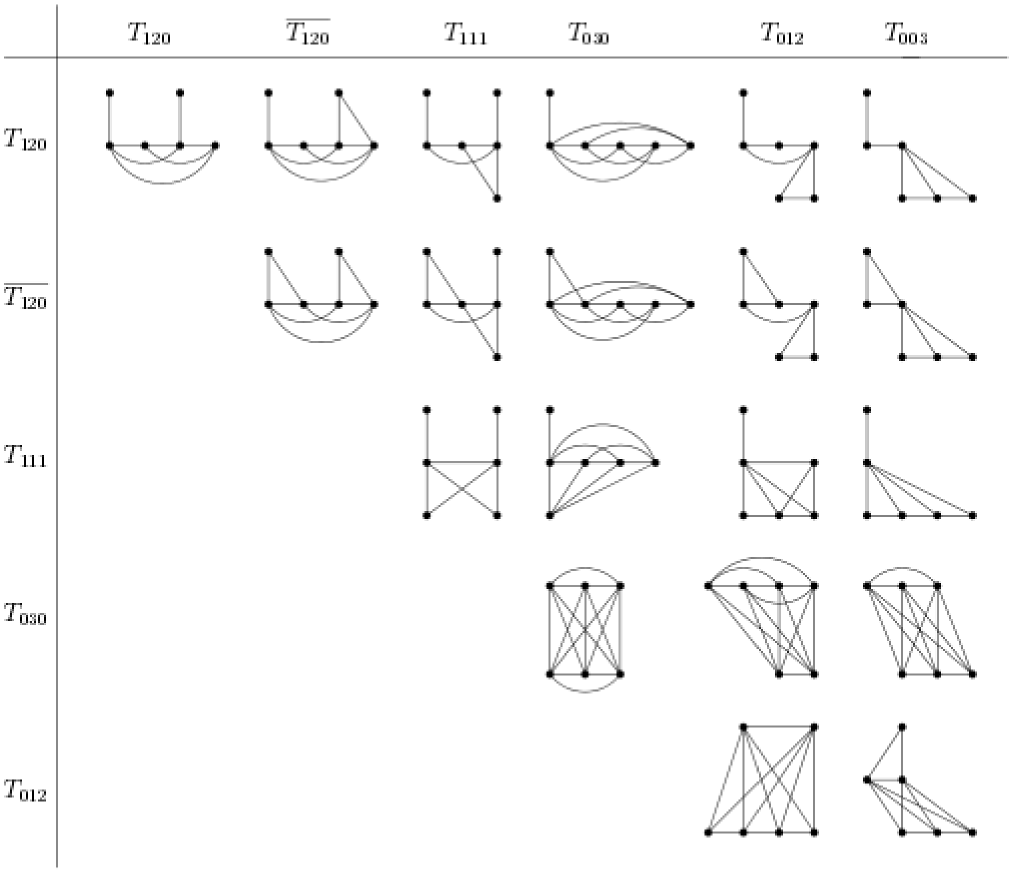


Figure 3. The subgraph of G induced by two 3-components of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$.

Next assume that there exists at most one 3-components of type in $\{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{003}\}$. Thus, at least $s-1$ 3-components have type T_{210}. Since $s \geq 4$, $|I| \geq 2(s-1) \geq 6$. Moreover, by the assumption that the canonical

2-primitive partition of n is realizable in G, $|C| \geq |I|-1 \geq 5$, and there exists two 3-components T' and T'' of type T_{210}, say $T' = \{u_1, u_1', v_1\}$, $T'' = \{u_2, u_2', v_2\}$ with $v_1, v_2 \in C$, and two vertices $w_1, w_2 \in C$ such that $u_1w_1 \in E(G)$ and $u_2w_2 \in E(G)$, and w_1, w_2 are lying in 2-component or 3-component contained in $C \cup C_5$.

First assume that at least one of w_1 and w_2 belongs to a 3-component. Without loss of generality, suppose that $\{w_1, w_0, w_0'\} = T_0$ is a 3-component contained in $C \cup C_5$. Then $T_0 \in \{T_{030}, T_{021}, T_{012}\}$. For the case when $T_0 \cong T_{030}$ or $T_0 \cong T_{021}$, we can decompose the subgraph of G induced by $T' \cup T_0$ into three 2-components. For the case when $T_0 \cong T_{012}$, we may assume that $w_0w_0' \in E(G)$. So, we can decompose the subgraph of G induced by $T' \cup T_0$ into three 2-components.

