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A NOTE ON ADDITIVE GROUPS OF SOME SPECIFIC TORSION-FREE RINGS OF RANK THREE AND MIXED ASSOCIATIVE RINGS

Alireza Najafizadeh

Department of mathematics
Payame Noor University, I.R. of Iran
e-mail: najafizadeh@pnu.ac.ir

and

Mateusz Woronowicz

Institute of Mathematics
University of Białystok, Poland
e-mail: mworonowicz@math.uwb.edu.pl

Abstract

It is studied how rank two pure subgroups of a torsion-free Abelian group of rank three influences its structure and type set. In particular, the criterion for such a subgroup B to be a direct summand of a torsion-free Abelian group of rank three with the finite type set containing the greatest element which does not belong to the type set of B, is presented. Some results for nil groups and the square subgroup of a decomposable torsion-free Abelian group are also achieved. Moreover, new results for mixed Abelian groups supporting only associative rings are obtained. In particular, the first example of an Abelian group supporting only associative rings but not only commutative rings is given.

Keywords: rank, type, square subgroup, additive groups of rings.

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1. Introduction and Preliminaries

In [18], Hasani, Karimi, Najafizadeh and Sadeghi have studied the square subgroup of a torsion-free Abelian group $A = A_1 \oplus A_2$ of rank three, assuming that

A_i is a group of rank i, A_2 is not a nil group and either $t(A_1) \in T(A_2)$ or $t(A_1)$ is incomparable to any type belonging to $T(A_2)$. This research was continued by Woronowicz in [26]. The aim of the first section of that note is to study torsion-free Abelian groups of rank three with a bit more general assumptions. Namely, we investigate how rank two pure subgroups of a torsion-free Abelian group of rank three influences its structure and type set. In particular, we present the criterion for such a subgroup B to be a direct summand of a torsion-free Abelian group of rank three with the finite type set containing the greatest element which does not belong to the type set of B. Some new results for nil groups and the square subgroup of a decomposable torsion-free Abelian group are also obtained.

The second section is inspired by papers [7, 16] concerning additive groups of associative and commutative rings. We indicate new examples of Abelian mixed groups which support only associative rings. In particular, we construct the first example of such a group which does not support only commutative rings. Furthermore, we slightly generalize some results from [7].

The topic has a long history in algebra. Its starting point can be localized in the middle of the 20^{th} century (see, e.g., [10, 21]). Several authors have followed this subject of study which resulted in next papers (see, e.g., [1,11,13,20]). Further research, conducted with the momentous contribution of Feigelstock, led to the monograph [14] and its complement [15]. Currently, torsion-free and mixed groups are generating renewed interest (see, e.g., [3–8,16,19,24]). However, many of basic aspects concerning the structure and the type set of a torsion-free Abelian group of rank greater than two remain unknown. Many of natural questions related to additive groups of associative rings are also unanswered. This is a main motivation for that note.

Symbols $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ stand for the the field of rationals, the ring of integers and the set of all positive integers, respectively. Throughout this paper all groups are Abelian and written additively. The square subgroup of an Abelian group A can be understood as a subgroup of A generated by the squares of all possible rings defined on A. It is denoted by $\square A$ or $\square_a A$ if we restrict our consideration to associative rings. The notion comes from [23] and has been examined in [2, 5]. In accordance with [14], by the type set of a torsion-free Abelian group A we mean the set $T(A) = \{t(a)\colon a \in A,\ a \neq 0\}$. The rank and torsion-free rank of A are denoted by $r(A)$ and $r_0(A)$, respectively. A ring R is said to be semi-prime if it contains no nonzero nilpotent ideals. If $x \in R$, then the symbol $[x]$ stands for the subring of R generated by x. The additive group of a ring R is denoted by R^+. The notation $I \lhd R$ means that I is an ideal of R. All other designations are consistent with generally accepted standards (see, e.g., [17]).

