Bulletin of the **S**ection of **L**ogic
Volume 52/1 (2023), pp. 59–83
https://doi.org/10.18778/0138-0680.2023.09

Zalán Gyenis
Zalán Molnár
Övge Öztürk

THE MODELWISE INTERPOLATION PROPERTY OF SEMANTIC LOGICS

Abstract

In this paper we introduce the modelwise interpolation property of a logic that states that whenever $\models \phi \to \psi$ holds for two formulas ϕ and ψ, then for every model $\mathfrak{M}$ there is an interpolant formula χ formulated in the intersection of the vocabularies of ϕ and ψ, such that $\mathfrak{M} \models \phi \to \chi$ and $\mathfrak{M} \models \chi \to \psi$, that is, the interpolant formula in Craig interpolation may vary from model to model. We compare the modelwise interpolation property with the standard Craig interpolation and with the local interpolation property by discussing examples, most notably the finite variable fragments of first order logic, and difference logic. As an application we connect the modelwise interpolation property with the local Beth definability, and we prove that the modelwise interpolation property of an algebraizable logic can be characterized by a weak form of the superamalgamation property of the class of algebras corresponding to the models of the logic.

Keywords: interpolation, algebraic logic, amalgamation, superamalgamation.

1. The modelwise interpolation property

Interpolation properties have been intensively studied in the literature of (algebraic) logic ever since Craig proved that in classical propositional and first order logic, whenever $\models \phi \to \psi$ holds for two formulas ϕ and

Presented by: Michał Zawidzki
Received: September 9, 2022
Published online: April 21, 2023

ψ formulated respectively using the vocabularies (signatures) $\mathrm{Voc}(\phi)$ and $\mathrm{Voc}(\psi)$, then there is an interpolant formula χ formulated in the vocabulary $\mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\models \phi \to \chi$ and $\models \chi \to \psi$ hold.

This paper introduces the *modelwise interpolation* property of a logic which states that whenever $\models \phi \to \psi$ holds, then one can find an interpolant formula in every model, that is, the interpolant formula in Craig interpolation may vary from model to model. In order to make sense of this notion we have to work with logics that are semantically defined, e.g. a notion of model should be built in the definition of the logic.[1]

We discuss the relations between the modelwise interpolation, Craig interpolation, and local interpolation properties by providing examples in all logically possible combinations. Most importantly, we prove that while difference logic and the n-variable fragment of first-order logic ($n \geq 2$) lack the standard Craig interpolation property, the former has, while the latter does not have the the modelwise interpolation property. Using the case of difference logic as an example, we show that the modelwise interpolation property implies the local Beth definability property for difference logic.

The modelwise interpolation property might have possible further applications in philosophy of science. Craig original interpolation property (for first-order logic) stemmed from the question of using logic to clarify the relationship between theoretical constructs and observed data: the interpolant formula gives an axiomatization of the observational consequences of the theory in which only symbols of the observational vocabulary occur (cf. [24]). Scientific theories are sometimes axiomatized by logics other than classical first-order logic, for example, in [2] modal logic is used to axiomatize relativity theory (cf. [21]). Such logics may or may not have the Craig interpolation property. If the logic we make use has no Craig interpolation but turns out to have the modelwise interpolation property, and our scientific theories are formulated in this logic and evaluated in a model, then changing our background logic from first-order logic to this new logic still allows us to carry out arguments inside models similar to

[1] While providing the definitions and discussing examples, we employ a rather general notion of a logic. But in the last section of the paper when we provide the algebraic characterization, we adopt the Andréka–Németi–Sain approach [3, 1], cf. [20, 17, 18] which focuses on the semantic aspects of logics. The more mainstream Blok–Pigozzi framework (cf. [8, 10, 29, 9] and Czelakowski [14]) seems not to be (directly) applicable as in that approach the focus is rather on the relation $\vdash$ between sets of formulas and is missing the general notion of models.

Craig's. The previously introduced *local interpolation property* (the definition is provided later below) was motivated by similar considerations, however even very basic logics such as sentential logic, propositional modal logics, finite variable fragments of first order logic, etc. do not have the local interpolation property. Cf. the examples below. Also, the Craig interpolation (resp. modelwise interpolation) has a strong connection with Beth definability (resp. local Beth definability). The local interpolation property does not have such connections. In this respect, the modelwise interpolation property seems to be a "more interesting" property than the local interpolation property. We do not pursue these philosophical issues in this paper.

Interpolation properties of a logic are strongly related to various amalgamation properties of the classes of algebras corresponding to the logic. We refer to [12], [13], [23, 22], [37], [25], [28], [35], [32], [3]. In the last section we show that the modelwise interpolation property of an algebraizable logic can be characterized by a weak form of the superamalgamation property of the class of algebras corresponding to the models of the logic.

* * *

By a logic we understand a tuple $\mathcal{L}(P, Cn) = \langle F, M, \models \rangle$, where

- P is a set, called the set of atomic formulas, and Cn is a set of logical connectives, i.e. function symbols of finite arity.
- F, called the set of formulas, is the universe of the absolutely free algebra generated by P in similarity type Cn.
- M is an abstract, non-empty class, called the class of models.
- $\models$ is a relation between models and formulas: $\models\, \subset M \times F$. For $\mathfrak{M} \in M$ and $\phi \in F$ we write $\mathfrak{M} \models \phi$ instead of $(\mathfrak{M}, \phi) \in\, \models$.

As it is standard in logic we extend the consequence relation $\models$ to a relation in between (sets) of formulas: For $\Gamma, \{\phi\} \subseteq F$ we write $\Gamma \models \phi$ if whenever $\mathfrak{M} \models \Gamma$ for a model $\mathfrak{M} \in M$, then $\mathfrak{M} \models \phi$ as well. When it is clear from the context, we simply write $\mathcal{L}$ in place of $\mathcal{L}(P, Cn)$. For a formula $\alpha \in F$, the vocabulary of α, $\mathrm{Voc}(\alpha)$ denotes the set of atomic formulas occurring in α, i.e. the smallest subset of P such that α belongs to the absolutely free algebra generated by $\mathrm{Voc}(\alpha)$ in similarity type Cn.

For our main definition 1.2 below we assume that there is a distinguished binary (derived) connective $\rightsquigarrow$ and we write $(\star)_{\phi,\psi}$ for the property

$$\{\chi \in F : \mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)\} \neq \emptyset \qquad (\star)_{\phi,\psi}$$

Recall (e.g. from [3, Def.6.13]) that the Craig interpolation property ($\mathrm{IP}^{\rightsquigarrow}$, for short) is the property that whenever $\phi, \psi \in F$ for which $(\star)_{\phi,\psi}$ holds, if $\models \phi \rightsquigarrow \psi$, then there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\models \phi \rightsquigarrow \chi$ and $\models \chi \rightsquigarrow \psi$.

Remark 1.1. The extra condition $(\star)_{\phi,\psi}$ can be satisfied in two ways: either there is a constant connective in the language, or $\mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ is not empty. Consider classical propositional logic with connectives $\{\vee, \neg\}$ and with two atomic formulas p and q. As usual, $\phi \to \psi$ abbreviates $\neg\phi \vee \psi$. There is no interpolant for the tautology $\models p \to (q \to q)$, as $\mathrm{Voc}(p) \cap \mathrm{Voc}(q \to q)$ is empty, and there are no formulas over the empty vocabulary (we did not allowed $\perp$ or $\top$ as constants in the language). However, if $(\star)_{\phi,\psi}$ is satisfied, then $\models \phi \to \psi$ will always have an interpolant in this logic.

Let us now define the modelwise interpolation property.

Definition 1.2. We say that the logic $\mathcal{L} = \langle F, M, \models \rangle$ has the *modelwise interpolation property* ($\mathrm{mIP}^{\rightsquigarrow}$, for short) if for every formulas $\phi, \psi \in F$ for which $(\star)_{\phi,\psi}$ holds, if $\models \phi \to \psi$, then for all models $\mathfrak{M} \in M$ there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\mathfrak{M} \models \phi \to \chi$ and $\mathfrak{M} \models \chi \to \psi$.