If $\{w_1, w_2\}$ is a 2-component, then we can decompose the subgraph of G induced by $T' \cup T'' \cup \{w_1, w_2\}$ into four 2-components. In the following, we assume that w_1 and w_2 belong to different 2-components. Denote $\{w_1, w_1'\}$ and $\{w_2, w_2'\}$ are two 2-components. If $w_1'w_2' \in E(G)$, we can decompose the subgraph of G induced by $T' \cup T'' \cup \{w_1, w_1', w_2, w_2'\}$ into five 2-components. If $w_1'w_2' \notin E(G)$, then $w_1', w_2' \in V(C_5)$, there exists at least one 2-component v_0v_0' such that $v_0 \in V(C_5)$, $v_0w_1' \in E(G)$ and $w_2'v_0' \in E(G)$, and then we can decompose the subgraph of $G[T' \cup T'' \cup \{w_1, w_1', w_2, w_2', v_0, v_0'\}]$ into six 2-components. ■

Lemma 8. *Let $G = (I, C, C_5, E)$ be a $\{2K_2, C_4\}$-free graph of order n. If a canonical 2-3-primitive partition λ of n is realizable in G, then G is $(1,3)$-reducible.*

Proof. Suppose $\lambda = 3^k1$ is realizable in G for some $k \geq 3$. Let $\{v_0\} \cup \Lambda$ be a $\lambda = 3^k1$-decomposition of G, in which $\{v_0\}$ is the 1-component and Λ is the set of 3-components.

Case 1. $v_0 \in C$. By the assumption, every vertex of C_5 belongs to a 3-component, and hence there exists a 3-component $T = \{w, u, v\}$ such that $T \cap C_5 \neq \emptyset$ and $T \cap C \neq \emptyset$. Thus $T \in \{T_{021}, T_{012}, T_{111}\}$. We may assume that $w \in T \cap C_5$ and $u \in T \cap C$. Then $v_0w \in E(G)$ and $uv \in E(G)$, implying that G is $(1,3)$-reducible.

Case 2. $v_0 \in C_5$. Let $w \in C_5$ be a vertex with $v_0w \in E(C_5)$ and $T = \{w, u, v\}$ be a 3-component containing w. Then, $T \in \{T_{021}, T_{012}, T_{111}, T_{003}\}$, and so, $uv \in E(G)$, implying that G is $(1,3)$-reducible.

Case 3. $v_0 \in I$. By the assumption, there exists a vertex $w \in C$ and $T = \{w, u, v\}$. Clearly $T \in \{T_{120}, \overline{T_{120}}, T_{111}, T_{030}, T_{021}, T_{012}, T_{210}\}$.

For the case $T \cong T_{120}$, assume that $u \in C$ and $v \in I$. If $uv \in E(G)$, then $\{v_0, w\}$ and $\{u, v\}$ are two 2-components, and so, G is $(1,3)$-reducible. Otherwise, $uv \notin E(G)$ and $wv \in E(G)$. Then we can choose $\{u\}$ as the new 1-component, and $\{v_0, w, v\}$ as the 3-component. Then, this case is reduced to Case 1.

If $T \cong \overline{T_{120}}$, we may assume that $v \in I$. Without loss of generality, let $uv \in E(G)$. Clearly, the subgraph $G[\{v, w, u, v_0\}]$ can be partitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$. So, G is $(1,3)$-reducible.

For the case when $T \cong T_{111}$, let $v \in I$ and $u \in C_5$. We can choose $\{u\}$ as the new 1-component, and $\{v_0, w, v\}$ as the 3-component. Then it is reduced to Case 2.

If $T \cong T_{030}$, then $\{v_0\} \cup T$ can be repartitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$.

If $T \cong T_{021}$, assume that $u \in C_5$ and $v \in C$. Again, $\{v_0\} \cup T$ can be repartitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$.