2. ON SOME SPECIFIC TORSION-FREE ABELIAN GROUPS OF RANK THREE, RINGS ON THEM AND THEIR TYPE SETS

The complete preliminary knowledge of characteristics, types and type set of an Abelian torsion-free group is contained in [14, 17]. Retaining the notation of [17] we remind the reader only the most basic properties of characteristics and types which will be used often throughout the paper. They are listed in the following lemma.

Lemma 2.1. *Let A, B and C be torsion-free Abelian groups and let G be a pure subgroup of A.*

(1) $\chi(a+b) \geq \chi(a) \cap \chi(b)$ *and* $t(a+b) \geq t(a) \cap t(b)$ *for all* $a, b \in A$.

(2) *If* $A = B \oplus C$, $b \in B$ *and* $c \in C$, *then* $\chi(b+c) = \chi(b) \cap \chi(c)$ *and* $t(b+c) = t(b) \cap t(c)$.

(3) $\chi_{A/G}(a+G) = \bigcup_{x \in a+G} \chi_A(x)$ *for every* $a \in A$.

(4) $t_G(g) = t_A(g)$ *for every* $g \in G$.

(5) *If* a *and* b *are dependent elements of* A, *then* $t(a) = t(b)$.

(6) *If* $f \in \mathrm{Hom(A,B)}$, *then* $\chi_A\big(f(a)\big) \geq \chi_B(a)$ *and* $t\big(f(a)\big) \geq t(a)$ *for every* $a \in A$.

(7) $t(a) \cdot t(b) \geq t(a)$ *for all* $a, b \in A$.

(8) *If* $R = (A, \star)$ *is a ring, then* $t(a \star b) \geq t(a) \cdot t(b)$ *for all* $a, b \in A$.

Proof. The proofs of (1)–(7) can be found in [14, 17]. Property (8) is placed in [3, Lemma 1] (if $a \star b = 0$, then the assertion is obvious). ■

Proposition 2.2. *Let A be a torsion-free Abelian group of rank three with $T(A)$ containing distinct maximal types t_1, t_2, t_3 and let x_1, x_2, x_3 be elements of A respectively of types t_1, t_2, t_3. If $\Box\langle x_1, x_2\rangle_* \neq \{0\}$, then $A = \langle x_1, x_2, x_3\rangle_*$. Moreover, if $\Box A \neq \{0\}$, then $T(A)$ contains no more maximal types and $t_i^2 = t_i$ for some $i \in \{1, 2, 3\}$.*

Proof. Let $G = \langle x_1, x_2\rangle_*$. Suppose, contrary to our claim, that the system $\{x_1, x_2, x_3\}$ is dependent. Since elements x_1, x_2, x_3 are pairwise independent (cf. (5)), there exist nonzero integers k_1, k_2, k_3 such that $k_3x_3 = k_1x_1 + k_2x_2$. Hence $x_3 \in G$. Moreover, $t_G(x_3) = t_3$ by (4). Consequently, $T(G)$ contains three maximal elements, contrary to [22, Theorem 3.3]. Therefore $A = \langle x_1, x_2, x_3\rangle_*$. Now, the second assertion follows at once form [19, Theorems 3.5 and 3.7]. ■

A direct consequence of Proposition 2.2 and [22, p. 204] is the following:

Corollary 2.3. *Let A be a torsion-free Abelian group of rank three with $\Box A \neq \{0\}$. If there exists a rank two pure subgroup G of A such that $\Box G \neq \{0\}$, $\left|T(G)\right| = 3$ and every maximal type of $T(G)$ is maximal in $T(A)$, then $T(A)$ contains at most three maximal types and at least one of these types is idempotent.*

Proposition 2.4. *If A is a torsion-free Abelian group of rank three with $T(A)$ containing a chain $t_1 < t_2 < t_3$, then t_1 is the least element of $T(A)$. Moreover, if A supports a semi-prime ring, then $t_3^2 = t_3$.*

