Bulletin of the **S**ection of **L**ogic
Volume 52/2 (2023), pp. 187–237
https://doi.org/10.18778/0138-0680.2023.18

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ON SYNONYMY IN PROOF-THEORETIC SEMANTICS. THE CASE OF `2Int`

Abstract

We consider an approach to propositional synonymy in proof-theoretic semantics that is defined with respect to a bilateral G3-style sequent calculus `SC2Int` for the bi-intuitionistic logic `2Int`. A distinctive feature of `SC2Int` is that it makes use of two kinds of sequents, one representing proofs, the other representing refutations. The structural rules of `SC2Int`, in particular its cut rules, are shown to be admissible. Next, interaction rules are defined that allow transitions from proofs to refutations, and vice versa, mediated through two different negation connectives, the well-known *implies-falsity negation* and the less well-known *co-implies-truth negation* of `2Int`. By assuming that the interaction rules have no impact on the identity of derivations, the concept of inherited identity between derivations in `SC2Int` is introduced and the notions of positive and negative synonymy of formulas are defined. Several examples are given of distinct formulas that are either positively or negatively synonymous. It is conjectured that the two conditions cannot be satisfied simultaneously.

Keywords: bilateralism, bi-intuitionistic logic `2Int`, cut-elimination, identity of derivations, synonymy.

Presented by: Andrzej Indrzejczak
Received: October 31, 2022
Published online: July 18, 2023

1. Introduction

This paper is a sequel to [18], where an approach to synonymy of formulas in proof-theoretic semantics is explored that is not based on a structural isomorphism between derivation trees and that departs from the idea of identifying only derivations of one and the same formula. The approach is *bilateral* in the sense that a distinction is drawn between two kinds of derivations, namely proofs and refutations.[1] The identification of derivations of distinct formulas is arrived at in particular by considering any proof of a formula A as a refutation of A's negation, $\sim A$, and identifying a refutation of A with a proof of $\sim A$. Such a direct relationship between proofs and refutations understood as disproofs is given in the constructive paraconsistent logic N4 with strong negation due to Almukdad and Nelson [1], a system that was independently studied already by Prawitz (very briefly in Appendix B of [12]) and von Kutschera [14]. In [18] the notion of *inherited identity* of derivations is introduced for derivations in a cut-free sequent system for N4 with two kinds of sequents by considering sequent rules the application of which leaves the identity of derivations untouched. The relation of inherited identity is used to define a bilateralist notion of synonymy between formulas, which is a relation drawing more fine-grained distinctions between formulas than the relation of strong equivalence that obtains between two formulas A and B if both A and B and their strong negations $\sim A$ and $\sim B$ are interderivable.

In [18] the problem was left open, whether and how the explored bilateralist conception of propositional synonymy in proof-theoretic semantics can be applied to a system closely related to N4, namely the bi-intuitionistic system 2Int from [15], see also [4]. Like in the proof theory of N4, in proof systems for 2Int a distinction can be drawn between proofs and refutations, there called "dual proofs". Now, however, the relationship between proofs and refutations is more intricate since the transition between them is reflected in the logical vocabulary not by the presence of a single strong negation connective, but by making use of two negation operations, the familiar *implies-falsity negation* known from intuitionistic logic and the *co-implies-truth negation* from 2Int. These are defined on the basis of two

[1] In [19] we discuss the existing notions of bilateralism in the context of proof-theoretic semantics and propose, based on our understanding of bilateralism, an extension to logical *multilateralism* as a theory of multiple derivability relations, more specifically, as a theory of sequent calculi that make use of multiple sequent arrows.

dual implications, namely the intuitionistic conditional and a so-called "co-implication", which can be seen as the object language realizations of the two derivability relations.

This paper is devoted to applying the bilateralist approach of [18] to 2Int. For that purpose, first of all a suitable proof-theoretic presentation of 2Int is needed, and this is a central contribution of the present paper. For motivation of the bilateralist approach in the case of N4 based on a proof/disproof interpretation that amends the Brouwer-Heyting-Kolmogorov interpretation of the intuitionistic connectives and a comparison to other approaches to propositional synonymy in proof-theoretic terms we refer to [18]. In the present paper we introduce the basic ideas only to the extent of keeping the paper self-contained. In Section 2 we first present the bilateralist sequent calculus SC2Int for 2Int. Next, in Section 3, the admissibility of the structural rules of SC2Int is dealt with. A detailed proof of cut-elimination for SC2Int is given in the appendix, Section 6. Section 4 is devoted to inherited identity of derivations in SC2Int and the definition of propositional synonymy. We conclude the paper with a brief summary and outlook in Section 5.

2. The calculus SC2Int

The purpose of this section is to introduce a bi-intuitionistic sequent calculus and to give proofs of admissibility for its structural rules. The calculus we will present, called SC2Int, is a sequent calculus for the bi-intuitionistic logic 2Int from [15]. There a natural deduction system for this logic, N2Int, is given to which SC2Int is equivalent in terms of what is derivable. We spell out below what this amounts to exactly. What is important is that these calculi represent a kind of bilateralist reasoning, since they do not only internalize processes of verification or provability but also the dual processes in terms of falsification or what is called *dual provability*. In [17] a normal form theorem for N2Int is stated, here, we want to prove a cut-elimination theorem for SC2Int, which goes beyond the results existing so far.

The language $\mathscr{L}_{2Int}$ of 2Int, as given in [15], is defined in Backus-Naur form as follows:

$$A ::= p \mid \bot \mid \top \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \prec A).$$

As can be seen, we have a non-standard connective in this language, namely the operator of co-implication, $\prec$, which acts as a dual to implication, just like conjunction and disjunction can be seen as dual connectives.[2] With that, we are in the realms of so-called bi-intuitionistic logic, which is a conservative extension of intuitionistic logic with co-implication.[3] We read $A \prec B$ as 'B co-implies A'.

The general design of `SC2Int` resembles the intuitionistic sequent calculus `G3ip`. The distinguishing features of this calculus consist in the shared contexts for all the logical rules, the axiom (in our calculus the reflexivity rules) being restricted to atomic formulas and the admissibility of *all* structural rules (cf. [10, pp. 28–30] for more information about the origins of this calculus). Another distinguishing feature is the repetition of $A \to B$ in the left premise of the left introduction rule for implication, which is necessary for the proof of admissibility of contraction. Here, this happens in $\to L^a$ as well as with $A \prec B$ in $\prec L^c$.

We will use $p, q, r, ...$ for atomic formulas, $A, B, C, ...$ for arbitrary formulas, and Γ, Δ, Γ', ... for multisets of formulas. For a singleton multiset $\{A\}$ we usually write just A, and A, Γ as well as Γ, A (Δ, Γ as well as Γ, Δ) designates the union of the multisets Γ and $\{A\}$ (Δ and Γ). Sequents are of the form $(\Gamma; \Delta) \vdash^* C$ (with Γ and Δ being finite, possibly empty multisets), which are read as "From the verification of all formulas in Γ and the falsification of all formulas in Δ one can derive the verification (resp.

[2] An anonymous reviewer raised the question whether co-implication as the dual of implication is again an implication (and the co-negation defined in section 4 is indeed a negation), whereas conjunction as the dual of disjunction is not a disjunction, and disjunction as the dual of conjunction is not a conjunction. Thus, is the dual of a logical operation of a kind different from the kind of operation from which it is a dual? We cannot address this general question here, or the questions "What is an implication?" and "What is a negation?". As far as `2Int` is concerned, there is a clear sense in which implication, $\to$, and co-implication, $\prec$, are of the same kind. In a two-sorted term calculus for `2Int`, see [2], the rule for introducing $\to$ on the right of a sequent arrow in proofs (for introducing $\prec$ on the right of a sequent arrow in dual proofs) comes with λ-abstraction, and the rule for introducing $\to$ on the left of a sequent arrow in proofs (for introducing $\prec$ on the left of a sequent arrow in dual proofs) comes with functional application. The same holds for negation and co-negation in `2Int`.

[3] Note that there is also a use of *bi-intuitionistic logic* in the literature to refer to a specific system, namely `BiInt`, also called *Heyting-Brouwer logic* (e.g. [13, 6, 11, 9, 5]). Co-implication is there to be understood to internalize the preservation of non-truth from the conclusion to the premises in a valid inference. The system 2Int, which is treated here, uses the same language as `BiInt`, but the meaning of co-implication differs (cf. [17, p. 30f.] and [15, 16, 4]).

falsification) of C for $* = +$ (resp. $* = -$)".[4] Thus, we have a calculus in which more than one derivability relation is considered, not only the one of verification but also the one of falsification (or refutation).[5] The formulas in Γ can then be understood as *assumptions*, while the formulas in Δ can be understood as *counterassumptions*. SC2Int is equivalent to N2Int in that we have a proof in N2Int of A from the pair $(\Gamma; \Delta)$ of assumptions Γ and counterassumptions Δ, iff the sequent $(\Gamma; \Delta) \vdash^+ A$ is derivable in SC2Int and we have a dual proof of A from the pair $(\Gamma; \Delta)$ of assumptions Γ and counterassumptions Δ, iff the sequent $(\Gamma; \Delta) \vdash^- A$ is derivable in SC2Int.

In contrast to G3ip, there will be no distinction between axioms and logical rules but within the logical rules the zero-premise rules, which comprise Rf^+, Rf^-, $\bot L^a$, $\top L^c$, $\bot R^-$, and $\top R^+$, are distinguished from the non-zero-premise rules due to the special role of the former for the admissibility proofs below. Each of the logical rules has a *context* designated by Γ and Δ, *active formulas* designated by A and B and a *principal formula*, which is the one introduced on the left or right side of $\vdash^*$. Within the right introduction rules we need to distinguish whether the derivability relation expresses verification or falsification by using the superscripts $+$ and $-$. Within the left rules this is not necessary, but what is needed here is distinguishing an introduction of the principal formula into the *assumptions* from an introduction into the *counterassumptions*. The former are indexed by superscript a, while the latter are indexed by superscript c. The set of R^+ and L^a rules are the *proof rules*; the set of R^- and L^c rules are the *dual proof rules*.