The $\mathrm{mIP}^{\rightsquigarrow}$ thus differs from the $\mathrm{IP}^{\rightsquigarrow}$ in that the interpolant formula may vary from model to model. Note that it is crucial for the definition of $\mathrm{mIP}^{\rightsquigarrow}$ to have a notion of model built in the definition of the logic $\mathcal{L}$.

Motivated by model theoretic investigations of homogeneous structures [15, 27] the local interpolation property ($\mathrm{lIP}^{\rightsquigarrow}$, for short) has been introduced in [16] as the property that whenever $\phi, \psi \in F$ for which $(\star)_{\phi,\psi}$ holds, for all $\mathfrak{M} \in M$ if $\mathfrak{M} \models \phi \rightsquigarrow \psi$, then there exists $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $\mathfrak{M} \models \chi \rightsquigarrow \psi$. Notice that the $\mathrm{lIP}^{\rightsquigarrow}$ differs from the $\mathrm{mIP}^{\rightsquigarrow}$ in that in the former the implication $\phi \rightsquigarrow \psi$ is also "localized" to models, making it a rather weak property of a logic.

Claim 1.3. Both the $\mathrm{IP}^{\rightsquigarrow}$ and the $\mathrm{lIP}^{\rightsquigarrow}$ imply the $\mathrm{mIP}^{\rightsquigarrow}$.

PROOF: Straightforward from the definitions. □

Remark 1.4. We note that the modelwise interpolation property could be defined for many other types of logics too. For example, one could allow for infinite formulas, or infinite connectives, or restrictions on the syntactic shape of formulas, etc. Adapting the definition to such cases seems to be straightforward and thus we do not pursue such a generalization. Also, all our examples, and in fact the most traditional propositional and first-order logics, fit to the notion of logic given above.

In the rest of this section we give examples for logics having or not having the discussed interpolation properties in all possible combinations. Even thought our definitions so far were employed for logics in a very broad sense, our examples below are all algebraizable and in fact well-studied in the literature (except for $\mathcal{L}_\infty$ which is algebraizable but not well-studied). The following table summarizes the examples given below.

	$\mathrm{IP}^{\to}$	$\mathrm{lIP}^{\to}$	$\mathrm{mIP}^{\to}$
$\mathcal{L}_{Prop}$	✓	✓	✓
$\mathcal{L}_{Sent}$	✓	×	✓
$\mathcal{L}_\infty$	×	✓	✓
$\mathcal{L}_D$	×	×	✓
$\mathcal{L}_n, \pounds_n,\ n > 2$	×	×	×

Note that there are 8 theoretically possible combinations of the three logical properties, but Claim 1.3 rules out three of them. This is why the table above consists of 5 rows only.

Propositional logic $\mathcal{L}_{Prop}$. Let P be an arbitrary set of propositional letters. Let $Cn(\mathcal{L}_{Prop}) = \{\wedge, \neg, \bot\}$ be the set of connectives and let F be the set of formulas generated by P in type $Cn(\mathcal{L}_{Prop})$. Models are evaluations $\mathfrak{M} : P \to \{0, 1\}$ that extend to the set of formulas by the usual $\mathfrak{M}(\bot) = 0$, $\mathfrak{M}(\phi \wedge \psi) = \mathfrak{M}(\phi) \cdot \mathfrak{M}(\psi)$, and $\mathfrak{M}(\neg\phi) = 1 - \mathfrak{M}(\phi)$. The validity relation is defined as

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \mathfrak{M}(\varphi) = 1. \tag{1.1}$$

We use the derived connectives $\vee$, $\to$ and $\top$ in the standard way. By Craig's result, $\mathcal{L}_{Prop}$ has the $\mathrm{IP}^{\to}$ and thus the $\mathrm{mIP}^{\to}$ as well. That $\mathcal{L}_{Prop}$

has the lIP$^{\rightarrow}$ follows from that whenever $\mathfrak{M} \models \phi \rightarrow \psi$, then either $\perp$ or $\top$ is a suitable interpolant formula inside the model $\mathfrak{M}$.

Sentential logic $\mathcal{L}_{Sent}$. The set of connectives and the set of formulas are as in the previous case. The class of models is

$$M = \big\{\langle W, V\rangle : \ W \neq \emptyset, \ V : P \rightarrow \mathcal{P}(W)\big\}. \tag{1.2}$$

For a model $\mathfrak{M} = \langle W, V\rangle$, $w \in W$ and a formula φ one defines $\mathfrak{M}, w \Vdash \varphi$ by

$$\mathfrak{M}, w \nVdash \perp \tag{1.3}$$
$$\mathfrak{M}, w \Vdash p \quad \Leftrightarrow \quad w \in V(p) \tag{1.4}$$
$$\mathfrak{M}, w \Vdash \phi \wedge \psi \quad \Leftrightarrow \quad \mathfrak{M}, w \Vdash \phi \text{ and } \mathfrak{M}, w \Vdash \psi \tag{1.5}$$
$$\mathfrak{M}, w \Vdash \neg\phi \quad \Leftrightarrow \quad \mathfrak{M}, w \nVdash \phi. \tag{1.6}$$

Finally, we set

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \{w \in W : \ \mathfrak{M}, w \Vdash \phi\} = W. \tag{1.7}$$

Craig's original result applies to this presentation of classical logic too, i.e. $\mathcal{L}_{Sent}$ has the IP$^{\rightarrow}$ and thus the mIP$^{\rightarrow}$ too. In contrast, however, $\mathcal{L}_{Sent}$ does not have the lIP$^{\rightarrow}$ in general. For, assume that there are (at least) two atomic formulas p and q. Take a model $\mathfrak{M}$ in which $\emptyset \neq V(p) \subsetneq V(q) \neq W$ holds for the atomic propositions p and q. Then $\mathfrak{M} \models p \rightarrow q$ holds by the definition of truth in a model. However, $\mathrm{Voc}(p) \cap \mathrm{Voc}(q)$ is empty, therefore the possible interpolant formulas are Boolean combinations of the constant symbol $\perp$. Each such formula is equivalent either to $\perp$ or to $\top$, but neither can be an interpolant in the model $\mathfrak{M}$, as p is not false in $\mathfrak{M}$, and q is not true in $\mathfrak{M}$.

Difference logic $\mathcal{L}_D$. Difference logic is discussed e.g. in Sain [33, 34], Venema [38], Roorda [31], but see also Segerberg [36] who traces this logic back to von Wright. The set of connectives of difference logic is $\{\wedge, \neg, D, \perp\}$. The set of formulas is defined as that of propositional logic together with the following clause: if $\phi \in F$, then $D\phi \in F$. The class of models and the definition of $\mathfrak{M}, w \Vdash \varphi$ are the same as in the sentential case but we also have the case of D:

$$\mathfrak{M}, w \Vdash D\phi \quad \Leftrightarrow \quad (\exists w' \in W \setminus \{w\})\ \mathfrak{M}, w' \Vdash \phi. \tag{1.8}$$

Truth in a model is defined in the same way as in the sentential case:[2]

$$\mathfrak{M} \models \varphi \quad \Leftrightarrow \quad \{w \in W : \mathfrak{M}, w \Vdash \phi\} = W. \tag{1.9}$$

That difference logic does not have the lIP$^{\rightarrow}$ can be seen exactly in the same way as in the case of sentential logic: Assuming p and q are atomic formulas, take a model $\mathfrak{M}$ in which p is not false, q is not true, and p implies q, that is, $\emptyset \neq V(p) \subsetneq V(q) \neq W$ holds. The common vocabulary of p and q is empty. Now, every formula of difference logic over the empty vocabulary is either true or false in a model: As for the Boolean combinations this is straightforward. As for the difference operator, it is enough to check that $D\bot$ cannot be satisfied in any world, and $D\top$ is true in all worlds (provided there are at least two worlds).