Proof. Since $t_1 < t_2 < t_3$, we get $A(t_3) \subsetneq A(t_2) \subsetneq A(t_1)$. Take any $x_1, x_2, x_3 \in A$ satisfying $t(x_i) = t_i$ for each $i \in \{1, 2, 3\}$. Then x_1, x_2, x_3 are pairwise independent. Hence $r\big(A(t_1)\big) = 2$ or $r\big(A(t_1)\big) = 3$. Since $A(t_2)$ is a pure subgroup of A, we infer that the first eventuality implies $A(t_2) = A(t_1)$ so it is impossible. Therefore $r\big(A(t_1)\big) = 3$. Now, the purity of $A(t_1)$ in A implies $A = A(t_1)$ and, consequently, $T(A) = T\big(A(t_1)\big)$. Thus $t_1 \leq t(a)$ for every $a \in A$. It follows from [9, Theorem 9.1] that t_3 is a maximal element of $T(A)$ so the second assertion is a consequence of [22, Proposition 4.1]. ■

It is a well-known fact that if I is an ideal in associative ring R and the ring I is unital, then $R = I \oplus J$ for some $J \lhd R$. We will use this observation in the following remark related to the group A described in Proposition 2.4.

Remark 2.5. Let $I = A(t_3)$. Suppose that $R = (A, *)$ is an associative semi-prime ring. Then $I * I \neq \{0\}$ because $I \lhd R$. By similar argument as in the proof of Proposition 2.4 we infer that $r(I) = 1$. Therefore I can be treated as a subgroup of $\mathbb{Q}^+$. Then, it follows from [25, Remark 4.2] that there exists $q \in I \setminus \{0\}$ such that for all $x, y \in I$ we have $x * y = x \cdot q \cdot y$. If $q^{-1} \in I$, then the ring I is unital and, consequently, I is a direct summand of A.

Proposition 2.6. *Let A be a torsion-free Abelian group of rank n such that $T(A)$ contains distinct maximal types $t_0, t_1, \ldots, t_n$, let $x_0, x_1, \ldots, x_n$ be elements of A respectively of these types and let $S = \{x_0, x_1, \ldots, x_n\}$. If every subset of S of cardinality n is independent, then A is a nil group.*

Proof. As $r(A) = n$ we infer that $\{x_1, x_2, \ldots, x_n\}$ is a maximal independent system of A. Hence, there exist $k_0, k_1, \ldots, k_n \in \mathbb{Z}$ such that $k_0 \neq 0$ and

$$k_0 x_0 = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n. \tag{1}$$

Since every subset of S of cardinality n is independent, we obtain $k_i \neq 0$ for each $i \in \{1, 2, \ldots, n\}$. Take any $i \in \{1, 2, \ldots, n\}$. Consider an arbitrary ring $(A, \cdot)$. Then (8) and (7) of Lemma 2.1 imply that $t(x_0 x_i) \geq t_i$. Suppose contrary to our claim that $x_0 x_i \neq 0$. Then, the maximality of t_i in $T(A)$ implies that $t(x_0 x_i) = t_i$. But it is impossible, because types $t_1, t_2, \ldots, t_n$ are distinct and i has been chosen arbitrarily. Moreover, $A = \langle x_1, x_2, \dots, x_n\rangle_*$ so $x_0 \cdot A = A \cdot x_0 = \{0\}$. By similar arguments, we get $x_r x_s = 0$ for all distinct $r, s \in \{1, 2, \dots, n\}$. Thus, by (1), we get $k_i x_i^2 = 0$. As $k_i \neq 0$ we obtain $x_i^2 = 0$. Consequently, the arbitrary choice of i implies that $A^2 = \{0\}$. Therefore A is a nil group. ■

Theorem 2.7. *Let A be a torsion-free Abelian group of rank three contains elements x_1, x_2, x_3 respectively of types t_1, t_2, t_3 satisfying $t_1 \neq t_2$, $t_1 < t_3$, $t_2 < t_3$ and let $B = \langle x_1, x_2\rangle_*$. If $|T(A)| < \infty$ and t_3 is the greatest element of $T(A)$, then the following conditions are equivalent:*

(i) $\left|\{b \in B \colon \chi_A(x_3 + b) \nleq \chi_A(x_3)\}\right| < \infty$*; and*

(ii) *B is a direct summand of A.*

In particular, if B is a direct summand of A, $\square B \neq \{0\}$ and $\square(A/B) \neq \{0\}$, then $\square_{(a)} A = \langle x_3\rangle_* \oplus \square B$.