SC2Int

For $* \in \{+, -\}$:

$$\frac{}{(\Gamma, p; \Delta) \vdash^+ p}\ Rf^+ \qquad \frac{}{(\Gamma; \Delta, p) \vdash^- p}\ Rf^-$$

[4]Note that the notation for sequents in [18] is different and follows the presentation of the subformula calculus for N4 in [7, 8]. In particular, expressions $\Gamma : \Delta \Rightarrow^* C$ (with Γ and Δ being finite, possibly empty multisets) are read as "From the falsification of all formulas in Γ and the verification of all formulas in Δ one can derive the verification (resp. falsification) of C for $* = +$ (resp. $* = -$)". The notation in the present paper is taken from [3]

[5]In N2Int this is indicated by using single lines for verification and double lines for falsification.

$$\frac{}{(\Gamma, \bot; \Delta) \vdash^* C}\ \bot L^a \qquad \frac{}{(\Gamma; \Delta, \top) \vdash^* C}\ \top L^c$$

$$\frac{}{(\Gamma; \Delta) \vdash^- \bot}\ \bot R^- \qquad \frac{}{(\Gamma; \Delta) \vdash^+ \top}\ \top R^+$$

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \wedge B}\ \wedge R^+ \qquad \frac{(\Gamma, A, B; \Delta) \vdash^* C}{(\Gamma, A \wedge B; \Delta) \vdash^* C}\ \wedge L^a$$

$$\frac{(\Gamma; \Delta) \vdash^- A}{(\Gamma; \Delta) \vdash^- A \wedge B}\ \wedge R_1^- \qquad \frac{(\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^- A \wedge B}\ \wedge R_2^-$$

$$\frac{(\Gamma; \Delta, A) \vdash^* C \qquad (\Gamma; \Delta, B) \vdash^* C}{(\Gamma; \Delta, A \wedge B) \vdash^* C}\ \wedge L^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ A}{(\Gamma; \Delta) \vdash^+ A \vee B}\ \vee R_1^+ \qquad \frac{(\Gamma; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \vee B}\ \vee R_2^+$$

$$\frac{(\Gamma, A; \Delta) \vdash^* C \qquad (\Gamma, B; \Delta) \vdash^* C}{(\Gamma, A \vee B; \Delta) \vdash^* C}\ \vee L^a$$

$$\frac{(\Gamma; \Delta) \vdash^- A \qquad (\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^- A \vee B}\ \vee R^- \qquad \frac{(\Gamma; \Delta, A, B) \vdash^* C}{(\Gamma; \Delta, A \vee B) \vdash^* C}\ \vee L^c$$

$$\frac{(\Gamma, A; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \to B}\ {\to} R^+ \qquad \frac{(\Gamma, A \to B; \Delta) \vdash^+ A \quad (\Gamma, B; \Delta) \vdash^* C}{(\Gamma, A \to B; \Delta) \vdash^* C}\ {\to} L^a$$

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^- A \to B}\ {\to} R^- \qquad \frac{(\Gamma, A; \Delta, B) \vdash^* C}{(\Gamma; \Delta, A \to B) \vdash^* C}\ {\to} L^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^- B}{(\Gamma; \Delta) \vdash^+ A \prec B}\ {\prec} R^+ \qquad \frac{(\Gamma, A; \Delta, B) \vdash^* C}{(\Gamma, A \prec B; \Delta) \vdash^* C}\ {\prec} L^a$$

$$\frac{(\Gamma; \Delta, B) \vdash^- A}{(\Gamma; \Delta) \vdash^- A \prec B}\ {\prec} R^- \qquad \frac{(\Gamma; \Delta, A \prec B) \vdash^- B \qquad (\Gamma; \Delta, A) \vdash^* C}{(\Gamma; \Delta, A \prec B) \vdash^* C}\ {\prec} L^c$$

Note that the rules for $\wedge L^a$, $\vee L^c$, $\to L^c$ and $\prec L^a$ could also be given in the form of two rules, each with only one active formula A or B, as it is for example done in Gentzen's original calculus for the left conjunction rule. We need this single rule formulation, however, in order to get the invertibility of these rules (cf. Lemma 3.3 below), which is important for the proof of admissibility of contraction. As said above, the structural rules do not have to be taken as primitive in the calculus but can be shown to be admissible.

We want to consider rules for weakening, contraction and cut. Due to the dual nature of the calculus, we need two rules for each of these rules:

$$\frac{(\Gamma; \Delta) \vdash^* C}{(\Gamma, A; \Delta) \vdash^* C}\ W^a \qquad \frac{(\Gamma; \Delta) \vdash^* C}{(\Gamma; \Delta, A) \vdash^* C}\ W^c$$

$$\frac{(\Gamma, A, A; \Delta) \vdash^* C}{(\Gamma, A; \Delta) \vdash^* C}\ C^a \qquad \frac{(\Gamma; \Delta, A, A) \vdash^* C}{(\Gamma; \Delta, A) \vdash^* C}\ C^c$$

$$\frac{(\Gamma; \Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^a$$

$$\frac{(\Gamma; \Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^c$$

3. Proving admissibility of the structural rules

3.1. Preliminaries

The proofs of admissibility of the structural rules and especially of cut-elimination are conducted analogously to the respective proofs of [10, pp. 30–40] for `G3ip`. The proofs will use induction on **weight of formulas** and **height of derivations**.

Definition 3.1. The weight w(A) of a formula A is defined inductively by
$w(\bot) = w(\top) = 0$,
$w(p) = 1$ for atoms p,
$w(A \# B) = w(A) + w(B) + 1$ for $\# \in \{\wedge, \vee, \rightarrow, \prec\}$.

Definition 3.2. A derivation in `SC2Int` is either an instance of a zero-premise rule, or an application of a logical rule to derivations concluding its premises. The *height of a derivation* is the greatest number of successive applications of rules in it, where zero-premise rules have height 0.

First, we will show that the reflexivity rules can be generalized to instances with arbitrary formulas, not only atomic formulas.

Lemma 3.3. *The sequents* $(\Gamma, C; \Delta) \vdash^+ C$ *and* $(\Gamma; \Delta, C) \vdash^- C$ *are derivable for an arbitrary formula* C *and arbitrary context* $(\Gamma; \Delta)$.

Proof: The proof is by induction on weight of C. If $w(C) \leq 1$, we have the 19 cases listed below. Note that for some of the derivations there is more than one possibility to derive the desired sequent and also some of the conclusions of zero-premise rules are conclusions of more than one of those rules. We will just show one exemplary derivation for each case, since this is enough for the proof.

$C = \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ is an instance of $\bot L^a$ and $(\Gamma; \Delta, C) \vdash^- C$ is an instance of $\bot R^-$.

$C = \top$. Then $(\Gamma, C; \Delta) \vdash^+ C$ is an instance of $\top R^+$ and $(\Gamma; \Delta, C) \vdash^- C$ is an instance of $\top L^c$.

$C = p$ for some atom p. Then $(\Gamma, C; \Delta) \vdash^+ C$ is an instance of Rf^+ and $(\Gamma; \Delta, C) \vdash^- C$ is an instance of Rf^-.

$C = \bot \land \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot, \bot; \Delta) \vdash^+ \bot \land \bot}\,{\scriptstyle \bot L^a}}{(\Gamma, \bot \land \bot; \Delta) \vdash^+ \bot \land \bot}\,{\scriptstyle \land L^a} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \bot \land \bot) \vdash^- \bot}\,{\scriptstyle \bot R^-}}{(\Gamma; \Delta, \bot \land \bot) \vdash^- \bot \land \bot}\,{\scriptstyle \land R^-}$$

$C = \bot \lor \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot; \Delta) \vdash^+ \bot \lor \bot}\,{\scriptstyle \bot L^a} \qquad \dfrac{}{(\Gamma, \bot; \Delta) \vdash^+ \bot \lor \bot}\,{\scriptstyle \bot L^a}}{(\Gamma, \bot \lor \bot; \Delta) \vdash^+ \bot \lor \bot}\,{\scriptstyle \lor L^a} \quad \text{and}$$

$$\dfrac{\dfrac{}{(\Gamma; \Delta, \bot \lor \bot) \vdash^- \bot}\,{\scriptstyle \bot R^-} \qquad \dfrac{}{(\Gamma; \Delta, \bot \lor \bot) \vdash^- \bot}\,{\scriptstyle \bot R^-}}{(\Gamma; \Delta, \bot \lor \bot) \vdash^- \bot \lor \bot}\,{\scriptstyle \lor R^-}$$

$C = \bot \to \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot \to \bot, \bot; \Delta) \vdash^+ \bot}\,{\scriptstyle \bot L^a}}{(\Gamma, \bot \to \bot; \Delta) \vdash^+ \bot \to \bot}\,{\scriptstyle \to R^+} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma, \bot; \Delta, \bot) \vdash^- \bot \to \bot}\,{\scriptstyle \bot L^a}}{(\Gamma; \Delta, \bot \to \bot) \vdash^- \bot \to \bot}\,{\scriptstyle \to L^c}$$

$C = \bot \prec \bot$. Then $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot; \Delta, \bot) \vdash^{+} \bot \prec \bot}\ {\scriptstyle \bot L^{a}}}{(\Gamma, \bot \prec \bot; \Delta) \vdash^{+} \bot \prec \bot}\ {\scriptstyle \prec L^{a}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \bot \prec \bot, \bot) \vdash^{-} \bot}\ {\scriptstyle \bot R^{-}}}{(\Gamma; \Delta, \bot \prec \bot) \vdash^{-} \bot \prec \bot}\ {\scriptstyle \prec R^{-}}$$

