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b-COLORING OF THE MYCIELSKIAN OF SOME CLASSES OF GRAPHS

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Abstract

The b-chromatic number $b(G)$ of a graph G is the maximum k for which G has a proper vertex coloring using k colors such that each color class contains at least one vertex adjacent to a vertex of every other color class. In this paper, we have mainly investigated on the b-chromatic number of the Mycielskian of regular graphs. In particular, we have obtained the exact value of the b-chromatic number of the Mycielskian of some classes of graphs. This includes a few families of regular graphs, graphs with $b(G) = 2$ and split graphs. In addition, we have found bounds for the b-chromatic number of the Mycielskian of some more families of regular graphs in terms of the b-chromatic number of their original graphs. Also we have found b-chromatic number of the generalized Mycielskian of some regular graphs.

Keywords: b-coloring, b-chromatic number, Mycielskian of graphs, regular graphs.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let G be a graph with vertex set V and edge set E. A b-coloring of a graph G using k colors is a proper coloring of the vertices of G using k colors in which each color class has a color dominating vertex (c.d.v.), that is, a vertex that has a neighbor in each of the other color classes. The b-chromatic number, $b(G)$ of G is the largest k such that G has a b-coloring using k colors. For a given b-coloring of a graph, a set of c.d.vs., one from each class, is known as a color dominating system (c.d.s.) of that b-coloring. The concept of b-coloring was introduced by

Irving and Manlove [10] in analogy to the achromatic number of a graph G (the maximum number of color classes in a complete coloring of G).

It is clear from the definition of $b(G)$ that the chromatic number, $\chi(G)$ of G is the least k for which G admits a b-coloring using k colors and hence $\chi(G) \leq b(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. Graphs for which there exists a b-coloring using k colors for every integer k such that $\chi(G) \leq k \leq b(G)$ are known as b-continuous graphs. It can be observed that not all graphs are b-continuous. For instance, it is shown in [15] that Q_3 has a b-coloring using 2 colors and 4 colors but none using 3 colors, and therefore Q_3 is not b-continuous. Hence the natural question that arises is to characterize graphs which are b-continuous. There are a few papers in this direction. Also recently there has been a survey on b-coloring of graphs. For instance, see [2–4, 6–8, 11, 12, 20]. The b-spectrum of a graph G is the set of positive integers k for which G has a b-coloring using k colors and is denoted by $S_b(G)$, that is, $S_b(G) = \{k : G$ has a b-coloring using k colors$\}$. Clearly, $\{\chi(G), b(G)\} \subseteq S_b(G)$ and G is b-continuous if and only if $S_b(G) = \{\chi(G), \chi(G) + 1, \ldots, b(G)\}$.

Let the vertices of a graph G be ordered as $v_1, v_2, \ldots, v_n$ such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$. Then the m-degree, $m(G)$ of G is defined by $m(G) = \max\{i : d(v_i) \geq i - 1, 1 \leq i \leq n\}$. For any graph G, $b(G) \leq m(G) \leq \Delta(G) + 1$. Also for any regular graph, $m(G) = \Delta(G) + 1$.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [19] developed an interesting graph construction as follows. For a graph $G = (V, E)$, the Mycielskian of G, denoted by $\mu(G)$, is the graph with vertex set $V(\mu(G)) = V \cup V' \cup \{u\}$ where $V' = \{x' : x \in V\}$ and the edge set $E(\mu(G)) = E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex x (and x the twin of x') and the vertex u is known as the root of $\mu(G)$. In $\mu(G)$, if $A \subseteq V$, let A' denotes the set of twin vertices of A in $\mu(G)$ and for every $x \in V$ and any non-negative integer i, define $N_i(x) = \{y \in V : d_G(x, y) = i\}$ where $d_G(x, y)$ is the length of the shortest path joining the vertices x and y in the graph G.

The generalized Mycielskian is defined as follows [17, 18]. Let G be a graph with vertex set $V_0 = \left\{v_1^0, v_2^0, \ldots, v_n^0\right\}$ and edge set E_0. Given an integer $m \geq 1$, the m-Mycielskian (also known as the generalized Mycielskian) of G, denoted by $\mu_m(G)$, is the graph whose vertex set is the disjoint union $V^0 \cup V^1 \cup \cdots \cup V^m \cup \{u\}$, where $V^i = \left\{v_j^i : v_j^0 \in V^0\right\}$ is the i-th copy of V^0, for $i = 1, 2, \ldots, m$, and the edge set $E^0 \cup \left(\bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\}\right) \cup \left\{v_j^m u : v_j^m \in V^m\right\}$. For every pair $i, j \in \{0, 1, \ldots, m\}$, $i \neq j$, and $s \in \{0, 1, \ldots, n-1\}$, the vertices $v_s^i \in V^i$ and $v_s^j \in V^j$ are considered as twins of each other. Also if $S \subseteq V^0$, then $S^i \subseteq V^i$ denotes the twins of the vertices of S in V^i.

In this paper, we have mainly investigated on the b-chromatic number of the Mycielskian of regular graphs. In particular, we have shown that, if G is a k-regular graph ($k \geq 3$) with girth at least 7 or with girth 5 whose diameter is at least 5 and which contain no C_6, then $b(\mu(G)) = 2k + 1 = 2b(G) - 1$. Further, if G is a k-regular graph with girth 6, we have shown that $k + \lfloor \frac{k+1}{2} \rfloor \leq b(\mu(G)) \leq 2k + 1$. In addition, we have proved that if G is a k-regular graph with girth at least 8, then $\mu(G)$ is b-continuous. Also, we have found the b-chromatic number of the Mycielskian of split graphs and graphs G with $b(G) = 2$. Finally, we have determined on the b-chromatic number of the generalized Mycielskian of some families of regular graphs.

For notation and terminologies not mentioned in this paper, see [21].

2. b-Coloring of the Mycielskian of Regular Graphs

In [1], it has been shown that if G is a graph with b-chromatic number b and for which the number of vertices of degree at least b is at most $2b - 2$, then $b(\mu(G))$ lies in the interval $[b + 1, 2b - 1]$. While considering regular graphs G, in [13, 14] it has been shown that $b(G) = \Delta(G) + 1$, when the girth of G is at least 6 or when the girth is at least 5 with no induced C_6. For these regular graphs, the number of vertices of degree at least b is 0 and hence $b(\mu(G))$ lies in the interval $[b + 1, 2b - 1]$. What we intend to do in Section 2 is to find the exact value of $b(\mu(G))$ or at least find some better bounds for these families of regular graphs. Also, we would like to investigate on the Mycielskian of k-regular graphs which are b-continuous.

The following are the notations that will be used throughout Section 2.

Let G be a k-regular graph. For $v, w \in V$:

(i) $N_1(v) = \{v_1, v_2, \ldots, v_k\}$ and $N_1(w) = \{w_1, w_2, \ldots, w_k\}$.

(ii) For $1 \leq i \leq k$, let $M(v_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,k-1}\}$ denote the neighbors of v_i other than v in G. Similarly, for $1 \leq i \leq k$, let $M(w_i) = \{w_{i,1}, w_{i,2}, \ldots, w_{i,k-1}\}$ denote the neighbors of w_i other than w in G.

(iii) For $1 \leq i \leq k$, and $1 \leq j \leq k - 1$, let $M(v_{i,j}) = \{v_{i,j,1}, v_{i,j,2}, \ldots, v_{i,j,k-1}\}$ be the neighbors of $v_{i,j}$ other than v_i in G. Similarly, for $1 \leq i \leq k$, and $1 \leq j \leq k - 1$, $M(w_{i,j})$ is defined.