If $T \cong T_{012}$, then $\{u, v\} \subseteq C_5$. Actually, we may assume that $uv \in E(G)$. Then $\{v, w, u, v_0\}]$ can be partitioned into two 2-components $\{v_0, w\}$ and $\{u, v\}$, as we desired.

Now we deal with the last case when $T \cong T_{210}$. Since the canonical 2-primitive partition of n is realizable in G, $|C| \geq |I| - 1$. It means that there must exist a 3-component T' with type distinct from T_{210}. Take a sequence of 3-components $T_1, \ldots, T_j$ of $\mathcal{T}$ (Let $T_i = \{u_i, w_i, v_i\}$ with $u_i, v_i \in I$ and $w_i \in C$), such that $v_i w_{i+1} \in E(G)$ and $T_i \cong T_{210}$ for each $i < j$, and $T_j \not\cong T_{210}$. Let $T'_i = (T_i \setminus \{v_i\}) \cup \{v_{i-1}\}$ for each $i \in \{1, \ldots, j\}$. Replacing the components $\{v_0\}$, $T_1, \ldots, T_j$ of $\mathcal{T}$ with $\{w_j\}$, $T'_1, \ldots, T'_j$, we obtain a new realization $\mathcal{T}'$ of 3^s1 in which 1-component $\{w_j\}$ does not belong to I. By Cases 1 and 2, G is $(1, 3)$-reducible. ■

Theorem 9. *Let $G = (I, C, C_5, E)$ be a connected $\{2K_2, C_4\}$-free graph of order n. If every canonical 2-3-primitive partition of n is realizable in G, then every 2-3-primitive partition of n is realizable in G.*

Proof. Let $\lambda = 2^r 3^s$ be a 2-3-primitive partition of n. Since every canonical 2-3-primitive partition of n is realizable in G, we may assume that $r \geq 2$ and $s \geq 2$.

If $r = 2$, we are able to obtain a λ-decomposition from the λ'-realization of the canonical primitive partition 13^{s+1}, because G is $(1, 3)$-reducible by Lemma 8. If $r \geq 3$, we can obtain a λ-decomposition from the realization of the 2-3-primitive partition $2^{r-3}3^{s+2}$, because G is $(3, 3)$-reducible by Lemma 11. ■

By Theorem 6 and Theorem 9, we obtain the following result.

Theorem 10. *A $\{2K_2, C_4\}$-free graph G on n vertices is AP if and only if every canonical 2-3-primitive partition of n is realizable in G.*

4. $2K_2$-FREE BIPARTITE GRAPHS

Lemma 11. *Let $G = (X, Y)$ be a connected $2K_2$-free bipartite graph. If G has a perfect matching or a near perfect matching, then every 2-3-primitive partition λ of n is realizable in G.*

Proof. Since G has a perfect matching or a near perfect matching, λ^* is realizable in G, where

$$\lambda^* = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-1}{2}}1, & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, since G is connected, $(2, \ldots, 2, 3)$ is realizable in G if n is odd. So, let Λ_0 be a λ_0-decomposition of G, where

$$\lambda_0 = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even,} \\ 2^{\frac{n-3}{2}}3, & \text{if } n \text{ is odd.} \end{cases}$$

To prove every 2-3-primitive partition λ of n is realizable in G, it suffices to show that

(i) the subgraph induced by any three 2-components of Λ_0 can be decomposed into two 3-components; and

(ii) the subgraph induced by any two 2-components of Λ_0 can be decomposed into one 1-component and one 3-component.

We first prove (i). Let x_1y_1, x_2y_2 and x_3y_3 be three 2-components of Λ_0, where $x_i \in X$, $y_i \in Y$, $1 \leq i \leq 3$. Since G is $2K_2$-free, the subgraph induced by any two 2-components is connected. Without loss of generality, we assume that $x_1y_2 \in E(G)$. If $x_2y_3 \in E(G)$, then $\{x_1, y_1, y_2\}$ and $\{x_2, x_3, y_3\}$ are two 3-components. Otherwise, $x_3y_2 \in E(G)$. If $x_1y_3 \in E(G)$, then $\{x_1, y_1, y_3\}$ and $\{x_2, x_3, y_2\}$ are two 3-components. If $x_1y_3 \notin E(G)$, then $x_3y_1 \in E(G)$, and so $\{x_1, x_2, y_2\}$ and $\{x_3, y_1, y_3\}$ are two 3-components. For each case, we can decompose three 2-components into two 3-components. Thus, (i) holds.