Proof. Suppose, contrary to our claim, that $x_3 \in B$. Then $t_3 \in T(B)$, by (4). It follows from [20, Corollary 1.9] that $t_1 \cap t_2 \in T(B)$. Hence, $T(B)$ contains a chain of length three, in contradiction to [9, Theorem 9.1]. Thus $x_3 \notin B$ and, consequently, A/B is a rank one group of type $t_{A/B}(x_3 + B)$. Define $\Omega = \{b \in B \colon \chi_A(x_3 + b) \nleq \chi_A(x_3)\}$. First suppose that $|\Omega| < \infty$. We will show that $t_{A/B}(x_3 + B) = t_3$. Since t_3 is the greatest element of $T(A)$, for each $b \in \Omega$ there exists $n_b \in \mathbb{N}$ such that $\chi_A(x_3 + b) \leq \chi_A(n_b x_3)$. Define $N = \prod_{b\in\Omega} n_b$. Then $\chi_A(x_3 + b) \leq \chi_A(N x_3)$ for each $b \in B$. Hence, by (3), we obtain $\chi_{A/B}(x_3 + B) \leq \chi_A(x_3)$. Therefore, $t_{A/B}(x_3 + B) \leq t_A(N x_3)$ and, consequently, $t_{A/B}(x_3 + B) \leq t_3$. Moreover, the opposite inequality follows at once from (6) if we put the canonical epimorphism $f \colon A \to A/B$. Thus, B is a direct summand of A by [17, Theorem 86.5]. Conversely, if B is a direct summand of A, then $A = \langle x_3\rangle_* \oplus B$ because of $x_3 \notin B$. Hence, by (2), we obtain $\chi_A(x_3 + b) = \chi_A(x_3) \cap \chi_A(b) \leq \chi_A(x_3)$ for each $b \in B$. Consequently, $\Omega = \emptyset$. This completes the proof of $(i) \Leftrightarrow (ii)$.

If $\square(A/B) \neq \{0\}$, then $t_3^2 = t_3$ (cf. [17, Theorems 85.1 and 121.1]). Obviously, $A/B \cong \langle x_3\rangle_*$ so it follows from [25, Theorem 4.8] that $\square_a \langle x_3\rangle_* = \langle x_3\rangle_*$. Moreover, $\square B = \square_a B$, by [3, Thorem 4] or [26, Theorem 3.6]. Combining this with [5, Proposition 1.4 and Remark 1.10] we obtain $\langle x_3\rangle_* \oplus \square B \subseteq \square_a A$. Consider an arbitrary ring $R = (A, \star)$. Let π be the natural projection of A on B. Then the multiplication $b_1 \circledast b_2 = \pi(b_1 \star b_2)$ for all $b_1, b_2 \in B$, induces a ring structure on B. Thus, if $a \in \square A$, then $\pi(a) \in \square B$. Next, $A(t_3) \subseteq \langle x_3\rangle_*$ because of $t_3 \notin T(B)$, t_3 is the greatest element of $T(A)$, $A = \langle x_3\rangle_* \oplus B$ and (2). The opposite inclusion is obvious so $\langle x_3\rangle_*$ is an ideal in every ring on A. Therefore $\square A \subseteq \langle x_3\rangle_* \oplus \square B$. Of course, $\square_a A \subseteq \square A$ so, finally, $\square_a A = \square A = \langle x_3\rangle_* \oplus \square B$. ■

Remark 2.8. The existence of an Abelian group A of rank three with the finite type set $T(A)$ containing distinct types t_1, t_2 and t_3 where t_3 is the greatest element of $T(A)$ is proved in [20] (see, Example 1.10 and Theorem 2.1 with the Remark placed under it). Please note that in contrast to [14] and this article, in the mentioned paper the type t_∞ of 0 belongs to the type set of A.