$C = \bot \wedge \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot, \top; \Delta) \vdash^{+} \bot \wedge \top}\ {\scriptstyle \bot L^{a}}}{(\Gamma, \bot \wedge \top; \Delta) \vdash^{+} \bot \wedge \top}\ {\scriptstyle \wedge L^{a}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \bot \wedge \top) \vdash^{-} \bot}\ {\scriptstyle \bot R^{-}}}{(\Gamma; \Delta, \bot \wedge \top) \vdash^{-} \bot \wedge \top}\ {\scriptstyle \wedge R_{1}^{-}}$$

$C = \bot \vee \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot \vee \top; \Delta) \vdash^{+} \top}\ {\scriptstyle \top R^{+}}}{(\Gamma, \bot \vee \top; \Delta) \vdash^{+} \bot \vee \top}\ {\scriptstyle \vee R_{2}^{+}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \bot, \top) \vdash^{-} \bot \vee \top}\ {\scriptstyle \top L^{c}}}{(\Gamma; \Delta, \bot \vee \top) \vdash^{-} \bot \vee \top}\ {\scriptstyle \vee L^{c}}$$

$C = \bot \to \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot \to \top, \bot; \Delta) \vdash^{+} \top}\ {\scriptstyle \top R^{+}}}{(\Gamma, \bot \to \top; \Delta) \vdash^{+} \bot \to \top}\ {\scriptstyle \to R^{+}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma, \bot; \Delta, \top) \vdash^{-} \bot \to \top}\ {\scriptstyle \top L^{c}}}{(\Gamma; \Delta, \bot \to \top) \vdash^{-} \bot \to \top}\ {\scriptstyle \to L^{c}}$$

$C = \bot \prec \top$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \bot; \Delta, \top) \vdash^{+} \bot \prec \top}\ {\scriptstyle \bot L^{a}}}{(\Gamma, \bot \prec \top; \Delta) \vdash^{+} \bot \prec \top}\ {\scriptstyle \prec L^{a}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \bot \prec \top, \top) \vdash^{-} \bot}\ {\scriptstyle \top L^{c}}}{(\Gamma; \Delta, \bot \prec \top) \vdash^{-} \bot \prec \top}\ {\scriptstyle \prec R^{-}}$$

$C = \top \wedge \bot$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \top, \bot; \Delta) \vdash^{+} \top \wedge \bot}\ {\scriptstyle \bot L^{a}}}{(\Gamma, \top \wedge \bot; \Delta) \vdash^{+} \top \wedge \bot}\ {\scriptstyle \wedge L^{a}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \top \wedge \bot) \vdash^{-} \bot}\ {\scriptstyle \bot R^{-}}}{(\Gamma; \Delta, \top \wedge \bot) \vdash^{-} \top \wedge \bot}\ {\scriptstyle \wedge R_{2}^{-}}$$

$C = \top \vee \bot$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \top \vee \bot; \Delta) \vdash^{+} \top}\ {\scriptstyle \top R^{+}}}{(\Gamma, \top \vee \bot; \Delta) \vdash^{+} \top \vee \bot}\ {\scriptstyle \vee R_{1}^{+}} \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma; \Delta, \top, \bot) \vdash^{-} \top \vee \bot}\ {\scriptstyle \top L^{c}}}{(\Gamma; \Delta, \top \vee \bot) \vdash^{-} \top \vee \bot}\ {\scriptstyle \vee L^{c}}$$

$C = \top \to \bot$. Then $(\Gamma, C; \Delta) \vdash^{+} C$ and $(\Gamma; \Delta, C) \vdash^{-} C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma, \top \to \bot; \Delta) \vdash^{+} \top}\ {\scriptstyle \top R^{+}} \qquad \dfrac{}{(\Gamma, \bot; \Delta) \vdash^{+} \top \to \bot}\ {\scriptstyle \bot L^{a}}}{(\Gamma, \top \to \bot; \Delta) \vdash^{+} \top \to \bot}\ {\scriptstyle \to L^{a}} \quad \text{and}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta,\top\to\bot)\vdash^+\top}\,\top R^+ \qquad \dfrac{}{(\Gamma;\Delta,\top\to\bot)\vdash^-\bot}\,\bot R^-}{(\Gamma;\Delta,\top\to\bot)\vdash^-\top\to\bot}\,{\to}R^-$$

$C=\top\prec\bot$. Then $(\Gamma,C;\Delta)\vdash^+ C$ and $(\Gamma;\Delta,C)\vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\prec\bot;\Delta)\vdash^+\top}\,\top R^+ \qquad \dfrac{}{(\Gamma,\top\prec\bot;\Delta)\vdash^-\bot}\,\bot R^-}{(\Gamma,\top\prec\bot;\Delta)\vdash^+\top\prec\bot}\,{\prec}R^+ \quad \text{and}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta,\top\prec\bot)\vdash^-\bot}\,\bot R^- \qquad \dfrac{}{(\Gamma;\Delta,\top)\vdash^-\top\prec\bot}\,\top L^c}{(\Gamma;\Delta,\top\prec\bot)\vdash^-\top\prec\bot}\,{\prec}L^c$$

$C=\top\wedge\top$. Then $(\Gamma,C;\Delta)\vdash^+ C$ and $(\Gamma;\Delta,C)\vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\wedge\top;\Delta)\vdash^+\top}\,\top R^+ \qquad \dfrac{}{(\Gamma,\top\wedge\top;\Delta)\vdash^+\top}\,\top R^+}{(\Gamma,\top\wedge\top;\Delta)\vdash^+\top\wedge\top}\,{\wedge}R^+ \quad \text{and}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta,\top)\vdash^-\top\wedge\top}\,\top L^c \qquad \dfrac{}{(\Gamma;\Delta,\top)\vdash^-\top\wedge\top}\,\top L^c}{(\Gamma;\Delta,\top\wedge\top)\vdash^-\top\wedge\top}\,{\wedge}L^c$$

$C=\top\vee\top$. Then $(\Gamma,C;\Delta)\vdash^+ C$ and $(\Gamma;\Delta,C)\vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\vee\top;\Delta)\vdash^+\top}\,\top R^+}{(\Gamma,\top\vee\top;\Delta)\vdash^+\top\vee\top}\,{\vee}R^+ \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma;\Delta,\top,\top)\vdash^-\top\vee\top}\,\top L^c}{(\Gamma;\Delta,\top\vee\top)\vdash^-\top\vee\top}\,{\vee}L^c$$

$C=\top\to\top$. Then $(\Gamma,C;\Delta)\vdash^+ C$ and $(\Gamma;\Delta,C)\vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top\to\top,\top;\Delta)\vdash^+\top}\,\top R^+}{(\Gamma,\top\to\top;\Delta)\vdash^+\top\to\top}\,{\to}R^+ \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma,\top;\Delta,\top)\vdash^-\top\to\top}\,\top L^c}{(\Gamma;\Delta,\top\to\top)\vdash^-\top\to\top}\,{\to}L^c$$

$C=\top\prec\top$. Then $(\Gamma,C;\Delta)\vdash^+ C$ and $(\Gamma;\Delta,C)\vdash^- C$ are derived by

$$\dfrac{\dfrac{}{(\Gamma,\top;\Delta,\top)\vdash^+\top\prec\top}\,\top L^c}{(\Gamma,\top\prec\top;\Delta)\vdash^+\top\prec\top}\,{\prec}L^a \quad \text{and} \quad \dfrac{\dfrac{}{(\Gamma;\Delta,\top\prec\top,\top)\vdash^-\top}\,\top L^c}{(\Gamma;\Delta,\top\prec\top)\vdash^-\top\prec\top}\,{\prec}R^-$$

The inductive hypothesis is that $(\Gamma, C; \Delta) \vdash^+ C$ and $(\Gamma; \Delta, C) \vdash^- C$ are derivable for all formulas C with $w(C) \leq n$, and we have to show that $(\Gamma, D; \Delta) \vdash^+ D$ and $(\Gamma; \Delta, D) \vdash^- D$ are derivable for formulas D of weight $\leq n + 1$. There are four cases:

$D = A \wedge B$. By the definition of weight and our inductive hypothesis, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \wedge B; \Delta) \vdash^+ A \wedge B$ by

$$\dfrac{\dfrac{(\Gamma, A, B; \Delta) \vdash^+ A}{(\Gamma, A \wedge B; \Delta) \vdash^+ A}\,{\wedge L^a} \qquad \dfrac{(\Gamma, A, B; \Delta) \vdash^+ B}{(\Gamma, A \wedge B; \Delta) \vdash^+ B}\,{\wedge L^a}}{(\Gamma, A \wedge B; \Delta) \vdash^+ A \wedge B}\,{\wedge R^+}$$

and $(\Gamma; \Delta, A \wedge B) \vdash^- A \wedge B$ by

$$\dfrac{\dfrac{(\Gamma; \Delta, A) \vdash^- A}{(\Gamma; \Delta, A) \vdash^- A \wedge B}\,{\wedge R_1^-} \qquad \dfrac{(\Gamma; \Delta, B) \vdash^- B}{(\Gamma; \Delta, B) \vdash^- A \wedge B}\,{\wedge R_2^-}}{(\Gamma; \Delta, A \wedge B) \vdash^- A \wedge B}\,{\wedge L^c}$$

$(\Gamma; \Delta, A) \vdash^- A$ and $(\Gamma; \Delta, B) \vdash^- B$ are derivable by the inductive hypothesis and since the context is arbitrary, so are $(\Gamma', A; \Delta) \vdash^+ A$ and $(\Gamma'', B; \Delta) \vdash^+ B$, for $\Gamma' = \Gamma, B$ and $\Gamma'' = \Gamma, A$.