Let us start with the following observations on k-regular graphs with girth at least 7.

Observation 2.1. *Let G be a k-regular graph with girth at least 7. For $v \in V$, we have the following.*

(i) *$N_1(v)$ and $N_2(v)$ are independent sets.*

(ii) *For $y, z \in N_2(v)$, $[N_1(y) \cap N_1(z)] \cap N_3(v) = \emptyset$ and there exists at most one edge between $N_1(y)$ and $N_1(z)$ (otherwise, we will get a C_6 or a C_4).*

(iii) *For* $w \in N_1(v)$ *and* $x \in N_3(v)$*, there exists at most one edge between* x *and* $N_2(w)$*.*

Theorem 2.2. *For* $k \geq 3$*, if* G *is a* k*-regular graph with girth at least* 7*, then* $b(\mu(G)) = 2k+1 = 2b(G)-1$*.*

Proof. Let G be a k-regular graph with girth at least 7 and $k \geq 3$. It can be easily seen that $m(\mu(G)) = 2k+1$. Hence it is enough to produce a b-coloring using $2k+1$ colors. Let $\{0, 1, 2, \ldots, 2k\}$ be the set of $2k+1$ colors. Let $v \in V$.

Let us first partially color the graph to get c.d.vs. for each of the color classes. This is done by defining a coloring c for $\mu(G)$ as follows.

(i) $c(u) = k$, $c(v) = 0$, $c(v') = 2k$, $c(v_{1,1}) = 2k$.

(ii) For $1 \leq i \leq k$
$$c(v_i) = i,$$
$$c(v_i') = k+i.$$

(iii) For $2 \leq i \leq k-1, 1 \leq j \leq k-1$
$$c(v_{i,j}) = \begin{cases} j & \text{for } i > j, \\ j+1 & \text{for } i \leq j, \end{cases}$$
$$c\big(v'_{i,j}\big) = k+j,$$
$$c\big(v_{k,j}\big) = k+j,$$
$$c\big(v'_{k,j}\big) = j.$$

This partial coloring makes $v, v_2, v_3, \ldots, v_k$ as c.d.vs. for the color classes $0, 2, 3, \ldots, k$, respectively. We have to extend this partial coloring in such a way that we get c.d.vs. for the remaining color classes, namely $1, k+1, k+2, \ldots, 2k$. Let us do this by making $v_{1,1}, v_{2,1}, v_{k,1}, v_{k,2}, \ldots, v_{k,k-1}$ as c.d.vs. for the color classes $2k, 1, k+1, k+2, \ldots, 2k-1$, respectively. Now, let us divide the proof into 2 cases.

Case 1. $k \geq 4$. Let us assign the colors $\{2, 3, \ldots, k\}$ to the vertices of $M(v_{1,1})$ and the colors $\{k+2, k+3, \ldots, 2k-1, 0\}$ to the vertices of $(M(v_{1,1}))'$ in any order. Next, let us assign the colors $\{3, 4, \ldots, k, 0\}$ to the vertices of $M(v_{2,1})$ and the colors $\{k+1, k+3, k+4, \ldots, 2k\}$ to the vertices of $(M(v_{2,1}))'$ (in any order), in such a way that the color 0 and $2k$ are assigned to a vertex in $M(v_{2,1})$ and its twin, respectively. Note that by using (ii) of Observation 2.1, there can be at most one edge between $M(v_{1,1})$ and $M(v_{2,1})$ and thereby a possibility of an edge between two vertices with the same color. Even in such a situation, we can permute the colors given for $M(v_{1,1})$ and $(M(v_{1,1}))'$ in such a way that the given partial coloring becomes proper. The coloring c has been given in Figure 1.

Next, for $1 \leq i \leq k-1$, let us assign the colors $\{k+1, k+2, \ldots, 2k-1, 0\} \backslash \{k+i\}$ to the vertices of $M(v_{k,i})$ (in any order) in G. Note that by using (i) of Observation 2.1 for the vertex v_k, for $1 \leq i, j \leq k-1$, there will be no edge between the vertices of $M(v_{k,i})$ and $M(v_{k,j})$ and by using (ii) of Observation 2.1,

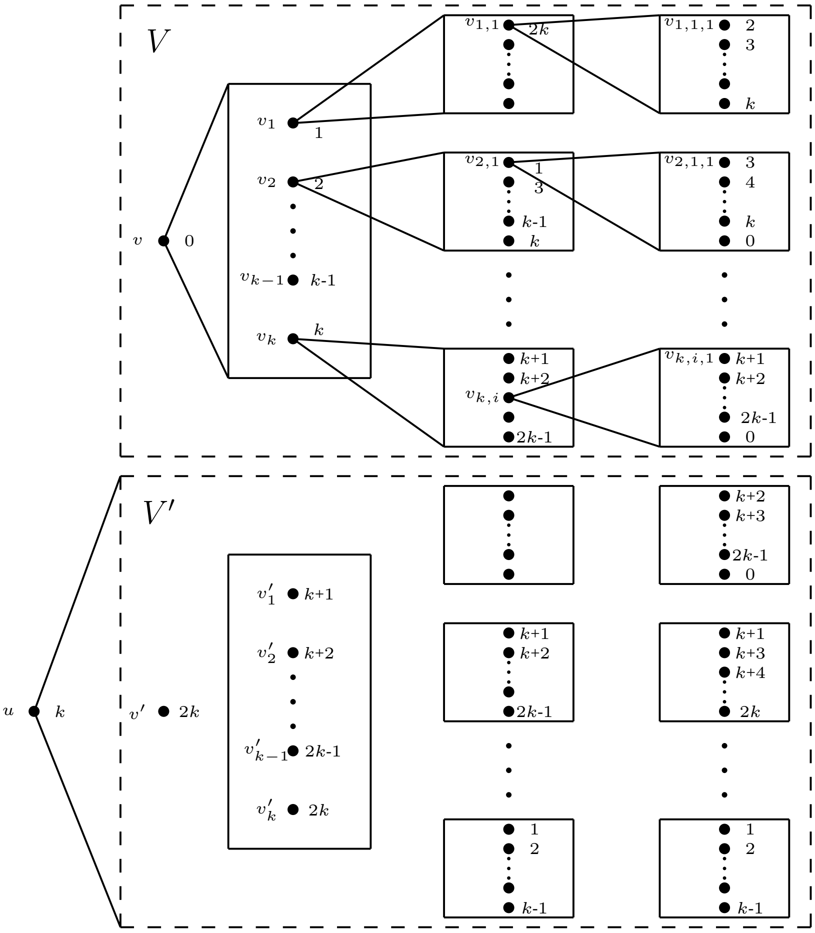

Figure 1. b-coloring of k-regular graph with girth at least 7 where $k \geq 4$.

for $1 \leq i \leq k-1$, there exist at most one edge between $M(v_{1,1})$ and $M(v_{k,i})$ and one edge between $M(v_{2,1})$ and $M(v_{k,i})$. Even in the worst case, a vertex in $M(v_{k,i})$ can have at most 4 colored neighbors in $N_3(v)$. Namely, one in $M(v_{1,1})$ and its twin in $(M(v_{1,1}))'$ and one in $M(v_{2,1})$ and its twin in $(M(v_{2,1}))'$. But the colors given to $M(v_{1,1})$ will never create a problem while coloring $M(v_{k,i})$, $1 \leq i \leq k-1$ and none of the colors in $M(v_{2,1})$ will create a problem except 0. Since the twin of the vertex with color 0 is given color $2k$, the color of at most one vertex from $M(v_{2,1}) \cup (M(v_{2,1}))'$ creates a problem for a vertex in $M(v_{k,i})$. Thus altogether, the color of at most 2 vertices from $M(v_{1,1}) \cup (M(v_{1,1}))' \cup M(v_{2,1}) \cup (M(v_{2,1}))'$ can create a problem for a vertex in $M(v_{k,i})$, $1 \leq i \leq k-1$. Since $k \geq 4$, for $1 \leq i \leq k-1$, we can permute the colors of $M(v_{k,i})$ to get a proper partial coloring.