It is known that $\mathcal{L}_D$ does not have the IP$^{\rightarrow}$ either (see e.g. [11]). Let us briefly recall the argument. Let $E\phi$ abbreviate $\phi \vee D\phi$. The following implication is a logical validity of difference logic:

$$\models_{\mathcal{L}_D} \big(Dp \wedge D\neg p\big) \ \longrightarrow\ \big(E(r \wedge \neg Dr) \ \rightarrow\ E(\neg r \wedge D\neg r)\big). \tag{1.10}$$

The reason is that in a model $\mathfrak{M}$ and a world w, $w \Vdash Dp \wedge D\neg p$ implies that there are at least two other worlds not equal to w, while $E(r \wedge \neg Dr) \rightarrow E(\neg r \wedge D\neg r)$ expresses that if there is only one world satisfying r, then there must be at least two different worlds satisfying $\neg r$. The common vocabulary of the subformulas on the two sides of the implication is empty, and it is not hard to check that neither $\top$ nor $\bot$ nor any formulas built up from $\top$ and $\bot$ can be a global interpolant ([11] contains a detailed proof).

However, $\mathcal{L}_D$ has the modelwise interpolation property as the following theorem shows.

THEOREM 1.5. *Difference logic has the mIP$^{\rightarrow}$.*

PROOF: Suppose $\models \phi(\vec{p}, \vec{q}) \longrightarrow \psi(\vec{q}, \vec{r})$ is a logical validity where the formulas ϕ and ψ use the atomic formulas $\vec{p}$, $\vec{q}$ and $\vec{r}$ as denoted. We need to find an interpolant formula using the atomic formulas $\vec{q}$ only. Write $\vec{q} = \langle q_0, \ldots, q_{n-1}\rangle$ and $\vec{p} = \langle p_0, \ldots, p_{m-1}\rangle$. Take any model $\mathfrak{M} = \langle W, V\rangle$.

[2] Thus, $\mathfrak{M} \models \phi$ is what is standardly called "global truth" in modal logic (cf. [7, Def.1.21]).

Two worlds $v, w \in W$ are said to be $\vec{q}$-equivalent ($v \sim w$ in symbols) if for all $i < n$ we have

$$\mathfrak{M}, v \Vdash q_i \qquad \Longleftrightarrow \qquad \mathfrak{M}, w \Vdash q_i \tag{1.11}$$

Claim 1.6. If $\mathfrak{M}, v \Vdash \phi$ and $w \sim v$, then $\mathfrak{M}, w \Vdash \psi$.

Proof: Assume $\mathfrak{M}, v \Vdash \phi$ and define a new model $\mathfrak{M}' = \langle W, V' \rangle$ on the same set of possible worlds as follows. For a world $u \in W$ let us use the notation

$$u' = \begin{cases} v & \text{if } u = w \\ w & \text{if } u = v \\ u & \text{if } u \neq v, u \neq w, \end{cases} \tag{1.12}$$

that is, we exchange v with w but keep everything fixed. Define the new evaluation V' by $V'(q_i) = V(q_i)$, $V'(r_i) = V(r_i)$ and

$$V'(p_i) = \{u' : \ u \in V(p_i)\}. \tag{1.13}$$

Lemma 1.7. *For any formula $\theta(\vec{p}, \vec{q})$ and world $u \in W$ we have*

$$\mathfrak{M}, u \Vdash \theta \quad \Leftrightarrow \quad \mathfrak{M}', u' \Vdash \theta.$$

Proof: Induction on the complexity of θ.

- For atomic propositions q_i: As $V'(q_i) = V(q_i)$, if $u \neq v$ and $u \neq w$, then $u = u'$ and thus the statement holds. For $u = v$ or $u = w$ we obtain the result by the assumption $v \sim w$.
- For atomic propositions p_i the statement follows directly from the definition of V': $\mathfrak{M}, u \Vdash p_i$ if and only if $\mathfrak{M}', u' \Vdash p_i$.
- For Boolean combinations the induction is straightforward.
- For formulas of the form $D\theta$: Assume (inductive hypothesis) that the statement holds for θ. Then

$$\begin{aligned} \mathfrak{M}, u \Vdash D\theta \quad &\Leftrightarrow \quad (\exists x \neq u)\ \mathfrak{M}, x \Vdash \theta && (1.14)\\ &\Leftrightarrow \quad (\exists x \neq u)\ \mathfrak{M}', x' \Vdash \theta && (1.15)\\ &\Leftrightarrow \quad (\exists x' \neq u')\ \mathfrak{M}', x' \Vdash \theta && (1.16)\\ &\Leftrightarrow \quad \mathfrak{M}', u' \Vdash D\theta. \quad \square && (1.17) \end{aligned}$$

Applying the lemma to v and ϕ we obtain $\mathfrak{M}', w \Vdash \phi$. As $\models \phi \to \psi$ holds we get $\mathfrak{M}', w \Vdash \psi$. But note that V and V' coincide on the elements of $\vec{q}$ and $\vec{r}$, therefore $\mathfrak{M}, u \Vdash \psi$ if and only if $\mathfrak{M}', u \Vdash \psi$ for any $u \in W$. It follows that $\mathfrak{M}, w \Vdash \psi$, completing the proof of the claim. $\square$

In what follows we use the notation $q^1 = q$ and $q^0 = \neg q$. For $v \in W$ write

$$\chi_v \;=\; \bigwedge_{i<n} q_i^{\varepsilon_i}, \tag{1.18}$$

where

$$\varepsilon_i = \begin{cases} 1 & \text{if } \mathfrak{M}, v \Vdash q_i \\ 0 & \text{if } \mathfrak{M}, v \Vdash \neg q_i \end{cases} \tag{1.19}$$

By the claim above for each v for which $\mathfrak{M}, v \Vdash \phi$ holds, the equivalence class $v/{\sim}$ is a subset of $\{u \in W : \mathfrak{M}, u \Vdash \psi\}$. As $\vec{q}$ is finite, there are only finitely many $\sim$ equivalence classes. Let $v_0, \ldots, v_\ell$ be representative elements of all the different equivalence classes such that $\mathfrak{M}, v_i \Vdash \phi$ and write

$$\chi \;=\; \bigvee_{i<\ell} \chi_{v_i}\,. \tag{1.20}$$

Then $\mathfrak{M} \models \phi \to \chi$ and $\mathfrak{M} \models \chi \to \psi$, that is, χ is a desired interpolant formula in $\mathfrak{M}$. $\square$

First-order logic with n variables $\mathcal{L}_n$. Let $\mathcal{L}_n$ denote standard first-order logic with the restriction that we are allowed to use n variables only (n is finite). It is not hard to see that given any first-order similarity type, $\mathcal{L}_n$ fits into our definition of a logic. The connectives are the standard $\wedge$, $\neg$, $\exists x$ (unary) and $x = y$ (constant) for variables x, y, and the set P is the set of first-order atomic formulas. Models, evaluations, $\models$, etc. are the usual.

For $n \geq 2$, $\mathcal{L}_n$ does not admit Craig's interpolation theorem $\mathrm{IP}^{\to}$, in general.[3] A proof can be found in [5, Theorem 3.5.1], here we briefly

[3]That is, there are similarity types for which the n-variables fragment of first-order logic does not have the Craig interpolation. [5, Theorem 3.5.1] shows the failure of interpolation with monadic predicates; [4] shows that interpolation still fails with one

sketch the argument. Let $n \geq 2$ and let $p_1, \ldots, p_n$ be unary predicates. The formula ϕ that states that there is a one-one correspondence between the elements of the domain of a model and the relations p_i can be expressed by the conjunction of the following formulas:

$$\forall x \bigvee_i p_i(x), \quad \bigwedge_i \exists x p_i(x), \quad \forall x \bigwedge_{i \neq j} (p_i(x) \to \neg p_j(x)), \tag{1.21}$$

$$\forall x \forall y \big(\bigwedge_i (x \neq y) \wedge p_i(x) \to \neg p_i(y) \big). \tag{1.22}$$

Thus, if ϕ is true in a model $\mathfrak{M}$, then $\mathfrak{M}$ has exactly n elements. Let ψ be a similar formula using relation symbols $r_1, \ldots, r_{n+1}$ expressing that the model has $n+1$ elements. Then clearly $\models \phi \to \neg\psi$, but there can be no interpolant formula as no n-variable formula using equality only can distinguish between n and $n+1$ elements. This latter statement follows from e.g. a standard back and forth argument to be recalled in the proof of Theorem 1.8 below.

In the next theorem we adapt this construction[4] to show that $\mathcal{L}_n$ does not always have the modelwise interpolation property, for $n \geq 3$. The $n = 2$ case remains open.