Remark 2.9. If the group B is indecomposable, then $\Box B$ is described in the proofs of Theorem 3.2 and Lemmas 4.2 and 4.3 placed in [4]. Otherwise, it is described in Theorem 3.6 of [26].

Remark 2.10. Notice that if $|T(B)| = 3$ and $\Box B \neq \{0\}$, then the assumption $|T(A)| \leq \infty$ is not needed. In fact, it follows from [22, p. 204] and (4) that $T(B) = \{t_0, t_1, t_2\}$ for some $t_0, t_1, t_2 \in T(A)$ where t_0 is a minimal type in $T(B)$ and t_1, t_2 are maximal types in $T(B)$. Hence $t_3 \notin T(B)$ and, consequently, $x_3 \notin B$.

3. Some new results for additive groups of associative rings

Abelian groups supporting only associative rings are called AR-groups. An Abelian group A is called a CR-group if every ring R with $R^+ = A$ is commutative. If A satisfies the condition CR restricted to the class of associative rings, then A is called an ACR-group. As was mentioned in the Introduction, these groups were partially examined in [7,16]. In this section we present some new results concerning AR-groups. Furthermore, we generalize some results related to $(A)CR$-groups placed in [7]. The symbol $\mathbb{P}(A)$ means the set of all primes p for which the p-component A_p of A is nontrivial.

Proposition 3.1. *Every CR-group is an AR-group.*

Proof. Consider an arbitrary CR-group A. Take any $* \in \mathrm{Mult}(A)$ and $a \in A$. An easy computation shows that the multiplication $x_1 \circledast x_2 = x_1 * (a * x_2)$ for all $x_1, x_2 \in A$, induces a ring structure on A. Take any $x, y \in A$. As A is a CR-group, we get $x \circledast y = y \circledast x$, i.e., $x * (a * y) = y * (a * x)$. Moreover, the multiplication $*$ is also commutative so $x * (a * y) = (a * x) * y = (x * a) * y$. Since x, y and a have been chosen arbitrarily, we infer that $(A, *)$ is an associative ring. Consequently, A is an AR-group. ∎

Remark 3.2. In [7], Abelian groups satisfying both conditions AR and CR were called $SACR$-groups. This abbreviation comes from *strongly associative and commutative additive groups of rings* and it is consistent with Feigelstock's suggestion concerning naming of Abelian groups satisfying some fixed ring properties (see, [14, p. 36]). In view of Proposition 3.1 the conditions CR and $SACR$ are equivalent. Moreover, for torsion Abelian groups all the conditions: CR, AR and ACR are equivalent (see, [7, Remark 2.3]). For all these reasons, we prefer the prefix $SACR$ to the prefix CR.

Theorem 3.3. *If C is a nontrivial torsion $SACR$-group and A is a torsion-free nil group satisfying $A = pA$ for each $p \in \mathbb{P}(C)$, then $G = C \oplus A$ is an AR-group.*

Proof. Let D be the greatest divisible subgroup of C and let K be a complement of D in C. Then $G = K \oplus D \oplus A$. The basic properties of the tensor product of Abelian groups and groups of homomorphism together with [7, Remark 2.3] and [16, Theorem 5] imply that $G \otimes G \cong K \oplus (A \otimes A)$. Consequently,

$$\mathrm{Mult}(G) \cong \mathrm{Mult}(K) \oplus \mathrm{Hom}\big((K \oplus A) \otimes (K \oplus A), D\big).$$