Finally, let us assign the colors $\{1, 2, \ldots, k-1\}$ to the vertices of $(M(v_{k,i}))'$ (in any order). Similar to the previous argument, for $1 \leq i \leq k-1$, every vertex in $(M(v_{k,i}))'$ has at most two neighbors in $M(v_{1,1}) \cup M(v_{2,1})$. For the same reason, since $k \geq 4$, for $1 \leq i \leq k-1$, we can permute the colors of $(M(v_{k,i}))'$ to get a proper partial coloring. This partial coloring will ensure that $v_{1,1}, v_{2,1}, v_{k,1}, v_{k,2}, \ldots, v_{k,k-1}$ are c.d.vs. for the color classes $2k, 1, k+1, k+2, \ldots, 2k-1$, respectively.

Case 2. $k = 3$. Let $c(v_{1,1,1}) = 5$ and $c(v_{1,1,2}) = 3$. Let us assign the colors $\{6, 3\}$ to the vertices of $M(v_{2,1})$ and the colors $\{5, 2\}$ to the vertices of $M(v_{3,1})$. By (ii) of Observation 2.1, $M(v_{1,1})$ can only be adjacent to at most one vertex in $M(v_{3,1})$ and one vertex in $M(v_{2,1})$ and hence we can permute the colors to get a proper partial coloring. Next, let us assign the colors $\{4, 1\}$, $\{0, 2\}$, $\{0, 4\}$, $\{0, 1\}$ and $\{0, 2\}$ to the vertices of $M(v_{3,2})$, $(M(v_{1,1}))'$, $(M(v_{2,1}))'$, $(M(v_{3,1}))'$ and $(M(v_{3,2}))'$, respectively. Again for the same reason, we can permute the colors to get a proper coloring. In this case also it can be seen that $v_{1,1}, v_{2,1}, v_{3,1}, v_{3,2}$ are c.d.vs. for the color classes $6, 1, 4, 5$, respectively.

In both cases, we have ensured that $v_{1,1}, v_{2,1}, v_{k,1}, v_{k,2}, \ldots, v_{k,k-1}$ are c.d.vs. for the color classes $2k, 1, k+1, k+2, \ldots, 2k-1$, respectively. For the remaining uncolored vertices, since the degree of each of the uncolored vertex is at most $2k$, we can apply greedy coloring to get a proper coloring for the whole $\mu(G)$ using $2k+1$ colors. ■

Let us recall the concept of System of Distinct Representatives (SDR) for a family of subsets of a given finite set. Let $\mathcal{F} = \{A_\alpha : \alpha \in J\}$ be a family of sets. An SDR for the family $\mathcal{F}$ is a set of elements $\{x_\alpha : \alpha \in J\}$ such that $x_\alpha \in A_\alpha$ for every $\alpha \in J$ and $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$. Theorem 2.3 gives a necessary and sufficient condition for the existence of an SDR for a given family of finite sets.

Theorem 2.3 [9]. *Let $\mathcal{F} = \{A_i : 1 \leq i \leq r\}$ be a family of finite sets. Then $\mathcal{F}$ has an SDR if and only if the union of any k members of $\mathcal{F}$, $1 \leq k \leq r$, contains at least k elements.*

Let us next consider k-regular graphs with girth at least 6.

Observation 2.4. *Let G be a k-regular graph with girth at least 6. For $v \in V$, we have the following.*

(i) *$N_1(v)$ and $N_2(v)$ are independent sets.*

(ii) *Any two vertices can have at most one common neighbor.*

Theorem 2.5. *If G is a k-regular graph with girth at least 6, then $k + \left\lfloor \frac{k+1}{2} \right\rfloor \leq b(\mu(G)) \leq 2k+1$.*

Proof. Let G be a k-regular graph with girth at least 6. For $k = 1, 2$, the result is trivial. So let us assume that $k \geq 3$. For graphs with girth at least 7, by using Theorem 2.2, we see that $b(\mu(G)) = 2k + 1$. So let us consider G to be a regular graph with girth exactly 6. Here it can be easily seen that $m(\mu(G)) = 2k + 1$. Let $\{0, 1, 2, \ldots, 2k\}$ be the set of $2k + 1$ colors. Let $v \in V$.

Let us start by defining a proper coloring using $2k + 1$ colors, in such a way that for each $i \in \left\{0, 1, 2, \ldots, k + \left\lfloor \frac{k-1}{2} \right\rfloor\right\}$ there exists a vertex with color i which has a neighbor in each of the other color classes.

Let us begin by defining c as done in Theorem 2.2.

(i) $c(u) = k$, $c(v) = 0$, $c(v') = 2k$.

For $1 \leq i \leq k$

$c(v_i) = i$,

$c(v'_i) = k + i$.

(ii) For $1 \leq i \leq k - 1, 1 \leq j \leq k - 1$

$c(v'_{i,j}) = k + j$,

$c(v_{k,j}) = k + j$,

$c(v'_{k,j}) = j$.

(iii) For $1 \leq i \leq \left\lfloor \frac{k-1}{2} \right\rfloor, 1 \leq j \leq k - 1$

$$c(v_{k,i,j}) = \begin{cases} j - 1 & \text{for } i > j - 1, \\ j & \text{for } i \leq j - 1. \end{cases}$$

This partial coloring is proper.

For, $1 \leq i \leq k - 1$, let $C_i = \{1, 2, \ldots, k\} \backslash \{i\}$ and for $1 \leq j \leq k - 1$, let A_{ij} denote the set of colors in C_i which are not assigned to the neighbors of $v_{i,j}$. That is, $A_{ij} = C_i \backslash \{\text{set of colors given to the neighbors of } v_{i,j}\}$.

For, $1 \leq i, j \leq k - 1$, it is easy to observe that, if a color of A_{ij} is assigned to the vertex $v_{i,j}$, then the coloring is proper. Thus we shall show that for $1 \leq i, j \leq k - 1$, a color of A_{ij} is available to the vertex $v_{i,j}$ and that the vertices in $M(v_i)$ receive distinct colors.

For $1 \leq i \leq k - 1$, let $\mathcal{F}_i = \{A_{ij} : 1 \leq j \leq k - 1\}$. Considering $\mathcal{F}_i$ as a family of finite sets, if we show that $\mathcal{F}_i$ has an SDR, then for $1 \leq i, j \leq k - 1$, we have proved that a color of A_{ij} is available to the vertex $v_{i,j}$ and that the vertices in $M(v_i)$ receive distinct colors.

By using Theorem 2.3, it is enough to prove that, for $1 \leq i \leq k - 1$, the union of any t $(1 \leq t \leq k - 1)$ members of $\mathcal{F}_i$ contains at least t elements. Let $\mathcal{E} = \{A_{i\alpha_1}, A_{i\alpha_2}, \ldots, A_{i\alpha_t}\}$ be a class of any t members of $\mathcal{F}_i$, $1 \leq i \leq k - 1$.