Since G is $2K_2$-free, the subgraph induced by any two 2-components is connected. So, it is easy to partition this subgraph into a subgraph of order 1 and a subgraph of order 3. Thus, (ii) holds. ■

By Theorem 6 and Lemma 11, we obtain the following result.

Theorem 12. *Let G be a connected $2K_2$-free bipartite graph. Then G is AP if and only if G has a perfect matching or a near perfect matching.*

5. $2K_2$-Free Nonbipartite Graphs with Clique Number 2

In this section, we consider $2K_2$-free nonbipartite graphs with clique number 2. Recall that $o(H)$ denotes the number of odd components in H. The well-known Tutte's 1-factor theorem says that a graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V(G)$. The following consequence can be derived easily from Tutte's 1-factor theorem.

Proposition 13. *Let G be a graph of odd order. Then G has a near perfect matching if and only if $o(G-S) \leq |S|+1$ for all $S \subset V$.*

For a vertex $v \in V(G)$ and a positive integer n, we say that H is obtained from G by multiplying v by n when H is formed by replacing the vertex v by an independent set of n vertices each having the same neighbors as v.

Theorem 14 (Chung, Gyárfás, Tuza and Trotter [13]). *Assume that G is $2K_2$-free, $\omega(G) = 2$ and G is not bipartite. Then G can be obtained from the cycle C_5 by vertex multiplication.*

So let G be a $2K_2$-free, nonbipartite graph with $\omega(G) = 2$. Then, by Theorem 14, we denote $G = (A_1, A_2, A_3, A_4, A_5)$, where the sets A_i are independent sets and form a partition of $V(G)$, and each vertex of A_i is adjacent to all vertices in $A_{i-1} \cup A_{i+1}$ for each $i = 1, 2, \ldots, 5$ where $i-1$ and $i+1$ are taken modulo 5.

Theorem 15. *Assume that G is a $2K_2$-free nonbipartite graph of even order with $\omega(G) = 2$. Then G has a perfect matching if and only if the following conditions are satisfied for each $i \in \{1, 2, \ldots, 5\}$,*

(1) $|A_i| + |A_{i+2}| \leq |A_{i-1}| + |A_{i+1}| + |A_{i-2}|$ *and*

(2) $|A_i| \leq |A_{i-1}| + |A_{i+1}|$, *with equality only if* $|A_{i-2}| = |A_{i+2}|$.

Proof. First assume that G has a perfect matching. The conclusions (1) and (2) can be deduced from Tutte's 1-factor theorem by taking $A_{i-1} \cup A_{i+1} \cup A_{i+3}$ and $A_{i-1} \cup A_{i+1}$ into S, respectively.

Conversely, let G be a $2K_2$-free nonbipartite graph of even order with $\omega(G) = 2$ satisfying conditions (1) and (2). Let $S \subset V(G)$. We shall show that $o(G-S) \leq |S|$. If $G-S$ is connected, then $o(G-S) \leq 1 \leq |S|$ for a nonempty set S, and $o(G-S) = o(G) = 0 = |S|$ for the empty set S, since $|V(G)|$ is even. Now assume that $G-S$ is disconnected. At least two nonadjacent parts of $\{A_1, A_2, A_3, A_4, A_5\}$ are contained in S. Without loss of generality, we assume that $A_2 \cup A_5 \subseteq S$.

Case 1. $S = A_2 \cup A_5$. If $|A_1| = |A_2| + |A_5|$, then by (2) $|A_3| = |A_4|$, and hence

$$o(G-S) = |A_1| = |A_2| + |A_5| = |S|.$$

If $|A_1| \leq |A_2| + |A_5| - 1$, then

$$o(G-S) \leq |A_1| + 1 \leq |A_2| + |A_5| - 1 + 1 = |A_2| + |A_5| = |S|.$$

Case 2. $A_2 \cup A_5 \subset S$ and $S \cap (A_1 \cup A_3 \cup A_4) \neq \emptyset$. If $A_3 \not\subseteq S$ and $A_4 \not\subseteq S$, then $o(G-S) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 \leq |S|$.