Moreover, it follows from [7, Remark 2.3] that K is an $SACR$-group so if $* \in \mathrm{Mult}(G)$, then there exist associative and commutative ring $(K, \diamond)$ and a homomorphism $\xi \colon (K \oplus A) \otimes (K \oplus A) \to D$ such that

$$(k_1, d_1, a_1) * (k_2, d_2, a_2) = \Big(k_1 \diamond k_2, \xi\big((k_1, a_1) \otimes (k_2, a_2)\big), 0\Big)$$

for all $k_1, k_2 \in K$, $d_1, d_2 \in D$ and $a_1, a_2 \in A$ (see, [17, Theorem 118.1]). For arbitrary $k_1, k_2, k_3 \in K$ and $a_1, a_2, a_3 \in A$ we get $\xi\big((k_1 \diamond k_2, 0) \otimes (k_3, a_3)\big) = \xi\big((k_1 \diamond k_2, 0) \otimes (k_3, 0)\big)$. Moreover, there exists a direct summand H of K such that $k_1, k_2, k_3 \in H$ and $H \cong \mathbb{Z}_m^+$ for some $m \in \mathbb{N}$. Let h be any generator of H. For $i = 1, 2, 3$ there exists $l_i \in \mathbb{Z}$ such that $k_i = l_i h$. Hence $\xi\big((k_1 \diamond k_2, 0) \otimes (k_3, 0)\big) = (l_1 l_2 l_3)\xi\big((h \diamond h, 0) \otimes (h, 0)\big)$. As $h \diamond h \in H$ we get $h \diamond h = lh$ for some $l \in \mathbb{Z}$. Thus $\xi\big((k_1 \diamond k_2, 0) \otimes (k_3, a_3)\big) = (l l_1 l_2 l_3)\xi\big((h, 0) \otimes (h, 0)\big)$. Analogously, $\xi\big((k_1, a_1) \otimes (k_2 \diamond k_3, 0)\big) = (l l_1 l_2 l_3)\xi\big((h, 0) \otimes (h, 0)\big)$. Therefore the ring $(G, *)$ is associative. Finally, G is an AR-group. ■

Remark 3.4. Suppose that A is a rank one group. Then $a_1 \otimes a_2 = a_2 \otimes a_1$ and $\xi\big((k_1, a_1) \otimes (k_2, a_2)\big) = \xi\big((k_1, 0) \otimes (k_2, 0)\big) + \xi\big((0, a_1) \otimes (0, a_2)\big)$. Combining this with the reasoning presented in the proof of Theorem 3.3 we infer that the ring $(G, *)$ is commutative. Hence, by Proposition 3.1 (or Theorem 3.3), we infer that G is an $SACR$-group.

In view of Theorem 3.3 and Remark 3.4, Proposition 2.17 from [7] can be somewhat generalized:

Proposition 3.5. *If C is a nontrivial torsion $SACR$-group and A is a subgroup of $\mathbb{Q}^+$ such that $A = pA$ for each $p \in \mathbb{P}(C)$, then $G = C \oplus A$ is an $SACR$-group.*

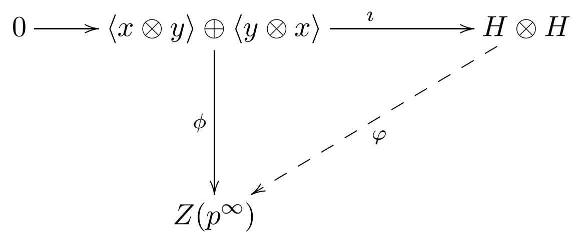
Theorem 3.3 is useful in indicating the first example of an AR-group which is not an $SACR$-group.

Example 3.6. Define $A = \left\langle \frac{1}{p} \colon p \in \mathbb{P} \right\rangle + \left[\frac{1}{2}\right]^+$, $H = A \oplus A$ and $G = Z(2^\infty) \oplus H$. Then H is a torsion-free Abelian group of rank two satisfying $H = 2H$. Furthermore, $\big(t(A)\big)^2 > t(A)$ so [14, Corollary 2.1.3] implies $\square H = \{0\}$. Hence, by Theorem 3.3, [16, Theorem 10] and Remark 3.2, we infer that G is an AR-group which is not an $SACR$-group.