$D = A \vee B$. As before, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \vee B; \Delta) \vdash^+ A \vee B$ by

$$\dfrac{\dfrac{(\Gamma, A; \Delta) \vdash^+ A}{(\Gamma, A; \Delta) \vdash^+ A \vee B}\,{\vee R_1^+} \qquad \dfrac{(\Gamma, B; \Delta) \vdash^+ B}{(\Gamma, B; \Delta) \vdash^+ A \vee B}\,{\vee R_2^+}}{(\Gamma, A \vee B; \Delta) \vdash^+ A \vee B}\,{\vee L^a}$$

and $(\Gamma; \Delta, A \vee B) \vdash^- A \vee B$ by

$$\dfrac{\dfrac{(\Gamma; \Delta, A, B) \vdash^- A}{(\Gamma; \Delta, A \vee B) \vdash^- A}\,{\vee L^c} \qquad \dfrac{(\Gamma; \Delta, A, B) \vdash^- B}{(\Gamma; \Delta, A \vee B) \vdash^- B}\,{\vee L^c}}{(\Gamma; \Delta, A \vee B) \vdash^- A \vee B}\,{\vee R^-}$$

Again, by inductive hypothesis we get the derivability of $(\Gamma, A; \Delta) \vdash^+ A$ and $(\Gamma, B; \Delta) \vdash^+ B$ and since the context is arbitrary, $(\Gamma; \Delta', A) \vdash^- A$ and $(\Gamma; \Delta'', B) \vdash^- B$ are derivable, for $\Delta' = \Delta, B$ and $\Delta'' = \Delta, A$.

$D = A \to B$. As before, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \to B; \Delta) \vdash^+ A \to B$ by

$$\dfrac{\dfrac{(\Gamma, A, A \to B; \Delta) \vdash^+ A \qquad (\Gamma, A, B; \Delta) \vdash^+ B}{(\Gamma, A, A \to B; \Delta) \vdash^+ B}\ {\to}L^a}{(\Gamma, A \to B; \Delta) \vdash^+ A \to B}\ {\to}R^+$$

and $(\Gamma; \Delta, A \to B) \vdash^- A \to B$ by

$$\dfrac{\dfrac{(\Gamma, A; \Delta, B) \vdash^+ A \qquad (\Gamma, A; \Delta, B) \vdash^- B}{(\Gamma, A; \Delta, B) \vdash^- A \to B}\ {\to}R^-}{(\Gamma; \Delta, A \to B) \vdash^- A \to B}\ {\to}L^c$$

The case of $(\Gamma, A, B; \Delta) \vdash^+ B$ was already mentioned in the case of conjunction and with the same reasoning $(\Gamma', A; \Delta) \vdash^+ A$ for $\Gamma' = \Gamma, A \to B$, $(\Gamma, A; \Delta') \vdash^+ A$ for $\Delta' = \Delta, B$ as well as $(\Gamma'; \Delta, B) \vdash^- B$ for $\Gamma' = \Gamma, A$ are derivable.

$D = A \prec B$. As before, $w(A) \leq n$ and $w(B) \leq n$.
We can derive $(\Gamma, A \prec B; \Delta) \vdash^+ A \prec B$ by

$$\dfrac{\dfrac{(\Gamma, A; \Delta, B) \vdash^+ A \qquad (\Gamma, A; \Delta, B) \vdash^- B}{(\Gamma, A; \Delta, B) \vdash^+ A \prec B}\ {\prec}R^+}{(\Gamma, A \prec B; \Delta) \vdash^+ A \prec B}\ {\prec}L^a$$

and $(\Gamma; \Delta, A \prec B) \vdash^- A \prec B$ by

$$\dfrac{\dfrac{(\Gamma; \Delta, B, A \prec B) \vdash^- B \qquad (\Gamma; \Delta, A, B) \vdash^- A}{(\Gamma; \Delta, B, A \prec B) \vdash^- A}\ {\prec}L^c}{(\Gamma; \Delta, A \prec B) \vdash^- A \prec B}\ {\prec}R^-$$

With the same reasoning as above $(\Gamma; \Delta', B) \vdash^- B$ is derivable for $\Delta' = \Delta, A \prec B$ and all other cases are already dealt with above. □

3.2. Admissibility of weakening

We will now start with the proof of admissibility of weakening by induction on height of derivations. The general procedure when proving admissibility of a rule with this is to prove it for applications of this rule to conclusions of zero-premise rules and then generalize by induction on the number of applications of the rule to arbitrary derivations. Thus, we can assume that there is only one instance – as the last step in the derivation – of the rule in question.

THEOREM 3.4 (Height-preserving weakening). *If* $(\Gamma; \Delta) \vdash^* C$ *is derivable with a height of derivation at most n, then* $(\Gamma, D; \Delta) \vdash^* C$ *and* $(\Gamma; \Delta, D) \vdash^* C$ *are derivable with a height of derivation at most n for arbitrary D.*

PROOF: If $n = 0$, then $(\Gamma; \Delta) \vdash^* C$ is a zero-premise rule, which means that one of the following six cases holds. C is an atom and 1) a formula in Γ with $* = +$ or 2) a formula in Δ with $* = -$. Otherwise it can be the case that 3) C is $\top$ with $* = +$ or 4) C is $\bot$ with $* = -$. Lastly, it could be that 5) $\bot$ is a formula in Γ or 6) $\top$ a formula in Δ. In either case, $(\Gamma, D; \Delta) \vdash^* C$ *and* $(\Gamma; \Delta, D) \vdash^* C$ are conclusions of the respective zero-premise rules. Our inductive hypothesis is now that height-preserving weakening is admissible up to derivations of height $\leq n$. Let $(\Gamma; \Delta) \vdash^* C$ be derivable with a height of derivation at most $n + 1$.
If the last rule applied is $\wedge L^a$, then $\Gamma = \Gamma', A \wedge B$ and the last step is

$$\frac{(\Gamma', A, B; \Delta) \vdash^* C}{(\Gamma', A \wedge B; \Delta) \vdash^* C}\ \wedge L^a$$

So $(\Gamma', A, B; \Delta) \vdash^* C$ is derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma', A, B, D; \Delta) \vdash^* C$ and $(\Gamma', A, B; \Delta, D) \vdash^* C$ are derivable in $\leq n$ steps. Thus, the application of $\wedge L^a$ gives a derivation of $(\Gamma', A \wedge B, D; \Delta) \vdash^* C$ and $(\Gamma', A \wedge B; \Delta, D) \vdash^* C$ in $\leq n + 1$ steps.
If the last rule applied is $\wedge L^c$, then $\Delta = \Delta', A \wedge B$ and the last step is

$$\frac{(\Gamma; \Delta', A) \vdash^* C \qquad (\Gamma; \Delta', B) \vdash^* C}{(\Gamma; \Delta', A \wedge B) \vdash^* C}\ \wedge L^c$$

So $(\Gamma; \Delta', A) \vdash^* C$ and $(\Gamma; \Delta', B) \vdash^* C$ are derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma, D; \Delta', A) \vdash^* C$, $(\Gamma; \Delta', A, D) \vdash^* C$, $(\Gamma, D; \Delta', B) \vdash^* C$ and $(\Gamma; \Delta', B, D) \vdash^* C$ are derivable in $\leq n$ steps. Thus, the application of $\wedge L^c$ to the first and the third premise and to the second and the fourth premise gives a derivation of $(\Gamma, D; \Delta', A \wedge B) \vdash^* C$ and $(\Gamma; \Delta', A \wedge B, D) \vdash^* C$, respectively, in $\leq n + 1$ steps.
If the last rule applied is $\wedge R^+$, then $C = A \wedge B$ and the last step is

$$\frac{(\Gamma; \Delta) \vdash^+ A \qquad (\Gamma; \Delta) \vdash^+ B}{(\Gamma; \Delta) \vdash^+ A \wedge B}\ \wedge R^+$$

So $(\Gamma; \Delta) \vdash^{+} A$ and $(\Gamma; \Delta) \vdash^{+} B$ are derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma, D; \Delta) \vdash^{+} A$, $(\Gamma; \Delta, D) \vdash^{+} A$, $(\Gamma, D; \Delta) \vdash^{+} B$ and $(\Gamma; \Delta, D) \vdash^{+} B$ are derivable in $\leq n$ steps. Thus, the application of $\wedge R^{+}$ to the first and the third premise and to the second and the fourth premise gives a derivation of $(\Gamma, D; \Delta) \vdash^{+} A \wedge B$ and $(\Gamma; \Delta, D) \vdash^{+} A \wedge B$, respectively, in $\leq n + 1$ steps.
If the last rule applied is $\wedge R_1^{-}$, then $C = A \wedge B$ and the last step is

$$\frac{(\Gamma; \Delta) \vdash^{-} A}{(\Gamma; \Delta) \vdash^{-} A \wedge B} \wedge R_1^{-}$$

So $(\Gamma; \Delta) \vdash^{-} A$ is derivable in $\leq n$ steps. By inductive hypothesis, also $(\Gamma, D; \Delta) \vdash^{-} A$ and $(\Gamma; \Delta, D) \vdash^{-} A$ are derivable in $\leq n$ steps. Thus, the application of $\wedge R_1^{-}$ gives a derivation of $(\Gamma, D; \Delta) \vdash^{-} A \wedge B$ and $(\Gamma; \Delta, D) \vdash^{-} A \wedge B$ in $\leq n + 1$ steps.