If $A_3 \subset S$, then $o(G-S) \leq |A_1| + |A_4| \leq |A_2| + |A_5| + |A_3| \leq |S|$.

If $A_4 \subset S$, then $o(G-S) \leq |A_1| + |A_3| \leq |A_2| + |A_5| + |A_4| \leq |S|$.

In either case, we obtain $o(G-S) \leq |S|$ for $S \subset V(G)$. By Tutte's 1-factor theorem, G has a perfect matching. ■

Theorem 16. *Assume that G is a $2K_2$-free nonbipartite graph of odd order with $\omega(G) = 2$. Then G has a near perfect matching if and only if the following conditions are satisfied for each $i \in \{1, 2, \ldots, 5\}$,*

(1) $|A_i| + |A_{i+2}| \leq |A_{i-1}| + |A_{i+1}| + |A_{i-2}| + 1$ *and*

(2) $|A_i| \leq |A_{i-1}| + |A_{i+1}| + 1$, *with equality only if* $|A_{i-2}| = |A_{i+2}|$.

Proof. First assume that G has a near perfect matching. The conclusions (1) and (2) can be deduced from Proposition 13 by taking $A_{i-1} \cup A_{i+1} \cup A_{i+3}$ and $A_{i-1} \cup A_{i+1}$ into S, respectively.

Conversely, let G be a $2K_2$-free nonbipartite graph of odd order with $\omega(G) = 2$ satisfying conditions (1) and (2). Let $S \subset V(G)$. We shall show that $o(G-S) \leq |S|+1$. If $G-S$ is connected, then $o(G-S) \leq 1 \leq |S|$ for a nonempty set S, and $o(G-S) = o(G) = 1 = |S|+1$ for the empty set S, since $|V(G)|$ is odd. If $G-S$ is disconnected, then at least two nonadjacent parts of $\{A_1, A_2, A_3, A_4, A_5\}$ are contained in S. Without loss of generality, we assume that $A_2 \cup A_5 \subseteq S$.

Case 1. $S = A_2 \cup A_5$. If $|A_1| \leq |A_2| + |A_5|$, then

$$o(G-S) = o(G - A_2 - A_5) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 = |S| + 1.$$

If $|A_1| = |A_2| + |A_5| + 1$, then by the assumption, $|A_3| = |A_4|$. Therefore,

$$o(G-S) = o(G - A_2 - A_5) = |A_1| = |A_2| + |A_5| + 1 = |S| + 1.$$

Case 2. $A_2 \cup A_5 \subset S$ and $S \cap (A_1 \cup A_3 \cup A_4) \neq \emptyset$. If $A_3 \nsubseteq S$ and $A_4 \nsubseteq S$, then $o(G-S) \leq |A_1| + 1 \leq |A_2| + |A_5| + 1 + 1 \leq |S| + 1$.

If $A_3 \subset S$, then $o(G-S) \leq |A_1| + |A_4| \leq |A_2| + |A_5| + |A_3| + 1 \leq |S| + 1$.

If $A_4 \subset S$, then $o(G-S) \leq |A_1| + |A_3| \leq |A_2| + |A_5| + |A_4| + 1 \leq |S| + 1$.

For each case, we obtain $o(G-S) \leq |S| + 1$ for $S \subset V(G)$. By proposition 13, G has a near perfect matching. ■

Theorem 17. *Let G be a $2K_2$-free nonbipartie graph G with $\omega(G) = 2$. Then G is AP if and only if it has a perfect matching or a near perfect matching.*

Proof. The necessity is obvious. We prove the sufficiency by induction on the order n of G. If $5 \leq n \leq 6$, then G is traceable, and so, G is AP. If $n = 7$, then

$$(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) \in \{(1, 1, 1, 2, 2), (1, 1, 2, 1, 2)\}.$$

It is easy to check that in the both cases, G is traceable, and thus it is AP. If $n = 8$, then $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 2, 3)$ or $(1, 1, 2, 2, 2)$ or $(1, 2, 1, 2, 2)$. For each case, it can be checked that G is traceable, and hence G is AP.