The next result is a complement of [7, Theorem 2.7] which is closely related to [16, Theorem 10]. The proof is inspired by the proof of [16, Theorem 10]. However, we present a complete reasoning for the transparency of the paper.

Theorem 3.7. *Let p be a prime. If the p-component of a mixed ACR-group A is not cyclic, then the torsion-free rank of its complement is equal to one.*

Proof. It follows from [7, Theorem 2.7] and [16, Theorem 10] that there exists a subgroup H of A such that $A = A_p \oplus H$. Suppose, contrary to our claim, that $r_0(H) > 1$. Then there exist independent and torsion-free elements x, y of H. Therefore $\langle x\rangle \oplus \langle y\rangle$ is a free subgroup of H. Hence, by [12, Theorem 2] and by basic properties of the tensor product of Abelian groups, we conclude that $\langle x\otimes y\rangle\oplus\langle y\otimes x\rangle$ is a free subgroup of $H\otimes H$ (we put $R = \mathbb{Z}$, $M' = N' = \langle x\rangle\oplus\langle y\rangle$ and $M = N = H$ in the mentioned theorem). Since A_p is not cyclic, it follows from [16, Theorem 3] that $Z(p^\infty)$ is a direct summand of A_p. Take any $d \in Z(p^\infty) \setminus \{0\}$. Then there exists $s \in \mathbb{N}$ such that $d \in Z(p^s)$. Let ϑ be the natural projection of $\langle x \otimes y\rangle \oplus \langle y \otimes x\rangle$ onto $\langle x \otimes y\rangle$. Since $\langle x \otimes y\rangle/p^s\langle x \otimes y\rangle \cong Z(p^s)$, there exists an epimorphism $\psi\colon \langle x\otimes y\rangle \to Z(p^s)$ such that $\psi(x\otimes y) = d$. Define $\phi = \psi\circ\vartheta$. Then ϕ is a homomorphism of $\langle x \otimes y\rangle \oplus \langle y \otimes x\rangle$ into $Z(p^\infty)$ satisfying $\phi(x\otimes y) = d$ and $\phi(y\otimes x) = 0$. Let $\imath$ be the natural injection of $\langle x\otimes y\rangle \oplus \langle y\otimes x\rangle$ into $H\otimes H$. Since $Z(p^\infty)$ is injective in the category of Abelian groups, there exists a homomorphism $\varphi\colon H \otimes H \to Z(p^\infty)$ for which $\phi = \varphi \circ \imath$.



Let $B = Z(p^\infty) \oplus H$. Then B is a direct summand of A. It is easily seen that the multiplication $(d_1, h_1) \star (d_2, h_2) = \big(\varphi(h_1 \otimes h_2), 0\big)$ for all $d_1, d_2 \in Z(p^\infty)$ and $h_1, h_2 \in H$, provides a ring structure on B. Moreover, $(0, x) \star (0, y) = (d, 0) \neq (0, 0)$, $(0, y) \star (0, x) = (0, 0)$ and $(B \star B) \star B = B \star (B \star B) = \{0\}$. Thus $(B, \star)$ is an associative ring which is not commutative and, consequently, B is not an ACR-group, contrary to [16, Lemma 1]. ∎

Corollary 3.8. *If A is a mixed ACR-group such that $\mathbb{P}(A) = \{p\}$ and A_p is not cyclic, then it follows from Theorem* 3.7 *and* [16, Theorem 3] *that either $A = Z(p^n)\oplus D\oplus H$ or $A = D\oplus H$ where n is a positive integer, D is a nontrivial divisible p-group and H is a torsion-free Abelian group of rank one. Moreover, if the first eventuality holds, then $H = pH$, by* [7, Theorem 2.7]. *Thus, all mixed*

ACR-groups such that $\mathbb{P}(A) = \{p\}$ and A_p is neither divisible nor reduced are described in Proposition 3.5. *In particular, they are $SACR$-groups.*

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