For the other logical rules the same can be shown with similar steps. □

Now we want to show one other thing related to weakening because we will need this result later in our proof for the admissibility of the cut rules, namely that for the special case that the weakening formula is $\top$ for W^a and respectively $\bot$ for W^c, the weakening rules are invertible, i.e.:

$$\frac{(\Gamma, \top; \Delta) \vdash^{*} C}{(\Gamma; \Delta) \vdash^{*} C} W_{inv}^{\top} \qquad \frac{(\Gamma; \Delta, \bot) \vdash^{*} C}{(\Gamma; \Delta) \vdash^{*} C} W_{inv}^{\bot}$$

Lemma 3.5 (Special case of inverted weakening). *If $(\Gamma, \top; \Delta) \vdash^{*} C$ or $(\Gamma; \Delta, \bot) \vdash^{*} C$ are derivable with a height of derivation at most n, then so is $(\Gamma; \Delta) \vdash^{*} C$.*

Proof: If $n = 0$, then exactly the same reasoning as for Theorem 3.4 can be applied here.
Now we assume height-preserving invertibility for these two special cases of weakening up to height n, and let $(\Gamma, \top; \Delta) \vdash^{*} C$ and $(\Gamma; \Delta, \bot) \vdash^{*} C$ be derivable with a height of derivation $\leq n + 1$. The proof works correspondingly to the proof of height-preserving weakening above. We will show it for the case of the $\rightarrow L^c$-rule this time, just to choose one that is not familiar in 'usual' calculi, but it works similar for all logical connectives and their rules.
If the last rule applied is $\rightarrow L^c$, then we have $\Delta = \Delta', A \rightarrow B$ and the last step is

$$\frac{(\Gamma, A, \top; \Delta', B) \vdash^* C}{(\Gamma, \top; \Delta', A \to B) \vdash^* C} \to L^c \text{ or respectively } \frac{(\Gamma, A; \Delta', B, \bot) \vdash^* C}{(\Gamma; \Delta', A \to B, \bot) \vdash^* C} \to L^c$$

So, $(\Gamma, A, \top; \Delta', B) \vdash^* C$ and $(\Gamma, A; \Delta', B, \bot) \vdash^* C$ are derivable in $\leq n$ steps. Then by inductive hypothesis, $(\Gamma, A; \Delta', B) \vdash^* C$ is derivable in $\leq n$ steps. If we apply $\to L^c$ to this, this gives us $(\Gamma; \Delta', A \to B) \vdash^* C$ in $\leq n + 1$ steps. □

3.3. Admissibility of contraction

Before we can prove the admissibility of the contraction rules, we need to prove the following lemma about the invertibility of premises and conclusions of the logical rules for the left introduction of formulas. Note that for $\to L^a$ and $\prec L^c$ the invertibility only holds for the right premises.[6]

Lemma 3.6 (Inversion).

(i_1) *If $(\Gamma, A \wedge B; \Delta) \vdash^* C$ is derivable with a height of derivation at most n, then $(\Gamma, A, B; \Delta) \vdash^* C$ is derivable with a height of derivation at most n.*

(i_2) *If $(\Gamma; \Delta, A \wedge B) \vdash^* C$ is derivable with a height of derivation at most n, then $(\Gamma; \Delta, A) \vdash^* C$ and $(\Gamma; \Delta, B) \vdash^* C$ are derivable with a height of derivation at most n.*

(ii_1) *If $(\Gamma, A \vee B; \Delta) \vdash^* C$ is derivable with a height of derivation at most n, then $(\Gamma, A; \Delta) \vdash^* C$ and $(\Gamma, B; \Delta) \vdash^* C$ are derivable with a height of derivation at most n.*

(ii_2) *If $(\Gamma; \Delta, A \vee B) \vdash^* C$ is derivable with a height of derivation at most n, then $(\Gamma; \Delta, A, B) \vdash^* C$ is derivable with a height of derivation at most n.*

(iii_1) *If $(\Gamma, A \to B; \Delta) \vdash^* C$ is derivable with a height of derivation at most n, then $(\Gamma, B; \Delta) \vdash^* C$ is derivable with a height of derivation at most n.*

(iii_2) *If $(\Gamma; \Delta, A \to B) \vdash^* C$ is derivable with a height of derivation at most n, then $(\Gamma, A; \Delta, B) \vdash^* C$ is derivable with a height of derivation at most n.*

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[6] [10, p. 33] give a counterexample for the implication rule. The analogous counterexamples for `SC2Int` would be the derivability of the sequents $(\bot \to \bot; \varnothing) \vdash^+ \bot \to \bot$ and $(\varnothing; \top \prec \top) \vdash^- \top \prec \top$.

(iv_1) *If* $(\Gamma, A \prec B; \Delta) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma, A; \Delta, B) \vdash^* C$ *is derivable with a height of derivation at most* n*.*

(iv_2) *If* $(\Gamma; \Delta, A \prec B) \vdash^* C$ *is derivable with a height of derivation at most* n*, then* $(\Gamma; \Delta, A) \vdash^* C$ *is derivable with a height of derivation at most* n*.*

Proof: The proof is by induction on n.
1) If $(\Gamma, A \# B; \Delta) \vdash^* C$ with $\# \in \{\wedge, \vee, \rightarrow, \prec\}$ is the conclusion of a zero-premise rule, then so are $(\Gamma, A, B; \Delta) \vdash^* C$, $(\Gamma, A; \Delta) \vdash^* C$, $(\Gamma, B; \Delta) \vdash^* C$, $(\Gamma; \Delta, B) \vdash^* C$ since $A \# B$ is neither atomic nor $\bot$ nor $\top$.
Now we assume height-preserving inversion up to height n, and let $(\Gamma, A \# B; \Delta) \vdash^* C$ be derivable with a height of derivation $\leq n + 1$.

(i_1) Either $A \wedge B$ is principal in the last rule or not. If $A \wedge B$ is the principal formula, the premise $(\Gamma, A, B; \Delta) \vdash^* C$ has a derivation of height n. If $A \wedge B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \wedge B; \Delta') \vdash^* C'$, $(\Gamma'', A \wedge B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A, B; \Delta') \vdash^* C'$, $(\Gamma'', A, B; \Delta'') \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, A, B; \Delta) \vdash^* C$ in at most $n + 1$ steps.

(ii_1) Either $A \vee B$ is principal in the last rule or not. If $A \vee B$ is the principal formula, the premises $(\Gamma, A; \Delta) \vdash^* C$ and $(\Gamma, B; \Delta) \vdash^* C$ have a derivation of height $\leq n$. If $A \vee B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \vee B; \Delta') \vdash^* C'$, $(\Gamma'', A \vee B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A; \Delta') \vdash^* C'$, $(\Gamma', B; \Delta') \vdash^* C'$ and $(\Gamma'', A; \Delta') \vdash^* C''$, $(\Gamma'', B; \Delta'') \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to the first and third premise to conclude $(\Gamma, A; \Delta) \vdash^* C$ and to the second and fourth premise to conclude $(\Gamma, B; \Delta) \vdash^* C$ in at most $n + 1$ steps.

(iii_1) Either $A \rightarrow B$ is principal in the last rule or not. If $A \rightarrow B$ is the principal formula, the premise $(\Gamma, B; \Delta) \vdash^* C$ has a derivation of height $\leq n$. If $A \rightarrow B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \rightarrow B; \Delta') \vdash^* C'$, $(\Gamma'', A \rightarrow B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', B; \Delta') \vdash^* C'$, $(\Gamma'', B; \Delta'') \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, B; \Delta) \vdash^* C$ in at most $n + 1$ steps.

(iv_1) Either $A \prec B$ is principal in the last rule or not. If $A \prec B$ is the principal formula, then the premise $(\Gamma, A; \Delta, B) \vdash^* C$ has a derivation of height n. If $A \prec B$ is not principal in the last rule, then there must be one or two premises $(\Gamma', A \prec B; \Delta') \vdash^* C'$, $(\Gamma'', A \prec B; \Delta'') \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A; \Delta', B) \vdash^* C'$, $(\Gamma'', A; \Delta'', B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, A; \Delta, B) \vdash^* C$ in at most $n + 1$ steps.

2) If $(\Gamma; \Delta, A \# B) \vdash^* C$ with $\# \in \{\wedge, \vee, \rightarrow, \prec\}$ is the conclusion of a zero-premise rule, then so are $(\Gamma; \Delta, A) \vdash^* C$, $(\Gamma; \Delta, B) \vdash^* C$, $(\Gamma; \Delta, A, B) \vdash^* C$, $(\Gamma, A; \Delta) \vdash^* C$ since $A \# B$ is neither atomic nor $\bot$ nor $\top$.
Now we assume height-preserving inversion up to height n, and let $(\Gamma; \Delta, A \# B) \vdash^* C$ be derivable with a height of derivation $\leq n + 1$.

(i_2) Either $A \wedge B$ is principal in the last rule or not. If $A \wedge B$ is the principal formula, the premises $(\Gamma; \Delta, A) \vdash^* C$ and $(\Gamma; \Delta, B) \vdash^* C$ have a derivation of height $\leq n$. If $A \wedge B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \wedge B) \vdash^* C'$, $(\Gamma''; \Delta'', A \wedge B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma'; \Delta', A) \vdash^* C'$, $(\Gamma'; \Delta', B) \vdash^* C'$, $(\Gamma''; \Delta'', A) \vdash^* C''$, $(\Gamma''; \Delta'', B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to the first and third premise to conclude $(\Gamma; \Delta, A) \vdash^* C$ and to the second and fourth premise to conclude $(\Gamma; \Delta, B) \vdash^* C$ in at most $n + 1$ steps.

(ii_2) Either $A \vee B$ is principal in the last rule or not. If $A \vee B$ is the principal formula, the premise $(\Gamma; \Delta, A, B) \vdash^* C$ has a derivation of height n. If $A \vee B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \vee B) \vdash^* C'$, $(\Gamma''; \Delta'', A \vee B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma'; \Delta', A, B) \vdash^* C'$, $(\Gamma''; \Delta'', A, B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma; \Delta, A, B) \vdash^* C$ in at most $n + 1$ steps.