Now let $n \geq 9$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition of n with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$. Since G is 2-connected, if $p \leq 2$, then λ is realizable in G by Theorem

1. So, we assume that $p \geq 3$. If there exists $V_1 \subseteq V(G)$ with $|V_1| = \lambda_1$, which come from some (at least two) consecutive parts of G, such that $G_1 = G - V_1$ is $2K_2$-free, $\omega(G_1) = 2$ and G_1 is not bipartite graph with a perfect matching or a near perfect matching, then by induction hypothesis, G_1 is $(\lambda_2, \ldots, \lambda_p)$-realizable, and hence G is λ-realizable. If such a set V_1 does not exist, we have the following result.

Claim 1. *There exist two nonadjacent parts of* $(A_1, A_2, A_3, A_4, A_5)$ *with cardinality* 1. *Moreover,* $\sum_{i=2}^{p} \lambda_i \geq 6$.

Proof. Since $n = \sum_{i=1}^{p} \lambda_i \geq 9$ with $\lambda_1 \leq \cdots \leq \lambda_p$ and $p \geq 3$, if $\sum_{i=2}^{p} \lambda_i \leq 5$, then $\lambda_1 \leq \frac{1}{2} \sum_{i=2}^{p} \lambda_i \leq \frac{1}{2} \times 5 = 2.5$. It follows that $\sum_{i=1}^{p} \lambda_i = \lambda_1 + \sum_{i=2}^{p} \lambda_i \leq 2.5 + 5 = 7.5 < 9$, a contradiction. Thus, $\sum_{i=2}^{p} \lambda_i \geq 6$. □

By Claim 1, suppose that $|A_1| = |A_3| = 1$, without loss of generality. Since G has a perfect matching or a near perfect matching, by Theorem 15 and Theorem 16,

$$|A_2| \leq \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Claim 2. $\min\{|A_4|, |A_5|\} \geq 2$.

Proof. Suppose that $\min\{|A_4|, |A_5|\} = 1$, and without loss of generality, let $|A_4| = 1$. Then by Theorem 16(1), $|A_2| + |A_5| \leq |A_1| + |A_3| + |A_4| + 1 = 4$. Thus, $n = \Sigma_{i=1}^{5} |A_i| \leq 7$, a contradiction. □

Claim 3. $|A_4| + |A_5| - 2 < \lambda_1 \leq \frac{n}{3}$ *and* $9 \leq n \leq 10$.

Proof. Since $p \geq 3$, $\sum_{i=1}^{p} \lambda_i = n$ and $\lambda_1 \leq \cdots \leq \lambda_p$, we have $\lambda_1 \leq \frac{n}{3}$.

If $|A_4| + |A_5| - 2 \geq \lambda_1$, then we can obtain G_1 from G by deleting λ_1 vertices from $A_4 \cup A_5$, a contradiction. Since $|A_4| + |A_5| - 2 < \lambda_1$,

$$n \leq |A_4| + |A_5| + 2 + 3 < \lambda_1 + 2 + 2 + 3 = \lambda_1 + 7 \leq \frac{n}{3} + 7,$$

implying that $n < \frac{21}{2}$, i.e., $9 \leq n \leq 10$. □

If $n = 10$ (n is even), then $|A_2| \leq 2$. It follows that $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 4, 3)$ or $(1, 2, 1, 3, 3)$. Since $\lambda_1 \leq \frac{10}{3}$, we can obtain G_1 by deleting λ_1 vertices from $|A_4|$ and $|A_5|$, again a contradiction.

If $n = 9$, then $|A_2| \leq 3$. It follows that $(|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (1, 1, 1, 3, 3)$ or $(1, 2, 1, 3, 2)$ or $(1, 3, 1, 2, 2)$. Since $\lambda_1 \leq \frac{9}{3}$, for the cases when $(1, 1, 1, 3, 3)$ and $(1, 2, 1, 3, 2)$, we can obtain G_1 from G by deleting λ_1 vertices from $A_4 \cup A_5$. For the case when $(1, 3, 1, 2, 2)$, if $\lambda_1 \leq 2$, we can obtain G_1 from G by deleting λ_1 vertices from $A_4 \cup A_5$. If $\lambda_1 = 3$, then $\lambda = (3, 3, 3)$. Denote $A_2 = \{u_2, v_2, w_2\}$ and $A_4 = \{u_4, v_4\}$. Then we take $V_1 = A_1 \cup \{u_2, v_2\}$, $V_2 = A_3 \cup \{w_2, u_4\}$, $V_3 =$