Case 1. $t \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. By (ii) of Observation 2.4, every vertex in $M(v_i)$ can be adjacent to at most $\left\lfloor \frac{k-1}{2} \right\rfloor$ colored neighbors in $N_3(v)$. So for any j, $1 \leq j \leq k - 1, |A_{ij}| \geq \left\lceil \frac{k-1}{2} \right\rceil$. Thus $\left| \bigcup_{p=1}^{t} A_{i\alpha_p} \right| \geq \left\lfloor \frac{k-1}{2} \right\rfloor \geq t$.

Case 2. $t \geq \left\lfloor \frac{k-1}{2} \right\rfloor + 1$. Suppose $\left| \bigcup_{p=1}^{t} A_{i\alpha_p} \right| \leq t - 1 \leq k - 2$. Then there exists at least one color say $s \in C_i$, such that $s \notin A_{i\alpha_p}$, for $1 \leq p \leq t$. Then for $1 \leq p \leq t$, v_{i,α_p} is adjacent to a vertex with the color s. But by (ii) of Observation 2.4, every vertex with color s in $N_3(v)$ has at most one neighbor in $M(v_i)$. Also there are only $\lfloor \frac{k-1}{2} \rfloor$ vertices in $N_3(v)$ with color s. This is a contradiction to the fact that for every $1 \leq p \leq t$, v_{i,α_p} is adjacent to a vertex with color s. Thus $\left| \bigcup_{p=1}^{t} A_{i\alpha_p} \right| \geq t$.

Thus $\mathcal{F}_i$ has a SDR and this is true for any i such that $1 \leq i \leq k-1$. Hence the coloring c can be extended to a proper coloring including the vertices in $\bigcup_{i=1}^{k-1} M(v_i)$.

Next, for $1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor$, let us color the vertices of $\left(M(v_{k,j})\right)'$ with the colors $\{j\} \cup \{k+1, k+2, \ldots, k+j-1, k+j+1, \ldots, 2k-1\}$ as follows. One can see that j is the only color that creates a problem while coloring the vertices of $\left(M(v_{k,j})\right)'$. Since no two vertex can have more than one common neighbor, for $1 \leq j \leq \lfloor \frac{k-1}{2} \rfloor$, there can be at most $k-2$ vertices of $\left(M(v_{k,j})\right)'$ that can be adjacent to the vertices of $N_2(v) \backslash M(v_k)$ which are colored j (the number of vertices with color j in $N_2(v)$ is $k-2$). Even in the worst case, there exists a vertex in $\left(M(v_{k,j})\right)'$ which is not adjacent to a vertex with color j and hence by assigning the color j to that vertex and the rest of the colors in any order, we can extend the coloring c to the vertices of $\bigcup_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \left(M(v_{k,j})\right)'$. This partial coloring guarantees that $\left\{v, v_1, v_2, \ldots, v_k, v_{k,1}, v_{k,2}, \ldots, v_{k,\lfloor \frac{k-1}{2} \rfloor}\right\}$ are c.d.vs. of the colors $\left\{0, 1, 2, \ldots, k + \lfloor \frac{k-1}{2} \rfloor\right\}$. Let us color the remaining uncolored vertices by using greedy coloring technique. For $k + \lfloor \frac{k+1}{2} \rfloor \leq q \leq 2k$, if each of the vertices with color q is non-adjacent to the vertices of some color class, then by assigning one of the available color to each of the vertices with color q, we can eliminate the color q. If not, there exists a c.d.v. for the color q. Repeat this process for every q, $k + \lfloor \frac{k+1}{2} \rfloor \leq q \leq 2k$. This will yield a b-coloring using at least $k + \lfloor \frac{k+1}{2} \rfloor$ colors. Thus $b(\mu(G)) \geq k + \lfloor \frac{k+1}{2} \rfloor$. ■

Let us next consider k-regular graphs with girth 5.

Theorem 2.6. *For $k \geq 3$, if G is a k-regular graph with girth 5, diameter at least 5 and which contains no cycle of length 6, then $b(\mu(G)) = 2k+1$.*

Proof. Let G be a k-regular graph with girth 5, diameter at least 5 and which contains no cycle of length 6. Here also it can be seen that $m(\mu(G)) = 2k+1$ and hence it is enough to produce a b-coloring using $2k+1$ colors. Let $\{0, 1, 2, \ldots, 2k\}$ be the set of colors.

First, let us consider the case when $diam(G) = 5$. Also, let $v, w \in V$ such that $d(v, w) = 5$.

As done in Theorem 2.2, let us first partially color the graph to get c.d.vs. for each of the color classes. Let us do this by making $v, v_1, v_2, \ldots, v_{k-1}, u, w_2, w_3, \ldots,$

w_k, w as c.d.vs. for the color classes $0, 1, 2, \ldots, 2k$. Let us start by defining a coloring c for $\mu(G)$ as follows.

(i) $c(u) = k$, $c(v) = 0$, $c(v') = 2k$, $c(w) = 2k$ and $c(w') = 0$.

(ii) For $1 \le i \le k$
$c(v_i) = i$,
$c(v'_i) = k + i$.

Now, let us color the vertices of $N_2(v) \backslash N(v_k)$. For $1 \le i \le k-1$, let us assign the colors $C_i = \{1, 2, \ldots, k\} \backslash \{i\}$ to the vertices of $M(v_i)$ by using induction on i. For $i = 1$, let us assign the colors $\{2, 3, \ldots, k\}$ to the vertices of $M(v_1)$ in any order. Let s be a positive integer such that $2 \le s \le k-1$. Let us assume that for each ℓ such that $1 \le \ell \le s-1$, the vertices of $M(v_\ell)$ are assigned distinct colors from C_ℓ such that the coloring is proper. Now, let us assign the colors of C_s to the vertices of $M(v_s)$ by using SDR technique as done in Theorem 2.5.

For $1 \le j \le k-1$, let $A_{sj} = C_s \backslash \{$set of colors given to the neighbors of $v_{s,j}\}$.

Let $\mathcal{F}_s = \{A_{sj} : 1 \le j \le k-1\}$. By using Theorem 2.3, it is enough to prove that the union of any t $(1 \le t \le k-1)$ members of $\mathcal{F}_i$ contains at least t elements. Let $\mathcal{E} = \left\{A_{s\alpha_1}, A_{s\alpha_2}, \ldots, A_{s\alpha_t}\right\}$ be a class of t members of $\mathcal{F}_s$.

Since the girth of G is 5 and G contains no cycle of length 6, each of the vertices in $M(v_s)$ has at most one neighbor in $\bigcup_{\ell=1}^{s-1} M(v_\ell)$. So $|A_{sj}| \ge k-2$. Therefore for $t \le k-2$, the union of t members of $\mathcal{F}_s$ contains at least $k-2$ elements. Also the number of colored neighbors of $M(v_s)$ in $\bigcup_{\ell=1}^{s-1} M(v_\ell)$ is at most $k-2$. So there exists a vertex v_{sj_0} in $M(v_s)$ which has no colored neighbor in $\bigcup_{\ell=1}^{s-1} M(v_\ell)$ and hence $|A_{sj_0}| = k-1$. Thus even when $t = k-1$, the union contains at least $k-1$ elements. Thus, for $1 \le i \le k-1$, the vertices of $M(v_i)$ can be properly colored with distinct colors of C_i.