(iii_2) Either $A \to B$ is principal in the last rule or not. If $A \to B$ is the principal formula, the premise $(\Gamma, A; \Delta, B) \vdash^* C$ has a derivation of height n. If $A \to B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \to B) \vdash^* C'$, $(\Gamma''; \Delta'', A \to B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma', A; \Delta', B) \vdash^* C'$, $(\Gamma'', A; \Delta'', B) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, A; \Delta, B) \vdash^* C$ in at most $n + 1$ steps.

(iv_2) Either $A \prec B$ is principal in the last rule or not. If $A \prec B$ is the principal formula, the premise $(\Gamma; \Delta, A) \vdash^* C$ has a derivation of height $\leq n$. If $A \prec B$ is not principal in the last rule, then there must be one or two premises $(\Gamma'; \Delta', A \prec B) \vdash^* C'$, $(\Gamma''; \Delta'', A \prec B) \vdash^* C''$ with a height of derivation $\leq n$. Then, by inductive hypothesis, also $(\Gamma'; \Delta', A) \vdash^* C'$, $(\Gamma''; \Delta'', A) \vdash^* C''$ are derivable with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma; \Delta, A) \vdash^* C$ in at most $n + 1$ steps. □

Next, we will prove the admissibility of the contraction rules in `SC2Int`.

THEOREM 3.7 (Height-preserving contraction). *If* $(\Gamma, D, D; \Delta) \vdash^* C$ *is derivable with a height of derivation at most n, then* $(\Gamma, D; \Delta) \vdash^* C$ *is derivable with a height of derivation at most n and if* $(\Gamma; \Delta, D, D) \vdash^* C$ *is derivable with a height of derivation at most n, then* $(\Gamma; \Delta, D) \vdash^* C$ *is derivable with a height of derivation at most n.*

PROOF: The proof is again by induction on the height of derivation n.
If $(\Gamma, D, D; \Delta) \vdash^* C$ (resp. $(\Gamma; \Delta, D, D) \vdash^* C$) is the conclusion of a zero-premise rule, then either C is an atom and contained in the antecedent, in the assumptions for $\vdash^+$ or in the counterassumptions for $\vdash^-$, or $\bot$ is part of the assumptions, or $\top$ is part of the counterassumptions, or $C = \top$ for $\vdash^+$, or $C = \bot$ for $\vdash^-$. In either case, also $(\Gamma, D; \Delta) \vdash^* C$ (resp. $(\Gamma; \Delta, D) \vdash^* C$) is a conclusion of the respective zero-premise rule.
Let contraction be admissible up to derivation height n and let $(\Gamma, D, D; \Delta) \vdash^* C$ (resp. $(\Gamma; \Delta, D, D) \vdash^* C$) be derivable in at most $n + 1$ steps. Either the contraction formula is not principal in the last inference step or it is principal.
If D is not principal in the last rule concluding the premise of contraction $(\Gamma, D, D; \Delta) \vdash^* C$, there must be one or two premises $(\Gamma', D, D; \Delta') \vdash^* C'$,

$(\Gamma'', D, D; \Delta'') \vdash^* C'$ with a height of derivation $\leq n$. So by inductive hypothesis, we can derive $(\Gamma', D; \Delta') \vdash^* C'$, $(\Gamma'', D; \Delta'') \vdash^* C'$ with a height of derivation $\leq n$. Now the last rule can be applied to these premises to conclude $(\Gamma, D; \Delta) \vdash^* C$ in at most $n + 1$ steps. For the case of $(\Gamma; \Delta, D, D) \vdash^* C$ being the premise of contraction, the same argument applies respectively.
If D is principal in the last rule, we have to consider four cases for each contraction rule according to the form of D. We will show the cases for C^c this time; for C^a the same arguments apply respectively.

$D = A \wedge B$. Then the last rule applied must be $\wedge L^c$ and we have as premises $(\Gamma; \Delta, A \wedge B, A) \vdash^* C$ and $(\Gamma; \Delta, A \wedge B, B) \vdash^* C$ with a derivation height $\leq n$. By the inversion lemma this means that $(\Gamma; \Delta, A, A) \vdash^* C$ and $(\Gamma; \Delta, B, B) \vdash^* C$ are also derivable with a derivation height $\leq n$. Then by inductive hypothesis, we get $(\Gamma; \Delta, A) \vdash^* C$ and $(\Gamma; \Delta, B) \vdash^* C$ with a height of derivation $\leq n$ and by applying $\wedge L^c$ we can derive $(\Gamma; \Delta, A \wedge B) \vdash^* C$ in at most $n + 1$ steps.

$D = A \vee B$. Then the last rule applied must be $\vee L^c$ and $(\Gamma; \Delta, A \vee B, A, B) \vdash^* C$ is derivable with a height of derivation $\leq n$. By the inversion lemma, also $(\Gamma; \Delta, A, B, A, B) \vdash^* C$ is derivable with a derivation height $\leq n$. Then by inductive hypothesis (applied twice), we get $(\Gamma; \Delta, A, B) \vdash^* C$ with a height of derivation $\leq n$ and by applying $\vee L^c$ we can derive $(\Gamma; \Delta, A \vee B) \vdash^* C$ in at most $n + 1$ steps.

$D = A \rightarrow B$. Then the last rule applied must be $\rightarrow L^c$ and accordingly $(\Gamma, A; \Delta, B, A \rightarrow B) \vdash^* C$ is derivable with a height of derivation $\leq n$. By the inversion lemma, then also $(\Gamma, A, A; \Delta, B, B) \vdash^* C$ is derivable with a derivation height $\leq n$. By inductive hypothesis (applied twice), we get $(\Gamma, A; \Delta, B) \vdash^* C$ with a height of derivation $\leq n$ and by applying $\rightarrow L^c$ we can derive $(\Gamma; \Delta, A \rightarrow B) \vdash^* C$ in at most $n + 1$ steps.

$D = A \prec B$. Then the last rule applied must be $\prec L^c$ and we have as premises $(\Gamma; \Delta, A \prec B, A \prec B) \vdash^- B$ and $(\Gamma; \Delta, A \prec B, A) \vdash^* C$ with a derivation height $\leq n$. The inductive hypothesis applied to the first, gives us $(\Gamma; \Delta, A \prec B) \vdash^- B$ with a derivation height $\leq n$ and the inversion lemma applied to the second, also $(\Gamma; \Delta, A, A) \vdash^* C$ and again by inductive hypothesis $(\Gamma; \Delta, A) \vdash^* C$ with a derivation height $\leq n$. By applying $\prec L^c$ we can now derive $(\Gamma; \Delta, A \prec B) \vdash^* C$ in at most $n + 1$ steps. $\square$

3.4. Admissibility of cut

Now, we will come to the main result of this section, the proof of cut-elimination. The proof shows that cuts can be permuted upward in a derivation until they reach one of the zero-premise rules the derivation started with. When cut has reached zero-premise rules, the derivation can be transformed into one beginning with the conclusion of the cut, which can be shown by the following reasoning.

When both premises of cut are conclusions of a zero-premise rule, then the conclusion of cut is also a conclusion of one of these rules: If the left premise is $(\Gamma, \bot; \Delta) \vdash^* D$, then the conclusion also has $\bot$ in the assumptions of the antecedent. If the left premise is $(\Gamma; \Delta, \top) \vdash^* D$, then the conclusion also has $\top$ in the counterassumptions of the antecedent. If the left premise of Cut^a is $(\Gamma; \Delta) \vdash^+ \top$ or the left premise of Cut^c is $(\Gamma; \Delta) \vdash^- \bot$, then the right premise is $(\Gamma', \top; \Delta') \vdash^* C$ or $(\Gamma'; \Delta', \bot) \vdash^* C$ respectively. These are conclusions of zero-premise rules only in one of the following cases:

- C is an atom in Γ' for $* = +$ or C is an atom in Δ' for $* = -$
- $C = \top$ for $* = +$ or $C = \bot$ for $* = -$
- $\bot$ is in Γ' or $\top$ is in Δ'

In each case the conclusion of cut $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of the same zero-premise rule as the right premise. The last two possibilities are that the left premise is $(\Gamma, p; \Delta) \vdash^+ p$ for Cut^a or $(\Gamma; \Delta, p) \vdash^- p$ for Cut^c respectively. For the former case this means that the right premise is $(\Gamma', p; \Delta') \vdash^* C$. This is the conclusion of a zero-premise rule only in one of the following cases:

- For $* = +$: $C = p$, or C is an atom in Γ', or $C = \top$
- For $* = -$: C is an atom in Δ', or $C = \bot$
- $\bot$ is in Γ', or $\top$ is in Δ'

In each case the conclusion of cut $(\Gamma, p, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of the same zero-premise rule as the right premise. For the latter case this means that the right premise is $(\Gamma'; \Delta', p) \vdash^* C$. This is the conclusion of a zero-premise rule only in one of the following cases:

- For $* = +$: C is an atom in Γ', or $C = \top$
- For $* = -$: $C = p$, or C is an atom in Δ', or $C = \bot$
- $\bot$ is in Γ', or $\top$ is in Δ'

In each case the conclusion of cut $(\Gamma, \Gamma'; \Delta, p, \Delta') \vdash^* C$ is also a conclusion of the same zero-premise rule as the right premise. So, when cut has reached zero-premise rules as premises, the derivation can be transformed into one beginning with the conclusion of the cut by deleting the premises.

The proof is – as before – conducted in a manner corresponding to the proof of cut-elimination for `G3ip` by [10], which means that it is by induction on the weight of the cut formula and a subinduction on the cut-height, the sum of heights of derivations of the two premises of cut.

Definition 3.8. The *cut-height* of an application of one of the rules of cut in a derivation is the sum of heights of derivation of the two premises of the rule.