Theorem 1.8. *For $n \geq 3$, $\mathcal{L}_n$ does not have the mIP$^{\to}$, in general.*

Proof: Assume there are unary relation symbols p_1, …, p_n, and r_1, …, r_{n+1} and a binary relation symbol e in the similarity type.

Let $\phi(x)$ be the conjunction of the following formulas, having free variable x, using the relation symbols e, p_1, …, p_n only:

binary and two unary relation symbols. With only two non-logical symbols the question is open. The cases $n = 0$ and $n = 1$ can basically be reduced respectively to propositional logic and modal logic $S5$; both have the Craig interpolation property. Cf. p. 107 in [3].

[4]We would like to thank László Csirmaz for a similar idea.

$$\forall y\, \neg e(y,y), \tag{1.23}$$

$$\forall y\bigl(e(x,y) \to \bigvee_i p_i(y)\bigr), \tag{1.24}$$

$$\bigwedge_i \exists y(e(x,y) \wedge p_i(y)), \tag{1.25}$$

$$\forall y\bigl(e(x,y) \to \bigwedge_{i \neq j}(p_i(y) \to \neg p_j(y))\bigr), \tag{1.26}$$

$$\forall y \forall z\bigl(y \neq z \wedge e(x,y) \wedge e(x,z) \to \bigwedge_i (p_i(y) \to \neg p_i(z))\bigr). \tag{1.27}$$

In a model $\mathfrak{M}$, $e^{\mathfrak{M}}$ is a simple graph, and if $\mathfrak{M} \models \phi[a]$ holds for $a \in \mathfrak{M}$, then a has exactly n neighbours, as there is a bijection between the neighbours of a and the p_i's.

Let $\psi(x)$ be the similar formula but with the relation symbols $r_1, \ldots, r_{n+1}$ in place of the p_i's. Clearly, if $\mathfrak{M} \models \psi[a]$ holds for $a \in \mathfrak{M}$, then a has exactly $n+1$ neighbours.

As no vertex in a graph can have n and $n+1$ neighbours at the same time, we have $\models \phi \to \neg\psi$. The common vocabulary of the formulas ϕ and ψ contains the relation symbol e and the equalities only.

In what follows $\mathfrak{A}$ and $\mathfrak{B}$ denotes the following graphs:

$$A = \{a, c_1, \ldots, c_n\}, \quad e^{\mathfrak{A}} = \{(a, c_i) :\ 1 \leq i \leq n\} \tag{1.28}$$

$$B = \{b, d_1, \ldots, d_{n+1}\}, \quad e^{\mathfrak{B}} = \{(b, d_i) :\ 1 \leq i \leq n+1\}, \tag{1.29}$$

that is, $\mathfrak{A}$ is a "star" with center a, having n neighbours $c_1, \ldots, c_n$; and similarly, $\mathfrak{B}$ is a star with center b, having $n+1$ neighbours $d_1, \ldots, d_{n+1}$. We assume that A and B are disjoint.

Let $\mathfrak{M}$ be the disjoint union of the graphs $\mathfrak{A}$ and $\mathfrak{B}$, and interpret the relation symbols p_i and r_j as the respective neighbours of a and b:

$$M = A \cup B, \quad e^{\mathfrak{M}} = e^{\mathfrak{A}} \cup e^{\mathfrak{B}}, \tag{1.30}$$

$$p_j^{\mathfrak{M}} = \{c_j\} \quad \text{for } 1 \leq j \leq n, \tag{1.31}$$

$$r_k^{\mathfrak{M}} = \{d_k\} \quad \text{for } 1 \leq k \leq n+1. \tag{1.32}$$

The neighbours of a are in one-one correspondence with the p_j's, and the neighbours of b are in one-one correspondence with the r_k's. In this model, we have

$$\phi^{\mathfrak{M}} = \{m \in M : \mathfrak{M} \models \phi[m]\} = \{a\}, \tag{1.33}$$
$$\psi^{\mathfrak{M}} = \{m \in M : \mathfrak{M} \models \psi[m]\} = \{b\}, \tag{1.34}$$
$$(\neg\psi)^{\mathfrak{M}} = M \smallsetminus \{b\}. \tag{1.35}$$

Suppose χ is an interpolant for $\models \phi \to \neg\psi$ in the model $\mathfrak{M}$, formulated in the language using equality and e only. As $\phi^{\mathfrak{M}}$ is not empty, χ cannot be false in $\mathfrak{M}$. Similarly, as $(\neg\psi)^{\mathfrak{M}}$ is non-empty, χ cannot be true in $\mathfrak{M}$. Observe, that the set

$$I = \{g : g \subseteq f \text{ for some partial isomorphism} \tag{1.36}$$
$$f : \mathfrak{M} \to \mathfrak{M} \text{ with } f(a) = b\} \tag{1.37}$$

is an n-back-and-forth system between $\mathfrak{M}$ and $\mathfrak{M}$: it satisfies the properties

(i) $g \subseteq f \in I$ implies $g \in I$, and

(ii) if $f \in I$ and $|f| < n$, then for all $x \in A$ (resp. $y \in B$) there is a $g \in I$ with $f \subseteq g$ and $x \in \mathrm{dom}(g)$ (resp. $y \in \mathrm{ran}(g)$).

Therefore, by a standard back-and-forth argument (see e.g. Theorem 2.4 in [6]) $a \in M$ and $b \in M$ satisfy the same formulas with at most n variables.

It follows that no formula χ in the language of equality and e only can make a distinction between the elements a and b of $\mathfrak{M}$: either both or none of them satisfy χ in $\mathfrak{M}$. Consequently, χ cannot be the desired interpolant formula. □

In the light of Claim 1.3, Theorem 1.8 gives an alternative proof for that $\mathcal{L}_n$ does not admit Craig's interpolation theorem $\mathrm{IP}^{\to}$, and that it does not have the $\mathrm{lIP}^{\to}$ either.

Łukasiewicz's Ł_n for $n > 2$. Let $n > 2$ be finite and consider the n-element algebra

$$\mathfrak{A}_n = \big\langle \{\frac{i}{n-1} : i < n\}, \wedge, \vee, \neg, \to, 1 \big\rangle, \tag{1.38}$$

where the operations are given by

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}, \tag{1.39}$$
$$\neg x = 1 - x, \quad x \to y = \min\{1, 1 - x + y\}. \tag{1.40}$$

Łukasiewicz's logic Ł_n is defined as follows (cf. e.g. [30, 7.3.9]). The connectives $Cn(\text{Ł}_n) = \{\wedge, \vee, \neg, \rightarrow, \top\}$ are the usual. If P is a set of propositional variables, then the set of formulas F is generated by P using the connectives. Write $\mathcal{F}$ for the absolutely free formula algebra $\mathcal{F} = \langle F, \wedge, \vee, \neg, \rightarrow, \top\rangle$. The class of models is

$$M = \big\{h : \mathcal{F} \rightarrow \mathfrak{A}_n \ : \ h \text{ is a homomorphism}\big\}. \tag{1.41}$$

In a model $h \in M$, $h \models \phi$ holds if $h(\phi) = 1$. The definition of logical validity is then

$$\models_{\text{Ł}_n} \phi \quad \iff \quad (\forall h \in M)\; h(\phi) = 1. \tag{1.42}$$

Assume that there are at least two atomic formulas in P. The paper [19] showed that Ł_n does not have the Craig interpolation property $\text{IP}^{\rightarrow}$. A similar argument below reveals that Ł_n does not have the $\text{mIP}^{\rightarrow}$. Then, by Claim 1.3 then it cannot have the $\text{lIP}^{\rightarrow}$ either.

Truth tables show that the implication

$$\models_{\text{Ł}_n} p \wedge \neg p \;\longrightarrow\; q \vee \neg q \tag{1.43}$$

holds for any propositional variables $p, q \in P$. Every formula in the empty vocabulary is a Boolean combination of $\perp$ and $\top$, and therefore is equivalent to either $\perp$ or $\top$. However, in the model where both p and q are evaluated to $\frac{\lfloor n/2\rfloor}{n-1}$ neither $\top$ not $\perp$ can be an interpolant. This is because the truth value $\frac{\lfloor n/2\rfloor}{n-1}$ is neither 0 nor 1 if $n > 2$.