Next, let us color the vertices in $N_1(w) \cup N_2(w)$. Here also one can observe that, for $1 \le i \le k-1$, each of the vertices in $M(w_i)$ has at most one neighbor in $N_2(v)$ and similarly each of the vertices in $M(v_i)$ has at most one neighbor in $N_2(w)$. Hence the set of vertices in $N_2(w)$ has at most $k-1$ neighbors in $N_2(v)$ that are colored k. Without loss of generality, let $w_1, w_2, \ldots, w_k$ be the neighbors of w in G such that for $1 \le i \le k-1$, the number of neighbors of $M(w_i)$ in $N_2(v)$ which are colored k is at least the number of neighbors of $M(w_{i+1})$ in $N_2(v)$ which are colored k. Hence $M(v_k)$ has no neighbor in $N_2(v)$ which is colored k. Now, let us extend the coloring c as follows.

For $1 \le i \le k$
$c(w_i) = k + i - 1$,
$c(w'_i) = i - 1$.

For $2 \le i \le k$, let us assign the colors $D_i = \{k, k+1, \ldots, 2k-1\} \backslash \{k+i-1\}$ to the vertices of $M(w_i)$ by induction on i. For $i = 2$, let us assign the colors $\{k, k+2, k+3, \ldots, 2k-1\}$ to the vertices of $M(w_2)$ as follows. Since the number of neighbors of $M(w_1)$ in $N_2(v)$ which are colored k is the maximum, $M(w_2)$ will have at least a vertex which has no neighbor in $N_2(v)$ which is colored k and hence color k can be assigned to that vertex and the remaining colors can be assigned in any order to the remaining vertices of $M(w_2)$. Let r be a positive integer such that $3 \leq r \leq k-1$. Let us assume that for each ℓ such that $2 \leq \ell \leq r-1$, the vertices of $M(w_\ell)$ are assigned distinct colors from D_ℓ such that the coloring is proper. Now, let us assign the colors of D_r to the vertices of $M(w_r)$ by using SDR technique as done in Theorem 2.5.

For $1 \leq j \leq k-1$, let $B_{rj} = D_r \setminus \{\text{set of colors given to the neighbors of } w_{r,j}\}$.

Let $\mathcal{H}_r = \{B_{rj} : 1 \leq j \leq k-1\}$. By using Theorem 2.3, it is enough to prove that the union of any t $(1 \leq t \leq k-1)$ members of $\mathcal{H}_r$ contains at least t elements. Let $\mathcal{S} = \left\{B_{r\beta_1}, B_{r\beta_2}, \ldots, B_{r\beta_t}\right\}$ be a class of any t members of $\mathcal{H}_r$.

Since the girth of G is 5 and G contains no cycle of length 6, each of the vertices in $M(w_r)$ has at most one colored neighbor in $\bigcup_{\ell=2}^{r-1} M(w_\ell)$ and has at most one colored neighbor in $N_2(v)$. So $|B_{rj}| \geq k-3$. Therefore the union of any $t \leq k-3$ members of $\mathcal{H}_r$ contains at least $k-3$ elements. Also the number of colored neighbors of $M(w_r)$ in $\bigcup_{\ell=2}^{r-1} M(w_\ell)$ is at most $k-3$. So there exist two vertices w_{rj_1}, w_{rj_2} in $M(w_r)$ which have no colored neighbor in $\bigcup_{\ell=2}^{r-1} M(w_\ell)$ and hence $|B_{rj_1}| \geq k-2$ and $|B_{rj_2}| \geq k-2$. Hence the union of any $k-2$ members of $\mathcal{H}_r$ will also contain at least $k-2$ elements.

Suppose the union of all the $k-1$ members of $\mathcal{H}_r$ contains only $k-2$ elements. Then all the vertices of $M(w_r)$ have distinct neighbors with some particular color. Since $r-1 \leq k-2$, the only possibility for this color is k. Also $M(w_r)$ has at most $r-2$ vertices that have neighbors in $N_2(w)$ which are colored k. Depending on r, let us consider two cases.

Case 1. $r \leq \left\lceil \frac{k-1}{2} \right\rceil$. Then $M(w_r)$ has at least $(k-1)-(r-2) \geq (k-1) - \left(\left\lceil \frac{k-1}{2} \right\rceil - 2\right) = \left\lfloor \frac{k-1}{2} \right\rfloor + 2$ neighbors in $N_2(v)$ with color k. We know that, for $1 \leq j \leq r$, the number of neighbors of $M(w_j)$ in $N_2(v)$ with color k is at least the number of neighbors of $M(w_r)$ in $N_2(v)$ with color k. Since $r \geq 3$, the number of neighbors of $M_2(w)$ in $N_2(v)$ with color k is at least $3\left(\left\lfloor \frac{k-1}{2} \right\rfloor + 2\right) > k-1$, a contradiction.

Case 2. $r \geq \left\lceil \frac{k-1}{2} \right\rceil + 1$. Then $M(w_r)$ has at least $(k-1)-(r-2) \geq r-(r-2) = 2$ neighbors in $N_2(v)$ which are colored k. For the same reason as mentioned in Case 1, for $1 \leq j \leq r$, the number of neighbors of $M(w_j)$ in $N_2(v)$ with color k is at least 2 and hence $k-1 \geq 2r \geq 2\left(\left\lceil \frac{k-1}{2} \right\rceil + 1\right)$, a contradiction.

Finally, for $M(w_k)$, since none of the vertices in $M(w_k)$ has a neighbor in $N_2(v)$ which is colored k, argument similar to those given for coloring $M(v_{k-1})$ will also work in coloring the vertices of $M(w_k)$ with distinct color of D_k. Therefore for $2 \leq i \leq k$, we can assign the colors $\{k, k+1, \ldots, 2k-1\} \setminus \{k+i-1\}$ properly to the vertices of $M(w_i)$.

Also the SDR technique will ensures that, for $1 \le i \le k-1$ and $2 \le j \le k$, we can assign the colors $\{k+1, k+2, \ldots, 2k-1\}$ and $\{1, 2, \ldots, k-1\}$ to the vertices of $(M(v_i))'$ and $(M(w_j))'$, respectively and still the coloring is proper. This partial coloring ensures that $v, v_1, v_2, \ldots, v_{k-1}, u, w_2, w_3, \ldots, w_k, w$ are the c.d.vs. for the color classes $0, 1, 2, \ldots, 2k$, respectively. By using greedy coloring technique, the remaining uncolored vertices can be given a proper coloring using $2k+1$ colors. When the $diam(G) \ge 6$, it can be easily seen that none of the vertices in $N_2(v)$ can have a neighbor in $N_2(w)$ and hence a similar coloring will still yield a b-coloring using $2k+1$ colors. ■

Let us next find the b-spectrum of Mycielskian of k-regular graph with girth at least 7.

Theorem 2.7. *If G is a k-regular graph with girth at least 7, then $\{k+3, k+4, \ldots, 2k\} \subseteq S_b(\mu(G))$.*

Proof. Let G be a k-regular graph with girth at least 7. Let $s \in \{k+3, k+4, \ldots, 2k\}$ and $\{0, 1, 2, \ldots, s-1\}$ be the set of colors. Let us now define a b-coloring c for $\mu(G)$ using s colors as follows. Let $v \in V(G)$.

(i) $c(u) = k$, $c(v) = 0$.