$A_5 \cup \{v_4\}$. Note that $G[V_i]$ is connected for each $i \in \{1, 2, 3\}$. That is, $\lambda = (3, 3, 3)$ is realizable in G. ■

By Theorem 15, Theorem 16 and Theorem 17, we can obtain the following result. Let $\mathcal{G}$ be set of $2K_2$-free graphs G with $\omega(G) = 2$, satisfying the conditions (1) and (2) in Theorem 16 or Theorem 17 (depending whether G has even or odd order).

Theorem 18. *If G is $2K_2$-free, $\omega(G) = 2$ and G is not bipartite, then the following statements are equivalent.*

(i) *G is AP.*

(ii) *$G \in \mathcal{G}$.*

(iii) *G has a perfect matching or a near perfect matching.*

By Theorem 12 and Theorem 18 we obtain the following result.

Theorem 19. *If G is $2K_2$-free and $\omega(G) = 2$, then G is AP if and only if G has a perfect matching or a near perfect matching.*

Acknowledgments

We would like to thank the referees for helpful suggestions and comments. The research is supported by the National Natural Science Foundation of China (No. 11501487, No. 11571294, No. 11531011, No. 11961067.) and the Key Laboratory Project of Xinjiang (2018D04017, XJEDU2019I001.)

References

[1] D. Barth, O. Baudon and J. Puech, *Decomposable trees: a polynomial algorithm for tripodes*, Discrete Math. **119** (2002) 205-216.
https://doi.org/10.1016/S0166-218X(00)00322-X

[2] D. Barth and H. Fournier, *A degree bound on decomposable trees*, Discrete Math. **306** (2006) 469–477.
https://doi.org/10.1016/j.disc.2006.01.006

[3] D. Barth, H. Fournier and R. Ravaux, *On the shape of decomposable trees*, Discrete Math. **309** (2009) 3882–3887.
https://doi.org/10.1016/j.disc.2008.11.012

[4] O. Baudon, J. Bensmail, F. Foucaud and M. Pilśniak, *Structural properties of recursively partitionable graphs with connectivity* 2, Discus. Math. Graph Theory **37** (2017) 89–115.
https://doi.org/10.7151/dmgt.1925

[5] O. Baudon, J. Bensmail, R. Kalinowski, A. Marczyk, J. Przybyło and M. Woźniak, *On the Cartesian product of an arbitrarily partitionable graph and a traceable graph*, Discrete Math. Theor. Comput. Sci. **16** (2014) 225–232.

[6] O. Baudon, J. Bensmail, J. Przybyło and M. Woźniak, *Partitioning powers of traceable of Hamiltonian graphs*, Theoret. Comput. Sci. **520** (2014) 133–137.
https://doi.org/10.1016/j.tcs.2013.10.016

[7] O. Baudon, F. Foucaud, J. Przybyło and M. Woźniak, *On the structure of arbitrarily partitionable graphs with given connectivity*, Discrete Appl. Math. **162** (2014) 381–385.
https://doi.org/10.1016/j.dam.2013.09.007

[8] O. Baudon, F. Gilbert and M. Woźniak, *Recursively arbitrarily vertex-decomposable graphs*, Opuscula Math. **32** (2012) 689–706.
https://doi.org/10.7494/OpMath.2012.32.4.689

[9] O. Baudon, J. Przybyło and M. Woźniak, *On minimal arbitrarily partitionable graphs*, Inform. Process. Lett. **112** (2012) 697–700.
https://doi.org/10.1016/j.ipl.2012.06.010

[10] J. Bensmail, *On the complexity of partitioning a graph into a few connected subgraphs*, J. Comb. Optim. **30** (2015) 174–187.
https://doi.org/10.1007/s10878-013-9642-8

[11] Z. Blázsik, M. Hujter, A. Pluhár and Zs. Tuza, *Graphs with no induced C_4 and $2K_2$*, Discrete Math. **115** (1993) 51–55.
https://doi.org/10.1016/0012-365X(93)90477-B