The same argument carries over to the infinite Łukasiewicz logic Ł_∞. (for this logic, see [26]).

The logic $\mathcal{L}_\infty$. We design the logic $\mathcal{L}_\infty$ for the sake of giving an example for the case where the $\text{IP}^{\rightarrow}$ fails but the $\text{lIP}^{\rightarrow}$ and thus the $\text{mIP}^{\rightarrow}$ hold.

Let ω denote the ordered set of natural numbers and let ω^* be the reverse ordering. Consider the ordering $\omega + \omega^*$. We write $n \in \omega$ and $n \in \omega^*$ to denote that n belong to the ω or the ω^* part of the ordering $\omega + \omega^*$. Particularly, $0 \in \omega$ is the smallest element, and $0 \in \omega^*$ is the largest element of the ordering. Define the algebra

$$\mathfrak{A} = \big\langle \omega + \omega^*, E, L, \rightarrow, c_i\big\rangle_{i \in \omega + \omega^*}, \tag{1.44}$$

where E and L are the unary functions

$$E(n) = \begin{cases} n & \text{if } n \in \omega \\ 0 \in \omega & \text{if } n \in \omega^*, \end{cases} \qquad L(n) = \begin{cases} n & \text{if } n \in \omega^* \\ 0 \in \omega^* & \text{if } n \in \omega, \end{cases} \tag{1.45}$$

the binary $\to$ is given by

$$x \to y = \begin{cases} 0 \in \omega^* & \text{if } x \leq^{\omega+\omega^*} y \\ 0 \in \omega & \text{otherwise,} \end{cases} \tag{1.46}$$

and each c_i is a constant with value i for $i \in \omega + \omega^*$.

The connectives of the logic $\mathcal{L}_\infty$ are $\{E, L, \to, c_i\}_{i \in \omega+\omega^*}$. If P is a set of propositional variables, then the set of formulas F is generated by P using the connectives. Write $\mathcal{F}$ for the absolutely free formula algebra. The class of models is

$$M = \big\{h : \mathcal{F} \to \mathfrak{A} \ : \ h \text{ is a homomorphism}\big\}. \tag{1.47}$$

For $h \in M$ we let the meaning function mng_h to be equal to h. In a model $h \in M$, $h \models \phi$ holds if $h(\phi) = 0 \in \omega^*$. The definition of logical validity is then

$$\models_{\mathcal{L}_\infty} \phi \quad \Longleftrightarrow \quad (\forall h \in M)\ h(\phi) = 0 \in \omega^*. \tag{1.48}$$

It is easy to check that the implication

$$\models_{\mathcal{L}_\infty} E(p) \to L(q) \tag{1.49}$$

holds for any propositional variables $p, q \in P$. Every formula in the empty vocabulary is equivalent to one of the constants c_i, therefore in order to see that $\mathcal{L}_\infty$ has no $\mathrm{IP}^{\to}$, it is enough to check that none of the constants c_i can be a (global) interpolant for the formula $E(p) \to L(q)$. Indeed, for any c_i take a model h in which $h(c_i) < h(Ep)$ or $h(Lq) < h(c_i)$ holds. Then either $h \not\models E(p) \to c_i$ or $h \not\models c_i \to L(q)$.

However, $\mathcal{L}_\infty$ has the $\mathrm{lIP}^{\to}$ (and thus the $\mathrm{mIP}^{\to}$) because in any model h the formula $c_{h(E(p))}$ is a suitable interpolant.

2. Applications

The local Beth property of a logic $\mathcal{L}$ states that every implicitly definable relation is locally explicitly definable, that is, the explicit definition may vary from model to model (see [3, Definition 6.9]). To be more precise, let $\mathcal{L} = \langle F, M, \models \rangle$ be a logic, and write F^P to denote the set of formulas of the logic $\mathcal{L}$ that are generated by the propositional letters P, that is, $F^P = \{\phi \in F : \operatorname{Voc}(\phi) \subseteq P\}$, and let $\leftrightarrow$ be a distinguished binary connective. For a set of propositional letters R let R' be a disjoint copy of R and for $\Sigma \subseteq F^R$ we write Σ' to denote the formulas obtained from Σ be replacing each $r \in R$ by the corresponding $r' \in R'$. We say that $\Sigma \subseteq F^{P \cup R}$ *defines R implicitly* in terms of P if and only if $\Sigma \cup \Sigma' \models r \leftrightarrow r'$ for every $r \in R$. Further, Σ *defines R locally explicitly* in terms of P if for every model $\mathfrak{M} \models \Sigma$, for all $r \in R$ there is $\varphi_r \in F^P$ such that $\mathfrak{M} \models r \leftrightarrow \varphi_r$. That is, the usual explicit definition may vary from model to model.

We show that the modelwise interpolation property implies the local Beth definability property for a wide range of logics. In what follows we work with logics that extend classical propositional logic in the sense that the connectives $\wedge$ and $\rightarrow$ are available and satisfy

$$\models (\phi \wedge \psi) \rightarrow \theta \ \text{ iff } \ \models \phi \rightarrow (\psi \rightarrow \theta) \ \text{ iff } \ \models \psi \rightarrow (\phi \rightarrow \theta) \tag{2.1}$$

The logic $\mathcal{L}$ is said to be *consequence compact* if for every $\Gamma, \{\phi\} \subseteq F$, if $\Gamma \models \phi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \phi$. $\mathcal{L}$ is *conjunctive* if for any $\phi, \psi \in F$ we have

$$\{\theta : \ \phi, \psi \models \theta\} = \{\theta : \ \phi \wedge \psi \models \theta\}. \tag{2.2}$$

We say that $\mathcal{L}$ has deduction theorem if for all $\phi, \psi, \theta \in F$ we have

$$\phi, \psi \models \theta \quad \text{if and only if} \quad \phi \models \psi \rightarrow \theta. \tag{2.3}$$

THEOREM 2.1. *Suppose $\mathcal{L}$ is consequence compact, conjunctive, and has deduction theorem. If $\mathcal{L}$ has the mIP$^{\rightarrow}$ then it has the local Beth definability property.*

PROOF: The proof is standard. Suppose that $\Sigma \subseteq F^{P \cup \{r\}}$ defines r implicitly, that is

$$\Sigma \cup \Sigma' \models r \leftrightarrow r'. \tag{2.4}$$

By consequence compactness one can take a finite subset $\Sigma_0 \subset \Sigma$ such that $\Sigma_0 \cup \Sigma_0' \models r \leftrightarrow r'$, and by conjunctiveness if ϕ is the conjunction of the formulas in Σ_0, then

$$\phi, \phi' \models r \leftrightarrow r'. \tag{2.5}$$

By deduction and conjunctiveness

$$\models (\phi \wedge \phi') \to (r \leftrightarrow r'). \tag{2.6}$$

Using (2.1), from (2.6) we get the equivalent

$$\models \phi \to (\phi' \to (r \to r')) \tag{2.7}$$

$$\models \phi \to (r \to (\phi' \to r')) \tag{2.8}$$

$$\models (\phi \wedge r) \to (\phi' \to r') \tag{2.9}$$

For any model $\mathfrak{M}$, by mIP$^{\to}$, there is an interpolant formula $\theta_{\mathfrak{M}} \subseteq F^P$ such that

$$\mathfrak{M} \models (\phi \wedge r) \to \theta_{\mathfrak{M}}, \quad \text{and} \quad \mathfrak{M} \models \theta_{\mathfrak{M}} \to (\phi' \to r'), \tag{2.10}$$

hence, using (2.1) again, we get

$$\mathfrak{M} \models \phi \to (r \leftrightarrow \theta_{\mathfrak{M}}). \tag{2.11}$$

By deduction, for every $\mathfrak{M} \models \Sigma$ one has $\mathfrak{M} \models r \leftrightarrow \theta_{\mathfrak{M}}$, that is, Σ locally explicitly defines r. □

Corollary 2.2. Difference logic $\mathcal{L}_D$ has the local Beth definability property.