(ii) For $1 \le i \le k$

$$c(v_i) = i.$$

(iii) For $1 \le i \le s-k-1$

$$c(v'_i) = k+i.$$

(iv) For $1 \le i \le k-1$, $1 \le j \le k-1$

$$c(v_{i,j}) = \begin{cases} j & \text{for } i > j, \\ j+1 & \text{for } i \le j, \end{cases}$$

$$c(v'_{k,j}) = j.$$

(v) For $1 \le i \le k-1$, $1 \le j \le s-k-1$

$$c(v'_{i,j}) = k+j,$$

$$c(v_{k,j}) = k+j.$$

(vi) For $1 \le i \le s-k-1$, $2 \le j \le s-k-1$

$$c(v_{k,i,1}) = 0,$$

$$c(v_{k,i,j}) = \begin{cases} k+j-1 & \text{for } i \ge j, \\ k+j & \text{for } i < j. \end{cases}$$

(vii) For $1 \le i \le s-k-1$, $1 \le j \le k-1$

$$c(v'_{k,i,j}) = j.$$

Since girth of G is at least 7, the sets $N_1(v), N_2(v)$ and $N_2(v_k) \cap N_3(v)$ are independent. Also for $1 \leq i \leq 3$, every vertex in $N_i(v)$ will has exactly one neighbor in $N_{i-1}(v)$. This guarantees that the given partial coloring c is proper and that the vertices $v, v_1, v_2, \ldots, v_k, v_{k,1}, v_{k,2}, \ldots, v_{k,s-k-1}$ are c.d.vs. for the color classes $0, 1, 2, \ldots, s-1$, respectively.

Next, let us color the remaining uncolored vertices of V. Let w be an uncolored vertex in V. Note that an uncolored vertex in V can have at most k colored neighbors in V. Let us consider the number of colored neighbors of w in V'. Clearly $w \notin N_1(v)$. Let us assume that $w \in N_2(v)$. Since $N_2(v)$ is independent and no neighbors of w in $N_3(v)$ are colored, there can be at most one colored neighbor of w in $(N_1(v))'$ and hence in this case, there is at most 1 colored neighbor of w in V'. Next, let us assume that $w \in N_3(v)$. Recall that, every vertex in $N_3(v)$ has at most one neighbor in $N_2(v)$ and hence one colored neighbor in $(N_2(v))'$. In $N_3(v)$, by using (iii) of Observation 2.1, w can have at most one neighbor in $N_2(v_k) \cap N_3(v)$ and hence at most one colored neighbor in $(N_3(v))'$. Hence in this case, there are at most 2 colored neighbors of w in V'. When $w \in N_4(v)$, it is easy to observe that the number of colored neighbors in $(N_3(v))'$ is at most 1 and when $w \in N_i(v)$, $i \geq 5$, w has no colored neighbors in V'. Therefore, for any uncolored vertex in V, the number of colored neighbors in V is at most k and in V' is at most 2 and hence is at most $k+2$ in $\mu(G)$. Since $s \geq k+3$, we always have an available color for all the uncolored vertices of V. Finally, since the degree of any vertex in V' is $k+1$, we can extend this partial coloring c to a b-coloring of the whole graph $\mu(G)$ using s colors. Therefore, $\{k+3, k+4, \ldots, 2k\} \subseteq S_b(\mu(G))$. ■

Let us recall a sufficient condition for the b-continuity of regular graphs given in [4].

Theorem 2.8 [4]**.** *If G is a k-regular graph with girth at least 6 having no cycles of length 7, then G is b-continuous.*

As a consequence of Theorem 2.7 and Theorem 2.8, we see that the Mycielskian of all k-regular graphs with girth at least 8 are b-continuous.

Theorem 2.9. *If G is a k-regular graph with girth at least 8, then $\mu(G)$ is b-continuous.*

Proof. Let G be a k-regular graph with girth at least 8. By using Theorem 2.8, $S_b(G) = \{\chi(G), \chi(G)+1, \ldots, b(G) = k+1\}$ and hence for every $\ell \in S_b(G)$, there exists a b-coloring for G using ℓ colors. This can be extended to a b-coloring for $\mu(G)$ using $\ell+1$ colors by coloring each of the twin vertex with the color of its corresponding vertex and by coloring the root vertex with $\ell+1$.

Hence $\{\chi(\mu(G)) = \chi(G)+1, \chi(G)+2, \ldots, b(G)+1 = k+2\} \subseteq S_b(\mu(G))$. Also by using Theorem 2.2 and Theorem 2.7, we see that $\{k+3, k+4, \ldots, 2k, 2k+1 = b(\mu(G))\} \subseteq S_b(\mu(G))$ and hence $\mu(G)$ is b-continuous. ∎

3. Exact Value of $b(\mu(G))$ for Some Families of Graphs

In [1], it has been shown that the b-chromatic number of the Mycielskian of split graph and $K_{n,n}$ minus a perfect matching of the graph lies in the interval $[b+1, 2b-1]$. In Section 3, we find the exact values of $b(\mu(G))$ of these families of graphs. In addition, we find the exact value of $b(\mu(G))$ when $b(G) = 2$. For a vertex $v \in V$, let $\overline{N}(v) = \{w \in V : vw \notin E, w \neq v\}$.

Theorem 3.1 [16]**.** *Let G be bipartite and $G_1, G_2, \ldots, G_r$ be its connected components such that $|G_i| \geq 3$ for $1 \leq i \leq r$. Then $b(G) \geq 3$ if and only if*

(i) *$r = 1$ and $X \subseteq \bigcup_{v \in Y} \overline{N}(v)$ or $Y \subseteq \bigcup_{v \in X} \overline{N}(v)$ where X and Y are the bipartite classes of G_1, or*

(ii) *$r = 2$ and at least one of G_1, G_2 is not complete bipartite or*

(iii) *$r \geq 3$.*

Equivalently, we can say that for a bipartite graph G with connected components $G_1, G_2, \ldots, G_r$, $b(G) = 2$ if and only if

(i) *$r = 1$ and there exist vertices $x_0 \in X$ and $y_0 \in Y$ such that $N(x_0) = Y$ and $N(y_0) = X$ where X and Y are the bipartite classes of G_1 (we denote these graphs as type* (i)*), or*

(ii) *for $r \geq 2$*

   (a) *G_1 is of type* (i) *and every other component is either a K_2 or a K_1, or*

   (b) *for $1 \leq i \leq r$, G_i is a complete bipartite graph (with at least one G_i such that $|G_i| \geq 2$) and at least $r-2$ components being K_2 or K_1.*

Theorem 3.2. *If G is a graph with $b(G) = 2$, then $b(\mu(G)) = 3$.*

Proof. Let G be a graph with $b(G) = 2$. Then $3 \leq \chi(G) + 1 = \chi(\mu(G)) \leq b(\mu(G))$. Let us first assume that G is connected and let $V = X \cup Y$ where X and Y are the bipartite classes of G. Then by using (i) of the equivalent form of Theorem 3.1, there exist vertices $x_0 \in X$ and $y_0 \in Y$ such that $N(x_0) = Y$ and $N(y_0) = X$. Suppose $b(\mu(G)) = \ell \geq 4$. Then there exists a b-coloring say c of $\mu(G)$ using ℓ colors. Let $\{1, 2, \ldots, \ell\}$ be the set of colors. Without loss of generality, let 1 and 2 be the colors given to x_0 and y_0. Since the neighborhood set of any vertex in X (likewise Y) is a subset of $N(x_0)$ $(N(y_0))$, no vertices in X or Y can be c.d.vs. for any color class $m \geq 3$. So, the c.d.vs. for any color class $m \geq 3$ must be in $X' \cup Y' \cup \{u\}$.

Case 1. u is a c.d.v. Without loss of generality, let $c(u) = 3$. Since every vertex in $X' \cup Y'$ is not adjacent to either color 1 or color 2, none of the vertices in $\mu(G)$ can be a c.d.v. of any color class $m \geq 4$, a contradiction.