In the proof permutations are given that always reduce the weight of the cut formula or the cut-height of instances of the rules. When the cut formula is not principal in at least one (or both) of the premises of cut, cut-height is reduced. In the other cases, i.e. in which the cut formula is principal in both premises, it is shown that cut-height and/or the weight of the cut formula can be reduced. This process terminates since atoms cannot be principal formulas.

The difference between the height of a derivation and cut-height needs to be emphasized here, because it is essential to understand that if there are two instances of cut, one occurring below the other in the derivation, this does not necessarily mean that the lower instance has a greater cut-height than the upper. Let us suppose the upper instance of cut occurs in the derivation of the left premise of the lower cut. The upper instance can have a cut-height which is greater than the height of either its premises because the *sum* of the premises is what matters. However, the lower instance can have as a right premise one with a much shorter derivation height than either of the premises of the upper cut, making the sum of the derivation heights of those two premises lesser than the one from the upper cut. So, what follows is that it is not enough to show that occurrences of cut can be permuted upward in a derivation in order to show that cut-height decreases, but we need to calculate exactly the cut-height of each derivation in our proof. As before, it can be assumed that in a given derivation the last instance is the one and only occurrence of cut.

THEOREM 3.9. *The cut rules*

$$\frac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma', D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^a} \quad \textit{and} \quad \frac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma';\Delta', D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,{\scriptstyle Cut^c}$$

are admissible in **SC2Int***.*

The proof is organized as follows. First, we consider the case that at least one premise in a cut is a conclusion of one of the zero-premise rules and show how cut can be eliminated in these cases. Otherwise three cases can be distinguished: 1) The cut formula is not principal in either premise of cut, 2) the cut formula is principal in just one premise of cut, and 3) the cut formula is principal in both premises of cut. The proof is presented in detail in the appendix, Section 6.

COROLLARY 3.10. (Subformula property) If $(\Gamma;\Delta)\vdash^{*} A$ $(* \in \{+,-\})$ is derivable in SC2Int, then all subformulas occurring in the derivation are subformulas of Γ or Δ.
(Decidability) Derivability of sequents $(\Gamma;\Delta)\vdash^{*} A$ $(* \in \{+,-\})$ in SC2Int is decidable.

4. Synonymy of formulas through inherited identity between derivations

In order to define a certain notion of identity between derivations that is inspired by the bilateralist distinction between proofs and their duals, we consider (i) the following two negation operations defined in terms of implication and co-implication:

$$\neg A := A \to \bot \text{ (negation)}, \qquad -A := \top \prec A \text{ (co-negation)}.$$

and (ii) the following rules that state an interaction between proofs and dual proofs mediated through the two negation connectives:

$$\frac{(\Gamma;\Delta,A)\vdash^{*} B}{(-A,\Gamma;\Delta)\vdash^{*} B}\,-ai \qquad \frac{(-A,\Gamma;\Delta)\vdash^{*} B}{(\Gamma;\Delta,A)\vdash^{*} B}\,-ae$$

$$\frac{(\Gamma;\Delta)\vdash^{-} A}{(\Gamma;\Delta)\vdash^{+} -A}\,-ci \qquad \frac{(\Gamma;\Delta)\vdash^{+} -A}{(\Gamma;\Delta)\vdash^{-} A}\,-ce$$

$$\frac{(\Gamma,A;\Delta)\vdash^{*} B}{(\Gamma;\neg A,\Delta)\vdash^{*} B}\,\neg ai \qquad \frac{(\Gamma;\neg A,\Delta)\vdash^{*} B}{(\Gamma,A;\Delta)\vdash^{*} B}\,\neg ae$$

$$\frac{(\Gamma;\Delta)\vdash^{+} A}{(\Gamma;\Delta)\vdash^{-} \neg A}\,\neg ci \qquad \frac{(\Gamma;\Delta)\vdash^{-} \neg A}{(\Gamma;\Delta)\vdash^{+} A}\,\neg ce$$

One idea behind these interaction rules is that they are rules the application of which has no effect on the identity of derivations, so that a proof of A *is* a refutation of $\neg A$, and vice versa, and a refutation of A *is* a proof of $-A$, and vice versa. Whereas in the case of the sequent calculus for `N4` in [18], it is possible to identify derivations of different formulas because the strong negation marks a back and forth between proofs and refutations, in the case of the interaction rules of the sequent calculus `SCInt`, derivations of different formulas are identified because proving (refuting) A is seen as amounting to refuting (proving) $\neg A$ ($-A$). As mentioned in the introduction, we shall not delve into elaborating a motivation for this approach but are content to apply the idea of interaction rules having no effect on the identity of derivations to `SC2Int`.

The interaction rules are admissible in `SC2Int`:

$$\dfrac{\dfrac{(\Gamma;\Delta,A)\vdash^{*} B}{(\Gamma,\top;\Delta,A)\vdash^{*} B}\,W^{a}}{(\top\prec A,\Gamma;\Delta)\vdash^{*} B}\,\prec L^{a}$$

$$\dfrac{\dfrac{\dfrac{}{(\varnothing;A)\vdash^{+}\top}\,\top R^{+} \quad \dfrac{}{(\varnothing;A)\vdash^{-} A}\,\textit{Lemma 3.3}}{(\varnothing;A)\vdash^{+}\top\prec A}\,\prec R^{+} \qquad (\top\prec A,\Gamma;\Delta)\vdash^{*} B}{(\Gamma;\Delta,A)\vdash^{*} B}\,Cut^{a}$$

$$\dfrac{\dfrac{}{(\Gamma;\Delta)\vdash^{+}\top}\,\top R^{+} \quad (\Gamma;\Delta)\vdash^{-} A}{(\Gamma;\Delta)\vdash^{+}\top\prec A}\,\prec R^{+} \qquad \dfrac{(\Gamma;\Delta)\vdash^{+}\top\prec A \quad \dfrac{\dfrac{}{(\top;A)\vdash^{-} A}\,\textit{Lemma 3.3}}{(\top\prec A;\varnothing)\vdash^{-} A}\,\prec L^{a}}{(\Gamma;\Delta)\vdash^{-} A}\,Cut^{a}$$

$$\dfrac{\dfrac{(\Gamma, A; \Delta) \vdash^* B}{(\Gamma, A; \Delta, \bot) \vdash^* B}\, W^c}{(\Gamma; \Delta, A \to \bot) \vdash^* B}\, \to L^c$$

$$\dfrac{\dfrac{\dfrac{}{(A; \varnothing) \vdash^+ A}\, \textit{Lemma 3.3} \quad \dfrac{}{(A; \varnothing) \vdash^- \bot}\, \bot R^-}{(A; \varnothing) \vdash^- A \to \bot}\, \to R^- \quad (\Gamma; A \to \bot, \Delta) \vdash^* B}{(\Gamma, A; \Delta) \vdash^* B}\, Cut^c$$

$$\dfrac{\dfrac{(\Gamma; \Delta) \vdash^+ A \quad \dfrac{}{(\Gamma; \Delta) \vdash^- \bot}\, \bot R^-}{(\Gamma; \Delta) \vdash^- A \to \bot}\, \to R^- \quad \dfrac{\dfrac{}{(A; \bot) \vdash^+ A}\, \textit{Lemma 3.3}}{(\varnothing; A \to \bot) \vdash^+ A}\, \to L^c}{(\Gamma; \Delta) \vdash^+ A}\, Cut^c$$

In what follows, we will consider `SC2Int` without the admissible structural rules of contraction, weakening, and cut. We use $s, s_1, s_2, \ldots$ to stand for sequents. If $\mathscr{D}$ and $\mathscr{D}'$ are derivations in `SC2Int`, we shall write $\mathscr{D} \equiv \mathscr{D}'$ to express that $\mathscr{D}$ and $\mathscr{D}'$ are syntactically identical (as types of expressions, not as tokens).

Definition 4.1. The relation $\approx$ of inherited identity (in-identity) between derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ in `SC2Int` is defined inductively. It is the smallest binary relation on the set of derivations in `SC2Int` such that:

1. $\mathscr{D}_1 \approx \mathscr{D}_2$ if $\mathscr{D}_1 \equiv \mathscr{D}_2$.
2. $\mathscr{D}_1 \approx \mathscr{D}_2$ if either $\mathscr{D}_1 \approx \mathscr{D}$ and $\mathscr{D}_2 \equiv \dfrac{\mathscr{D}}{s}$ or $\mathscr{D}_2 \approx \mathscr{D}$ and $\mathscr{D}_1 \equiv \dfrac{\mathscr{D}}{s}$, where s is obtained from $\mathscr{D}$ by an application of an (instance of an) interaction rule.
3. $\mathscr{D}_1 \approx \mathscr{D}_2$ if $\mathscr{D}_1 \equiv \dfrac{\mathscr{D}_1^1 \ldots \mathscr{D}_n^1}{s_1}$, $\mathscr{D}_2 \equiv \dfrac{\mathscr{D}_1^2 \ldots \mathscr{D}_n^2}{s_2}$, and $\mathscr{D}_i^1 \approx \mathscr{D}_i^2$ $(1 \leq i \leq n \leq 2)$.

As in [18] it can be shown that the relation $\approx$ is an equivalence relation. Note that the third clause of Definition 4.1 allows one to identify, for example, proofs of $(A \vee B)$ and $(A \vee C)$, which is in accordance with the Brouwer-Heyting-Kolmogorov interpretation allowing for one and the same construction being a proof of both $(A \vee B)$ and $(A \vee C)$. Moreover, it is obvious that not any two cut-free derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ in `SC2Int` of a formula A are in-identical. There are, e.g., syntactically distinct cut-free derivations of the sequent $(\varnothing; \varnothing) \vdash^+ (p \wedge q) \to (p \vee q)$ that are not in-identical. We shall give examples of in-identical derivations in the proof of Proposition 4.3.