Proof: Combine Theorems 1.5 and Theorem 2.1. □

Next, we give an algebraic characterization of the modelwise interpolation property in terms of amalgamation of algebras. Algebraic characterizations of the IP and the lIP have been done respectively in the papers [20] and [16]. The definition of logic employed so far is too general to have an algebraic counterpart. Therefore we restrict our attention to a subclass of logics that are algebraizable. From now on in this section we work with algebraizable logics as defined in the Andréka–Németi–Sain framework [3]. We recall the indispensable definitions below, and for a brief and self-contained summary we refer the reader to [3], or [20, 17].

* * *

By an *algebraizable logic* we understand a tuple $\mathcal{L} = \langle F, M, \text{mng}, \models \rangle$ that satisfies the following requirements.

- $\langle F, M, \models \rangle$ is a logic as described at the beginning of the present paper. That is, the set of formulas F is the universe of the free algebra $\mathcal{F}$ generated by some set P of atomic formulas in similarity type Cn. M is a non-empty class of models, and $\models$ is a relation between models and formulas.
- mng, called the meaning function, is a function with domain $M \times F$. We write $\text{mng}_{\mathfrak{M}}(\phi)$ in place of $\text{mng}(\mathfrak{M}, \phi)$ and require that $(\forall \phi, \psi \in F)$ $(\forall \mathfrak{M} \in M)$
$$\big(\text{mng}_{\mathfrak{M}}(\phi) = \text{mng}_{\mathfrak{M}}(\psi) \text{ and } \mathfrak{M} \models \phi\big) \Longrightarrow \mathfrak{M} \models \psi. \tag{2.12}$$
- **Compositionality**: For every model $\mathfrak{M}$, the meaning function $\text{mng}_{\mathfrak{M}}$ is a homomorphism from the formula algebra $\mathcal{F}$ into some algebra.
- **Filter property**: There are connectives $\leftrightarrow$ (binary) and $\top$ (constant) such that
$$\mathfrak{M} \models \phi \leftrightarrow \psi \quad \text{iff} \quad \text{mng}_{\mathfrak{M}}(\phi) = \text{mng}_{\mathfrak{M}}(\psi) \tag{2.13}$$
and
$$\mathfrak{M} \models \phi \quad \text{iff} \quad \mathfrak{M} \models \phi \leftrightarrow \top. \tag{2.14}$$
- **Substitution property**: For every model $\mathfrak{M}$ and homomorphism $h : \mathcal{F} \to \text{mng}_{\mathfrak{M}}(\mathcal{F})$ there is a model $\mathfrak{N}$ (called the substituted version of $\mathfrak{M}$) such that $\text{mng}_{\mathfrak{N}} = h$.
- **Patchwork property**: Suppose $\mathfrak{M}, \mathfrak{N}$ are models and A and B are sets of atomic formulas. If $\text{mng}_{\mathfrak{M}}$ and $\text{mng}_{\mathfrak{N}}$ agree on formulas using vocabulary $A \cap B$, then there is a model $\mathfrak{P}$ such that $\text{mng}_{\mathfrak{P}}$ agrees with $\text{mng}_{\mathfrak{M}}$ on formulas over the vocabulary A, and $\text{mng}_{\mathfrak{P}}$ agrees with $\text{mng}_{\mathfrak{N}}$ on formulas over the vocabulary B.

We note that all our examples $\mathcal{L}_{Prop}$, $\mathcal{L}_{Sent}$, $\mathcal{L}_D$ and $\mathcal{L}_n$ are algebraizable logics with a proper choice of the meaning function. For a detailed discussion and for more examples we refer to [3]. We write

$$\begin{aligned} \mathsf{Alg}_m(\mathcal{L}) &= \{\text{mng}_{\mathfrak{M}}(\mathcal{F}') : \mathfrak{M} \in M, \mathcal{F}' \text{ is a subalgebra of } \mathcal{F}\} \\ \mathsf{Alg}_{\models}(\mathcal{L}) &= \{\mathfrak{A} : \mathfrak{A} \cong \mathcal{F}/{\sim_K}, K \subseteq M\}, \text{ where } \phi \sim_K \psi \text{ iff } K \models \phi \leftrightarrow \psi, \end{aligned}$$

for the class of meaning algebras and the class of Lindenbaum-Tarski algebras, respectively.

* * *

Let t be an algebraic similarity type. Given a set of equations $e(x, y)$ of type t and a t-type algebra $\mathfrak{A}$ we write

$$\leq_e^{\mathfrak{A}} = \{\langle a, b\rangle \in A \times A : \mathfrak{A} \models e(a, b)\} \tag{2.15}$$

Many cases $e(x, y)$ is a single equation, consider for example the Boolean case, where $x \leq y$ corresponds to the equation $x \wedge y = x$. Note that $\leq_e^{\mathfrak{A}}$ need not be a partial ordering, in general.

Next we define a variant of the superamalgamation property. The original superamalgamation property goes back to Maksimova [23, 22] and a slightly modified version of it has been introduced in [20]. For a class K of algebras and a set X, $\mathfrak{Fr}_{\mathsf{K}}(X)$ denotes the K-free algebra generated by X. For algebras $\mathfrak{A}$ and $\mathfrak{B}$ the relation $\mathfrak{A} \subseteq \mathfrak{B}$ means that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.

Definition 2.3. Let $e(x, y)$ be a set of equations. We say that K has the SUP$_e$ (weak superamalgamation property) if for every $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathsf{K}$ with $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$ there exists $\mathfrak{A}_3 \in \mathsf{K}$ such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_3$, $\mathfrak{A}_2 \subseteq \mathfrak{A}_3$ and whenever the diagram below commutes (for arbitrary sets X and Y),

$$\begin{tikzcd}
 & \mathfrak{Fr}_{\mathsf{K}}(X) \arrow[rrr, dashed, two heads] \arrow[dr, tail] & & & \mathfrak{A}_1 \arrow[dr, tail] & \\
\mathfrak{Fr}_{\mathsf{K}}(X \cap Y) \arrow[ur, tail] \arrow[dr, tail] \arrow[rrr, dashed, two heads, bend left] & & \mathfrak{Fr}_{\mathsf{K}}(X \cup Y) \arrow[rrr, dashed, two heads, bend right] & \mathfrak{A}_0 \arrow[ur, tail] \arrow[dr, tail, "h"'] & & \mathfrak{A}_3 \\
 & \mathfrak{Fr}_{\mathsf{K}}(Y) \arrow[rrr, dashed, two heads] \arrow[ur, tail] & & & \mathfrak{A}_2 \arrow[ur, tail] &
\end{tikzcd}$$

then $\forall x \in \mathfrak{Fr}_{\mathsf{K}}(X)$ and $\forall y \in \mathfrak{Fr}_{\mathsf{K}}(Y)$ we have

$$\Big(x \ \leq_e^{\mathfrak{Fr}_{\mathsf{K}}(X\cup Y)} \ y \quad \Longrightarrow \quad (\exists z \in \mathfrak{A}_0)(h(x) \ \leq_e^{\mathfrak{A}_3} \ z \text{ and } z \ \leq_e^{\mathfrak{A}_3} \ h(y))\Big)$$

(Here the embeddings between the K-free algebras are the embeddings induced by the inclusion maps between the sets of generators).