Case 2. u is not a c.d.v. The c.d.vs. of the color class 3 are in $X' \cup Y'$. Without loss of generality, let X' contain one of the c.d.vs. of the color class 3. Then $c(u) = 1$. Clearly $c(x_0')$ cannot be 1 or 2. Also the neighborhood set of any vertex in X' is a subset of $N(x_0')$. Thus, $c(x_0') = 3$ and this in turn implies that no vertex in X' can be c.d.v. of any color class $m \geq 4$. Thus the c.d.vs. of color class 4 are only in Y'. But this is not possible, as none of the vertices in Y' is adjacent with a vertex with color 2. Thus there exists no c.d.v. for the color class 4, a contradiction.

Thus we see that when G is connected, $b(\mu(G)) \leq 3$ and hence $b(\mu(G)) = 3$. Next, let us consider the case when G is not connected. Then by using (ii) of the equivalent form of Theorem 3.1, for some $r \geq 2$, $G = \bigcup_{1 \leq i \leq r} G_i$, such that (a) G_1 is of type (i) and every other component is either a K_2 or a K_1, or (b) for $1 \leq i \leq r$, G_i is a complete bipartite graph (with at least one G_i such that $|G_i| \geq 2$) and at least $r - 2$ components being K_2 or K_1.

Let us consider the first possibility. Here for $2 \leq i \leq r$, the degree of the vertices in $G_i \cup G_i'$ is at most 2. Thus by using arguments similar to those given in the connected case, G will have no b-coloring using 4 colors. Thus $b(\mu(G)) \leq 3$.

Let us next consider the second possibility. Suppose $b(\mu(G)) = p \geq 4$. Then there exists a b-coloring, say ϕ of $\mu(G)$ using p colors. Let $\{1, 2, \ldots, p\}$ be the set of colors. Let $\phi(u) = 1$. Since $G_j = K_2$ or K_1, for $j \geq 3$, none of the c.d.vs. will be in $G_j \cup G_j'$. So, either $G_1 \cup G_1'$ or $G_2 \cup G_2'$ will contain at least two c.d.vs. of distinct color classes (other than 1).

Without loss of generality, let $G_1 \cup G_1'$ contain two c.d.vs. say for color classes 2 and 3. Let $G_1 = X_1 \cup Y_1$, where X_1 and Y_1 are the bipartition classes of G_1. Let us first consider the case when the c.d.v. of either 2 or 3 is in $X_1 \cup Y_1$, say $x_1 \in X_1$ with $\phi(x_1) = 2$. Then every color other than 2 is present in $Y_1 \cup Y_1'$. In particular, 1 is present in Y_1. This guarantees that no vertex in $Y_1 \cup (X_1 \backslash \{x_1\})$ is a c.d.v of any color class $q \geq 3$. Since the neighbors (excluding u) of any vertex in X_1' is a subset of $N(x_1)$, no vertex in X_1' is a c.d.v. of a color class $m \geq 3$. Finally, no vertex in Y_1' can have a neighbor with color $m \geq 3$ and hence cannot be a c.d.v. of a color class $m \geq 3$. Thus the c.d.vs. of 2 and 3 are in $X_1' \cup Y_1'$. But even in this case, with similar techniques we can show that it is also not possible. Thus $b(\mu(G)) = 3$. ■

Theorem 3.3. *If G is a split graph, then $b(\mu(G)) = b(G) + 1 = \omega(G) + 1$.*

Proof. Let G be a split graph. Then $\omega(G) + 1 \leq \chi(\mu(G)) \leq b(\mu(G))$. Then the vertex set V can be partitioned into two sets, one inducing a clique and the other inducing an independent set. Let $V = A \cup B$ where A induces a maximum clique and B is an independent set. Clearly $|A| = \omega(G)$.

Suppose $b(\mu(G)) = \ell \geq \omega(G)+2$. Then there exists a b-coloring say c of $\mu(G)$ using ℓ colors. Let $\{1, 2, \ldots, \ell\}$ be the set of colors. Without loss of generality, let $1, 2, \ldots, \omega(G)$ be the colors assigned to the vertices of A. Since the degree of the vertices in B is at most $\omega(G) - 1$, none of the vertices in B' can be a c.d.v. in $\mu(G)$.

Case (i) B contains a c.d.v. Let $v \in B$ be a c.d.v. of the color, say $\omega(G) + 1$. Since A is a maximum clique, there exists at least one vertex $w \in A$ which is not adjacent to v. It can also be observed that $N(v) \subseteq N(w)$ and hence v cannot be adjacent to a vertex whose color is $c(w)$, a contradiction.

Case (ii) B contains no c.d.vs. This concludes that all the c.d.vs. must be in $A \cup A' \cup \{u\}$. Since $|A| = \omega(G)$ and $\ell \geq \omega(G) + 2$, it can be seen that A' contains at least one c.d.v., say, w_1' of a color $\omega(G) + 1$. Then $c(u) = c(w_1)$ and hence the c.d.v. of $\omega(G) + 2$ must also be in A', say w_2'. This again forces $c(u) = c(w_2)$, a contradiction. ■

It can be observed that, not all k-regular graphs of girth 4 have $b(G) = k+1$, see for instance [5]. While considering k-regular graphs with girth 4 and $b(G) = k + 1$, we shall show that these assumptions does not imply that $b(\mu(G))$ is very close to $2k + 1$.

Theorem 3.4. *If $G = K_{n,n} - PM$ where PM is a perfect matching of $K_{n,n}$, then $b(\mu(G)) = n + \left\lceil \frac{n-1}{2} \right\rceil$, for $n \geq 3$.*

Proof. Let $G = K_{n,n} - PM$ where PM is a perfect matching of $K_{n,n}$. Let $V = X \cup Y$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of G and $\{x_1y_1, x_2y_2, \ldots, x_ny_n\}$ be the PM.

Let us first show that $b(\mu(G)) \leq n + \left\lceil \frac{n-1}{2} \right\rceil$. On the contrary, let us suppose that $b(\mu(G)) = \ell \geq n + \left\lceil \frac{n-1}{2} \right\rceil + 1$ and let c be a b-coloring using ℓ colors. Let C denote a c.d.s. of c. Without loss of generality, let $c(u) = 1$. Let us start with the following observations on c.

(i) The c.d.vs. of c can only be present in $X \cup Y \cup \{u\}$.

(ii) Since $\ell \geq n + \left\lceil \frac{n-1}{2} \right\rceil + 1 \geq n + 2$, there exist at least one c.d.v. in X and at least one c.d.v. in Y.

(iii) There exists an $i \in \{1, 2, \ldots, n\}$, say $i = 1$, such that $c(x_1) = c(y_1) = 1$ (Otherwise, none of the vertices in X and Y can be adjacent to the color 1 and hence there will be no c.d.v. in X or Y or both, a contradiction).

Let $S = \{i \in \{1, 2, \ldots, n\} : x_i \text{ and } y_i \text{ belong to } C\}$. Suppose $|S| = p \leq \left\lceil \frac{n-1}{2} \right\rceil$. Then the number of c.d.vs. of distinct colors present in $X \cup Y$ is at most $n + \left\lceil \frac{n-1}{2} \right\rceil$.

By using observation (iii), we see that $c(x_1) = c(y_1) = c(u) = 1$ and hence the number of c.d.vs. of distinct color classes present in $X \cup Y \cup \{u\}$ is at most $n + \lceil \frac{n-1}{2} \rceil$, a contradiction. Thus $|S| \geq \lceil \frac{n-1}{2} \rceil + 1$.