[12] H. Broersma, D. Kratsch and G.J. Woeginger, *Fully decomposable split graphs*, European J. Combin. **34** (2013) 567–575.
https://doi.org/10.1016/j.ejc.2011.09.044

[13] F.R.K. Chung, A. Gyárfás, Zs. Tuza and W.T. Trotter, *The maximum number of edges in $2K_2$-free graphs of bounded degree*, Discrete Math. **81** (1990) 129–135.
https://doi.org/10.1016/0012-365X(90)90144-7

[14] V. Chvátal and P.L. Hammer, *Aggregation of inequalities in integer programming*, Ann. Discrete Math. **1** (1977) 145–162.
https://doi.org/10.1016/S0167-5060(08)70731-3

[15] S. Cichacz, A. Görlich, A. Marczyk and J. Przybyło, *Arbitrarily vertex decomposable caterpillars with four or five leaves*, Discuss. Math. Graph Theory **26** (2006) 291–305.
https://doi.org/10.7151/dmgt.1321

[16] S. Földes, P.L. Hammer, *Split graphs*, Congr. Numer. **XIX** (1977) 311–315.

[17] E. Győri, *On division of graphs to connected subgraphs*, in: Combinatorics, Proc. Fifth Hungarian Colloq., (Keszthely, 1976, Colloq. Math. Soc. János Bolyai **18**, 1976) 485–494.

[18] M. Horňák, A. Marczyk, I. Schiermeyer and M. Woźniak, *Dense arbitrarily vertex decomposable graphs*, Graphs Combin. **28** (2012) 807–821.
https://doi.org/10.1007/s00373-011-1077-3

[19] M. Horňák, Zs. Tuza and M. Woźniak, *On-line arbitrarily vertex decomposable trees*, Discrete Appl. Math. **155** (2007) 1420–1429.
https://doi.org/10.1016/j.dam.2007.02.011

[20] M. Horňák and M. Woźniak, *Arbitrarily vertex decomposable trees are of maximum degree at most six*, Opuscula Math. **23** (2003) 49–62.

[21] M. Horňák and M. Woźniak, *On arbitrarily vertex decomposable trees*, Discrete Math. **308** (2008) 1268–1281.
https://doi.org/10.1016/j.disc.2007.04.008

[22] R. Kalinowski, *Dense on-line arbitrarily partitionable graphs*, Discrete Appl. Math. **226** (2017) 71–77.
https://doi.org/10.1016/j.dam.2017.04.006

[23] R. Kalinowski, M. Pilśniak, I. Schiermeyer and M. Woźniak, *Dense arbitrarily partitionable graphs*, Discuss. Math. Graph Theory **36** (2016) 5–22.
https://doi.org/10.7151/dmgt.1833

[24] R. Kalinowski, M. Pilśniak, M. Woźniak and O. Zioło, *Arbitrarily vertex decomposable suns with few rays*, Discrete Math. **309** (2009) 3726–3732.
https://doi.org/10.1016/j.disc.2008.02.019

[25] R. Kalinowski, M. Pilśniak, M. Woźniak and O. Zioło, *On-line arbitrarily vertex decomposable suns*, Discrete Math. **309** (2009) 6328–6336.
https://doi.org/10.1016/j.disc.2008.11.025

[26] L. Lovász, *A homology theory for spanning trees of a graph*, Acta Math. Acad. Sci. Hungar. **30(3-4)** (1977) 241–251.
https://doi.org/10.1007/BF01896190

[27] A. Marczyk, *A note on arbitrarily vertex decomposable graphs*, Opuscula Math. **26** (2006) 109–118.

[28] A. Marczyk, *An Ore-type condition for arbitrarily vertex decomposable graphs*, Discrete Math. **309** (2009) 3588–3594.
https://doi.org/10.1016/j.disc.2007.12.066

[29] R. Ravaux, *Decomposing trees with large diameter*, Theoret. Comput. Sci. **411** (2010) 3068–3072.
https://doi.org/10.1016/j.tcs.2010.04.032

Received 2 December 2018
Revised 30 November 2019
Accepted 30 November 2019