THEOREM 2.4. *Let $\mathcal{L}$ be an algebraizable logic. Assume $\mathcal{L}$ has a derived binary connective $\rightsquigarrow$ and let $e(x,y)$ denote the equations $x \rightsquigarrow y = \top$. Then*

$$\mathcal{L} \text{ has the mIP}^{\rightsquigarrow} \iff \mathsf{Alg}_m(\mathcal{L}) \text{ has the SUP}_e. \tag{2.16}$$

PROOF: ($\Leftarrow$) Assume $\mathsf{Alg}_m(\mathcal{L})$ has the SUP$_e$ and let $\phi, \psi \in F$ be such that $\models \phi \rightsquigarrow \psi$. We need to find, for every $\mathfrak{M} \in M$, a formula $\chi \in F$ with $\mathrm{Voc}(\chi) \subseteq \mathrm{Voc}(\phi) \cap \mathrm{Voc}(\psi)$ such that $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $\mathfrak{M} \models \chi \rightsquigarrow \psi$. In what follows, $\mathcal{F}^V$ denotes the set of formulas in F whose vocabulary is in V. Let $\mathfrak{M} \in M$ be an arbitrary model, write $V = \mathrm{Voc}(\phi)$, $W = \mathrm{Voc}(\psi)$ and consider the following meaning algebras: $\mathfrak{A}_3 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{V \cup W})$, $\mathfrak{A}_1 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^V)$, $\mathfrak{A}_2 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^W)$, $\mathfrak{A}_0 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{V \cap W})$. Now, $\models \phi \rightsquigarrow \psi$ implies (see [3, Corollary 5.5])

$$\mathsf{Alg}_{\models}(\mathcal{L}) \models \phi \rightsquigarrow \psi = \top, \tag{2.17}$$

hence, considering ϕ and ψ as elements of the free algebra $\mathfrak{Fr}_{\mathsf{Alg}(\mathcal{L})}(V \cup W)$, we have

$$\phi \leq_e^{\mathfrak{Fr}(V \cup W)} \psi. \tag{2.18}$$

We note that free algebras of $\mathsf{Alg}_{\models}(\mathcal{L})$ and that of $\mathsf{Alg}_m(\mathcal{L})$ are the same as $\mathbf{SP}\mathsf{Alg}_m(\mathcal{L}) \supseteq \mathsf{Alg}_{\models}(\mathcal{L})$ (see [3, Thm 5.3]). Consider the diagram in Definition 2.3. By SUP$_e$ there must exist $z \in \mathfrak{A}_0$ such that $h(\phi) \leq_e^{\mathfrak{A}_3} z$ and $z \leq_e^{\mathfrak{A}_3} h(\psi)$. As $z \in \mathfrak{A}_0$ there is $\chi \in F^{V \cap W}$ with $z = \mathrm{mng}_{\mathfrak{M}}(\chi)$. Then $h(\phi) \leq_e^{\mathfrak{A}_3} z$ implies $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $z \leq_e^{\mathfrak{A}_3} h(\psi)$ implies $\mathfrak{M} \models \chi \rightsquigarrow \psi$.

($\Rightarrow$) Assume that $\mathcal{L}$ has the mIP$^{\rightsquigarrow}$. To show that $\mathsf{Alg}_m(\mathcal{L})$ has the SUP$_e$, take algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathsf{Alg}_m(\mathcal{L})$ such that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$.

LEMMA 2.5. *For every $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathsf{Alg}_m(\mathcal{L})$ with $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$ there is $\mathfrak{A}_3 \in \mathsf{Alg}_m(\mathcal{L})$ such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_3$ and $\mathfrak{A}_2 \subseteq \mathfrak{A}_3$.*

PROOF: Suppose $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathsf{Alg}_m(\mathcal{L})$ are such that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_2$. Let $f : A_1 \to A_1$ and $g : A_2 \to A_2$ be the identity mappings. Then f and g extend to homomorphisms $\bar{f} : \mathcal{F}^{A_1} \to \mathfrak{A}_1$ and $\bar{g} : \mathcal{F}^{A_2} \to \mathfrak{A}_2$. By the substitution property of $\mathcal{L}$ there are models $\mathfrak{M} \in M$ and $\mathfrak{N} \in M$ so that $\bar{f} = \mathrm{mng}_{\mathfrak{M}}$ and $\bar{g} = \mathrm{mng}_{\mathfrak{N}}$. By the patchwork property, for some model $\mathfrak{D} \in M$ we have $\mathrm{mng}_{\mathfrak{D}} \restriction F^{A_1} = \mathrm{mng}_{\mathfrak{M}}$ and $\mathrm{mng}_{\mathfrak{D}} \restriction F^{A_2} = \mathrm{mng}_{\mathfrak{N}}$. It follows that $\mathfrak{A}_1 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_1}) \subseteq \mathrm{mng}_{\mathfrak{D}}(\mathcal{F}^{A_1 \cup A_2})$ and $\mathfrak{A}_2 = \mathrm{mng}_{\mathfrak{N}}(\mathcal{F}^{A_2}) \subseteq \mathrm{mng}_{\mathfrak{D}}(\mathcal{F}^{A_1 \cup A_2})$. □

Let $\mathfrak{A}_3$ be as in Lemma 2.5. As $\mathfrak{A}_3 \in \mathsf{Alg}_m(\mathcal{L})$ it is the image of the meaning function with respect to some model $\mathfrak{M}$, i.e. $\mathfrak{A}_3 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_3})$. Then $\mathfrak{A}_1 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_1})$, $\mathfrak{A}_2 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_2})$ and $\mathfrak{A}_0 = \mathrm{mng}_{\mathfrak{M}}(\mathcal{F}^{A_0})$.

Consider the diagram in Definition 2.3 and suppose that for $x \in \mathfrak{Fr}(X)$ and $y \in \mathfrak{Fr}(Y)$ we have $\mathfrak{Fr}(X \cup Y) \models x \leq_e y$. There are formulas $\phi \in F^{A_1}$ and $\psi \in F^{A_2}$ such that $\mathrm{mng}_{\mathfrak{M}}(\phi) = h(x)$ and $\mathrm{mng}_{\mathfrak{M}}(\psi) = h(y)$. By the filter property, $\mathfrak{A}_3 \models h(x) \leq_e h(y)$ is equivalent to $\mathfrak{M} \models \phi \rightsquigarrow \psi$. Using the mIP$^{\rightsquigarrow}$ one finds a formula $\chi \in F^{A_1 \cap A_2}$ such that $\mathfrak{M} \models \phi \rightsquigarrow \chi$ and $\mathfrak{M} \models \chi \rightsquigarrow \psi$. Clearly, $z = \mathrm{mng}_{\mathfrak{M}}(\chi) \in \mathfrak{A}_0$ and it follows that $h(x) \leq_e^{\mathfrak{A}_3} z$ and $z \leq_e^{\mathfrak{A}_3} h(y)$. □

The weak superamalgamation property is kind of a direct translation of the modelwise interpolation property into an algebraic setting. Even thought this translation is very direct, nevertheless it needed a justification (the proof of Theorem 2.4). As the weak superamalgamation property explicitly mentions free algebras, the correspondence might not be as strong as one would expect. On the other hand, let us note that the algebraic characterization of the regular Craig interpolation property also directly mentions free algebras, as it is equivalent to the superamalgamation property of free algebras (see [20, Def.4.4] for the definition of "Free SUPAP" and [20, Prop.4.6] for the equivalence between the Free SUPAP and the Craig interpolation property). It is "only" certain varieties of Boolean algebras with operators where the free superamalgamation property implies a more general amalgamation property of the variety (for such results we also refer to Madarász [20]). We do not yet know whether our weak superamalgamation property can be strengthened in classes of algebras having additional properties.

There are several variants of the interpolation property, such as Lyndon's interpolation, uniform interpolation, etc. It could be interesting to see to what extent the "modelwise" variants of these properties are meaningful or useful. We did not make effort to investigate this systematically, but it could serve a possible direction for further research.[5]

---

[5] We would like to thank the anonymous referee for suggesting us to mention such possible further directions.

Acknowledgements. We wish to express our gratitude towards the anonymous referees for the careful reading of the manuscript and the helpful suggestions.

This research was supported by the grant 2019/34/E/HS1/00044, "Epistemic inaccuracy and foundational issues in formal epistemology", of the National Science Centre, Poland. We would like to acknowledge the grant of the Hungarian National Research, Development and Innovation Office, contract number: K-134275.

References

[1] H. Andréka, Z. Gyenis, I. Németi, I. Sain, **Universal Algebraic Logic**, Birkhauser (2022), DOI: https://doi.org/10.1007/978-3-031-14887-3.

[2] H. Andréka, J. X. Madarász, I. Németi, *Logic of Space-Time and Relativity Theory*, [in:] M. Aiello, I. Pratt-Hartmann, J. Van Benthem (eds.), **Handbook of Spatial Logics**, Springer Netherlands, Dordrecht (2007), pp. 607–711, DOI: https://doi.org/10.1007/978-1-4020-5587-4_11.

[3] H. Andréka, I. Németi, I. Sain, *Algebraic logic*, [in:] **Handbook of Philosophical Logic**, vol. 2, Kluwer Academic Publishers, Dordrecht (2001), pp. 133–247.