For $i, j \in S$, x_i must have a neighbor with the color of x_j and vice versa. The only possibility for this to happen is that $c(y'_i) = c(x_i)$ and $c(y'_j) = c(x_j)$. Thus, for every $i \in S$, $c(y'_i) = c(x_i)$ and for similar reasons $c(x'_i) = c(y_i)$. We know that, for every $i \in S$, x_i is a c.d.v. and hence must have a neighbor whose color is $c(y_i)$. Since for $i \in S$, $c(x'_i) = c(y_i)$, the color of the vertices in $Y \backslash \{y_i\}$ cannot be $c(y_i)$ and hence one of the vertices in X' must have received the color $c(y_i)$. Thus $n = |Y'| \geq 2|S| \geq 2\left(\lceil \frac{n-1}{2} \rceil + 1\right)$, a contradiction. Hence $b(\mu(G)) \leq n + \lceil \frac{n-1}{2} \rceil$. Let us now show that we can define a b-coloring ϕ using $n + \lceil \frac{n-1}{2} \rceil$ colors as follows. Let $\left\{1, 2, \ldots, n + \lceil \frac{n-1}{2} \rceil\right\}$ be the set of colors.

(i) $\phi(u) = 1$.

(ii) $\phi(x_i) = i$ for $1 \leq i \leq n$.

(iii) $\phi(y_i) = \begin{cases} n+i-1 & \text{for } 2 \leq i \leq \lceil \frac{n-1}{2} \rceil + 1, \\ i & \text{for } \lceil \frac{n-1}{2} \rceil + 2 \leq i \leq n \text{ and } i = 1. \end{cases}$

(iv) $\phi(x'_i) = \begin{cases} 2 & \text{for } i = 1, \\ n+i-1 & \text{for } 2 \leq i \leq \lceil \frac{n-1}{2} \rceil + 1, \\ i - \lfloor \frac{n-1}{2} \rfloor & \text{for } \lceil \frac{n-1}{2} \rceil + 2 \leq i \leq n. \end{cases}$

(v) $\phi(y'_i) = \begin{cases} n+1 & \text{for } i = 1, \\ i & \text{for } 2 \leq i \leq \lceil \frac{n-1}{2} \rceil + 1, \\ i + \lceil \frac{n-1}{2} \rceil & \text{for } \lceil \frac{n-1}{2} \rceil + 2 \leq i \leq n. \end{cases}$

In a routine way, one can check that the given coloring ϕ is proper and $x_1, x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_{\lceil \frac{n-1}{2} \rceil + 1}$ are the c.d.vs. of the color classes $1, 2, 3, \ldots, n + \lceil \frac{n-1}{2} \rceil$, respectively. ∎

4. b-COLORING OF GENERALIZED MYCIELSKIAN OF SOME GRAPHS

In Section 4, we show that the results in Section 2 can be generalized to the generalized Mycielskian of regular graphs. For $m \geq 2$, while considering the generalized Mycielskian of k-regular graphs, it can be seen that the number of vertices with degree $2k$ is $(m-1)n$ and hence it can be shown that $b(\mu_m(G)) = 2k + 1$ even when G is a k-regular graph with girth at least 6.

Theorem 4.1. *For $m \geq 2$, if G is a k-regular graph with girth at least 6, then $b(\mu_m(G)) = 2k + 1$.*

Proof. Let $G = (V^0, E^0)$ be a k-regular graph with girth at least 6. Here also

$m(\mu_m(G)) = 2k+1$ and hence it is enough to show that there exists a b-coloring using $2k+1$ colors. Let $\{0, 1, \ldots, 2k\}$ be the set of colors. Let us first partially color the graph to get c.d.vs. for each of the color classes. This is done by defining a coloring c for $\mu_m(G)$ as follows. Let $v^0 \in V^0$.

(i) $c(u) = k+1$, $c(v^0) = 0$, $c(v^1) = 2k$, $c(v^2) = k$,

(ii) for $1 \le i \le k$

$$c(v_i^0) = i,$$
$$c(v_i^1) = k+i,$$

(iii) for $1 \le i \le k-1, 1 \le j \le k-1$

$$c(v_{i,j}^0) = \begin{cases} k+j & \text{for } i \ne j, \\ j+1 & \text{for } i = j \text{ and } i \ne k-1, \end{cases}$$
$$c(v_{k-1,k-1}^0) = 1,$$
$$c(v_{k,j}^0) = k+j,$$
$$c(v_{i,j}^1) = \begin{cases} k+j & \text{for } i = j, \\ k & \text{for } i = j-1 \quad \text{or } (i,j) = (k-1,1), \\ j & \text{for } i \ne j, i \ne j-1 \quad \text{and } (i,j) \ne (k-1,1), \end{cases}$$
$$c(v_{k,j}^1) = j,$$
$$c(v_{i,j}^2) = \begin{cases} 2k & \text{for } i = j-1 \quad \text{or } (i,j) = (k-1,1), \\ j & \text{for } i \ne j-1 \quad \text{and } (i,j) \ne (k-1,1), \end{cases}$$
$$c(v_{k,j}^2) = j.$$

One can easily see that the given partial coloring is proper and the vertices $v^0, v_1^0, v_2^0, \ldots, v_k^0, v_1^1, v_2^1, \ldots, v_k^1$ are the c.d.vs. for the color classes $0, 1, 2, \ldots, 2k$, respectively. Since the degree of each of the uncolored vertex is at most $2k$, we can apply greedy coloring to get a proper coloring for the remaining vertices of $\mu_m(G)$ using $2k+1$ colors. ■

Theorem 4.2. *If G is a k-regular graph with girth at least 7, then $\{k+3, k+4, \ldots, 2k\} \subseteq S_b(\mu_m(G))$.*

Proof. Let $G = (V^0, E^0)$ be a k-regular graph with girth at least 7. Let $s \in \{k+3, k+4, \ldots, 2k\}$ and $\{0, 1, 2, \ldots, s-1\}$ be the set of colors. By Theorem 2.7, the result is true for $m = 1$. So, let us assume that $m \ge 2$. While coloring the vertices of $\mu_m(G)$, we can use the same technique used in Theorem 2.7 to color the vertices of $V^0 \cup V^1 \cup \{u\}$. Now color the vertices of $V^2, V^3, \ldots, V^m$ successively. For $2 \le p \le m-1$, the number of colored neighbors of any vertex in V^p is at most k and the number of colored neighbors of any vertex in V^m is at most $k+1$. Since $s \ge k+3$, all the vertices in $V^2, V^3, \ldots, V^m$ can be properly colored.

Therefore $\mu_m(G)$ has a b-coloring using s colors. Hence $\{k+3, k+4, \ldots, 2k\} \subseteq S_b(\mu_m(G))$. ■

As a consequence of Theorem 2.8, Theorem 4.2 and by using similar technique as used in Theorem 2.9, we see that the generalized Mycielskian of all k-regular graph with girth at least 8 are b-continuous.

Corollary 4.3. *If G is a k-regular graph with girth at least* 8*, then $\mu_m(G)$ is b-continuous.*

In a similar way, combining the techniques used in Theorem 2.6 and Theorem 4.2, we can establish Corollary 4.4.

Corollary 4.4. *If G is a k-regular graph with girth* 5*, diameter at least* 5 *and containing no cycles of length* 6*, then $b(\mu_m(G)) = 2k+1$.*

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