[4] H. Andréka, I. Németi, J. van Benthem, *Interpolation and Definability Properties of Finite Variable Fragments*, **Reports of the Mathematical Institute, Hungarian Academy of Sciences**, (1993).

[5] H. Andréka, I. Németi, J. van Benthem, *Modal languages and bounded fragments of predicate logic*, **Journal of Philosophical Logic**, vol. 27(3) (1998), pp. 217–274, DOI: https://doi.org/10.1023/A:1004275029985.

[6] J. Barwise, *On Moschovakis closure ordinals*, **Journal of Symbolic Logic**, vol. 42(2) (1977), pp. 292–296, DOI: https://doi.org/10.2307/2272133.

[7] P. Blackburn, M. d. Rijke, Y. Venema, **Modal Logic**, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press (2001), DOI: https://doi.org/10.1017/CBO9781107050884.

[8] W. J. Blok, D. Pigozzi, *Algebraizable logics*, **Memoirs of the American Mathematical Society**, vol. 77(396) (1989), pp. vi+78, DOI: https://doi.org/10.1090/memo/0396.

[9] W. J. Blok, D. Pigozzi, *Abstract Algebraic Logic*, **Lecture Notes of the Summer School "Algebraic Logic and the Methodology of Applying it", Budapest**, (1994).

[10] W. J. Blok, D. L. Pigozzi, *Local deduction theorems in algebraic logic*, [in:] **Algebraic logic (Budapest, 1988)**, vol. 54 of Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam (1991), pp. 75–109.

[11] W. Conradie, **Definability and changing perspectives**, Master's thesis, University of Amsterdam (2002).

[12] J. Czelakowski, *Logical matrices and the amalgamation property*, **Studia Logica**, vol. 41(4) (1982), pp. 329–341 (1983), DOI: https://doi.org/10.1007/BF00403332.

[13] J. Czelakowski, D. Pigozzi, *Amalgamation and interpolation in abstract algebraic logic*, [in:] **Models, algebras, and proofs (Bogotá, 1995)**, vol. 203 of Lecture Notes in Pure and Applied Mathematics, Dekker, New York (1999), pp. 187–265, DOI: https://doi.org/10.1201/9780429332890.

[14] J. Czelakowski, D. Pigozzi, *Fregean logics*, **Annals of Pure and Applied Logic**, vol. 127(1–3) (2004), pp. 17–76, DOI: https://doi.org/10.1016/j.apal.2003.11.008.

[15] Z. Gyenis, *Interpolation property and homogeneous structures*, **Logic Journal of the IGPL**, vol. 22(4) (2014), pp. 597–607, DOI: https://doi.org/10.1093/jigpal/jzt051.

[16] Z. Gyenis, *Algebraic characterization of the local Craig interpolation property*, **Bulletin of the Section of Logic**, vol. 47(1) (2018), pp. 45–58, DOI: https://doi.org/10.18778/0138-0680.47.1.04.

[17] E. Hoogland, **Algebraic characterizations of two Beth definability properties**, Master's thesis, Universiteit van Amsterdam (1996).

[18] E. Hoogland, **Definability and Interpolation, model-theoretic investigations**, Ph.D. thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam (2001).

[19] P. Krzystek, S. Zachorowski, *Lukasiewicz logics have not the interpolation property*, **Reports on Mathematical Logic**, vol. 9 (1977), pp. 39–40.

[20] J. X. Madarász, *Interpolation and amalgamation; pushing the limits. I*, **Studia Logica**, vol. 61(3) (1998), pp. 311–345, DOI: https://doi.org/10.1023/A:1005064504044.

[21] J. X. Madarász, I. Németi, G. Székely, *First-Order Logic Foundation of Relativity Theories*, [in:] D. M. Gabbay, M. Zakharyaschev, S. S. Goncharov (eds.), **Mathematical Problems from Applied Logic II: Logics for the XXIst Century**, Springer New York, New York, NY (2007), pp. 217–252, DOI: https://doi.org/10.1007/978-0-387-69245-6_4.

[22] L. Maksimova, *Amalgamation and interpolation in normal modal logic*, **Studia Logica**, vol. 50(3–4) (1991), pp. 457–471, DOI: https://doi.org/10.1007/BF00370682, algebraic logic.

[23] L. L. Maksimova, *Interpolation theorems in modal logics and amalgamable varieties of topological Boolean algebras*, **Algebra i Logika**, vol. 18(5) (1979), pp. 556–586, 632.

[24] P. Mancosu, *Introduction: Interpolations—essays in honor of William Craig*, **Synthese**, vol. 164(3) (2008), pp. 313–319, DOI: https://doi.org/10.1007/s11229-008-9350-6.

[25] G. Metcalfe, F. Montagna, C. Tsinakis, *Amalgamation and interpolation in ordered algebras*, **Journal of Algebra**, vol. 402 (2014), pp. 21–82, DOI: https://doi.org/10.1016/j.jalgebra.2013.11.019.

[26] D. Mundici, *Consequence and Interpolation in Lukasiewicz Logic*, **Studia Logica**, vol. 99(1/3) (2011), pp. 269–278, URL: http://www.jstor.org/stable/41475204.

[27] D. Nyiri, *The Robinson property and amalgamations of higher arities*, **Mathematical Logic Quarterly**, vol. 62(4–5) (2016), pp. 427–433, DOI: https://doi.org/10.1002/malq.201500027.

[28] D. Pigozzi, *Amalgamation, congruence-extension, and interpolation properties in algebras*, **Algebra Universalis**, vol. 1 (1971/72), pp. 269–349, DOI: https://doi.org/10.1007/BF02944991.

[29] D. J. Pigozzi, *Fregean algebraic logic*, [in:] **Algebraic logic (Budapest, 1988)**, vol. 54 of Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam (1991), pp. 473–502.

[30] G. Priest, **An Introduction to Non-Classical Logic: From If to Is**, Cambridge University Press (2008).

[31] D. Roorda, **Resource Logics. Proof-Theoretical Investigations**, Ph.D. thesis, Institute for Logic, Language and Computation, University of Amsterdam (1991).

[32] G. Sági, S. Shelah, *On weak and strong interpolation in algebraic logics*, **The Journal of Symbolic Logic**, vol. 71(1) (2006), pp. 104–118, DOI: https://doi.org/10.2178/jsl/1140641164.

[33] I. Sain, *Successor axioms for time increase the program verifying power of full computational induction*, **Mathematical Institute if the Hungarian Academy of Sciences**, vol. 23 (1983).

[34] I. Sain, *Is "some-other-time" sometimes better than "sometime" for proving partial correctness of programs?*, **Studia Logica**, vol. 47(3) (1988), pp. 279–301, DOI: https://doi.org/10.1007/BF00370557.

[35] I. Sain, *Beth's and Craig's properties via epimorphisms and amalgamation in algebraic logic*, [in:] **Algebraic Logic and Universal Algebra in Computer Science (Ames, IA, 1988)**, vol. 425 of Lecture Notes in Computer Science, Springer, Berlin (1990), pp. 209–225, DOI: https://doi.org/10.1007/BFb0043086.

[36] K. Segerberg, *"Somewhere else" and "Some other time"*, [in:] **Wright and Wrong—Mini essays in honor of Georg Henrik von Wright**, vol. 3 of Publications, Group in Logic and Methodology of Real Finland (1976), pp. 61–64.

[37] S. J. van Gool, G. Metcalfe, C. Tsinakis, *Uniform interpolation and compact congruences*, **Annals of Pure and Applied Logic**, vol. 168(10) (2017), pp. 1927–1948, DOI: https://doi.org/10.1016/j.apal.2017.05.001.

[38] Y. Venema, **Many-dimensional Modal Logic**, Ph.D. thesis, Institute for Logic, Language and Computation, University of Amsterdam (1992).

Zalán Gyenis
Jagiellonian University
Institute of Philosophy
ul. Grodzka 52
31-044 Kraków, Poland
e-mail: zalan.gyenis@uj.edu.pl

Zalán Molnár
Eötvös Loránd University
Department of Logic
Múzeum krt. 4.
1088 Budapest, Hungary
e-mail: mozaag@gmail.com

Övge Öztürk
Eötvös Loránd University
Department of Logic
Múzeum krt. 4.
1088 Budapest, Hungary
e-mail: ovgeovge@gmail.com