Definition 4.2. Two formulas A and B are said to be synonymous with respect to `SC2Int` iff

1. (positive condition) there exists a derivation $\mathscr{D}$ of $(A; \varnothing) \vdash^+ B$ and a derivation $\mathscr{D}'$ of $(B; \varnothing) \vdash^+ A$ with $\mathscr{D} \approx \mathscr{D}'$,
2. (negative condition) there exists a derivation $\mathscr{D}$ of $(\varnothing; A) \vdash^- B$ and a derivation $\mathscr{D}'$ of $(\varnothing; B) \vdash^- A$ with $\mathscr{D} \approx \mathscr{D}'$.

If the positive (negative) condition is satisfied, A and B are said to be positively (negatively) synonymous.

Accomplishing the interaction between proofs and refutations by means of two different negation connectives instead of a single strong negation, $\sim$, as in the sequent calculus `SN4` from [18], has a considerable effect on the notion of synonymy stated in Definition 4.2. While in `N4` all double negation and De Morgan laws hold and, for example, the following pairs of formulas turn out to be synonymous with respect to cut-free `SN4`

1. p and $\sim\sim p$,
2. $(p \wedge q)$ and $\sim(\sim p \vee \sim q)$,
3. $(p \vee q)$ and $\sim(\sim p \wedge \sim q)$,

not all double negation and De Morgan laws hold for $\neg$ and $-$ in `SC2Int`. We can observe a number of cases of positive or negative synonymy with respect to `SC2Int`.

Proposition 4.3. The following pairs of formulas are positively synonymous with respect to `SC2Int`:

1. p and $-\neg p$,
2. $-(p \to q)$ and $(p \wedge -q)$,
3. $-(\neg p \vee q)$ and $(p \wedge -q)$,
4. $-(p \to q)$ and $-(\neg p \vee q)$,

whereas the following pairs are negatively synonymous:

5. p and $\neg - p$,
6. $\neg(p \prec q)$ and $(\neg p \vee q)$,
7. $\neg(p \wedge -q)$ and $(\neg p \vee q)$,
8. $\neg(p \prec q)$ and $\neg(p \wedge -q)$.

Proof: 1. and 5.: The following pairs of derivations are in-identical by the first clause of Definition 4.1:

$$\dfrac{\dfrac{(p;\varnothing) \vdash^+ p}{(p;\varnothing) \vdash^- \neg p}\,\neg ci}{(p;\varnothing) \vdash^+ -\neg p}\,-ci \qquad\qquad \dfrac{\dfrac{(p;\varnothing) \vdash^+ p}{(\varnothing;\neg p) \vdash^+ p}\,\neg ai}{(-\neg p;\varnothing) \vdash^+ p}\,-ai$$

$$\dfrac{\dfrac{(\varnothing;p) \vdash^- p}{(\varnothing,p) \vdash^+ -p}\,-ci}{(\varnothing;p) \vdash^- \neg - p}\,\neg ci \qquad\qquad \dfrac{\dfrac{(\varnothing;p) \vdash^- p}{(-p;\varnothing) \vdash^- p}\,-ai}{(\varnothing;\neg - p) \vdash^- p}\,\neg ai$$

2. We shall demonstrate the in-identity of the following two derivations in detail. The demonstration for the cases 3., 6., and 7. is similar and left to the reader.

$$\dfrac{\dfrac{\dfrac{(p;q) \vdash^+ p}{(\varnothing;(p \to q)) \vdash^+ p}}{(-(p \to q);\varnothing) \vdash^+ p} \quad \dfrac{\dfrac{\dfrac{(p;q) \vdash^- q}{(\varnothing;(p \to q)) \vdash^- q}}{(-(p \to q);\varnothing) \vdash^- q}}{(-(p \to q);\varnothing) \vdash^+ -q}}{(-(p \to q);\varnothing) \vdash^+ (p \wedge -q)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(p;q) \vdash^+ p}{(p,-q;\varnothing) \vdash^+ p}}{((p \wedge -q);\varnothing) \vdash^+ p} \quad \dfrac{\dfrac{(p;q) \vdash^- q}{(p,-q;\varnothing) \vdash^- q}}{((p \wedge -q);\varnothing) \vdash^- q}}{((p \wedge -q);\varnothing) \vdash^- (p \to q)}}{((p \wedge -q);\varnothing) \vdash^+ -(p \to q)}$$

Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be the derivations

$$\dfrac{(p;q) \vdash^+ p}{(\varnothing;(p \to q)) \vdash^+ p} \quad \text{and} \quad \dfrac{(p;q) \vdash^+ p}{(p,-q;\varnothing) \vdash^+ p}$$

and let $\mathscr{D}_3$ and $\mathscr{D}_4$ be the derivations

$$\dfrac{\mathscr{D}_1}{(-(p \to q);\varnothing) \vdash^+ p} \quad \text{and} \quad \dfrac{\mathscr{D}_2}{(p \wedge -q);\varnothing) \vdash^+ p.}$$

By clauses 1. and 3. of Definition 4.1, $\mathscr{D}_1 \approx \mathscr{D}_2$, and by clause 3. of Definition 4.1, $\mathscr{D}_3 \approx \mathscr{D}_4$. Let $\mathscr{D}_5$ and $\mathscr{D}_6$ be the derivations

$$\dfrac{(p;q)\vdash^- q}{(\varnothing;(p\to q))\vdash^- q} \quad\text{and}\quad \dfrac{(p;q)\vdash^- q}{(p,-q;\varnothing)\vdash^- q}$$

and let $\mathscr{D}_7$ and $\mathscr{D}_8$ be the derivations

$$\dfrac{\dfrac{\mathscr{D}_5}{(-(p\to q);\varnothing)\vdash^- q}}{(-(p\to q);\varnothing)\vdash^+ -q} \quad\text{and}\quad \dfrac{\mathscr{D}_6}{(p\wedge -q);\varnothing)\vdash^- q}.$$

By clauses 1. and 3. of Definition 4.1, $\mathscr{D}_5 \approx \mathscr{D}_6$, and by clauses 3. and 2. of Definition 4.1, $\mathscr{D}_7 \approx \mathscr{D}_8$. Then, by clause 3. of Definition 4.1, we obtain that for the derivations $\mathscr{D}_9$ and $\mathscr{D}_{10}$, namely,

$$\dfrac{\mathscr{D}_3 \qquad \mathscr{D}_7}{(-(p\to q);\varnothing)\vdash^+ (p\wedge -q)} \quad\text{and}\quad \dfrac{\mathscr{D}_4 \qquad \mathscr{D}_8}{(p\wedge -q);\varnothing)\vdash^- (p\to q)}$$

it holds that $\mathscr{D}_9 \approx \mathscr{D}_{10}$. Let $\mathscr{D}_{11}$ be

$$\dfrac{\mathscr{D}_{10}}{(p\wedge -q);\varnothing)\vdash^+ -(p\to q).}$$

By clause 2. of Definition 4.1, $\mathscr{D}_9 \approx \mathscr{D}_{11}$.
3.:

$$\dfrac{\dfrac{\dfrac{\dfrac{(p;q)\vdash^+ p}{(\varnothing;\neg p,q)\vdash^+ p}}{(\varnothing;(\neg p\vee q))\vdash^+ p}}{(-(\neg p\vee q);\varnothing)\vdash^+ p} \quad \dfrac{\dfrac{\dfrac{\dfrac{(p;q)\vdash^- q}{(\varnothing;\neg p,q)\vdash^- q}}{(\varnothing;(\neg p\vee q))\vdash^- q}}{(-(\neg p\vee q);\varnothing)\vdash^- q}}{(-(\neg p\vee q);\varnothing)\vdash^+ -q}}{(-(\neg p\vee q);\varnothing)\vdash^+ (p\wedge -q)} \approx \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(p;q)\vdash^+ p}{((p,-q);\varnothing)\vdash^+ p}}{((p\wedge -q);\varnothing)\vdash^+ p}}{((p\wedge -q);\varnothing)\vdash^- \neg p} \quad \dfrac{\dfrac{(p;q)\vdash^- q}{(p,-q;\varnothing)\vdash^- q}}{((p\wedge -q);\varnothing)\vdash^- q}}{((p\wedge -q);\varnothing)\vdash^- (\neg p\vee q)}}{((p\wedge -q);\varnothing)\vdash^+ -(\neg p\vee q)}$$

4.: By 2., 3., and the transitivity of $\approx$.
6.:

$$\dfrac{\dfrac{\dfrac{\dfrac{(p;q))\vdash^+ p}{((p\prec q);\varnothing)\vdash^+ p}}{(\varnothing;\neg(p\prec q))\vdash^+ p}}{(\varnothing;\neg(p\prec q))\vdash^- \neg p} \quad \dfrac{\dfrac{(p;q)\vdash^- q}{((p\prec q);\varnothing)\vdash^- q}}{(\varnothing;\neg(p\prec q))\vdash^- q}}{(\varnothing;\neg(p\prec q))\vdash^- (\neg p\vee q)} \approx \dfrac{\dfrac{\dfrac{\dfrac{(p;q)\vdash^+ p}{(\varnothing;\neg p,q)\vdash^+ p}}{(\varnothing;(\neg p\vee q))\vdash^+ p} \quad \dfrac{\dfrac{(p;q)\vdash^- q}{(\varnothing;\neg p,q)\vdash^- q}}{(\varnothing;(\neg p\vee q))\vdash^- q}}{(\varnothing;(\neg p\vee q))\vdash^+ (p\prec q)}}{(\varnothing;(\neg p\vee q))\vdash^- \neg(p\prec q)}$$

7.:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(p;q) \vdash^{+} p}{(p,-q;\varnothing) \vdash^{+} p}}{((p \wedge -q);\varnothing) \vdash^{+} p}}{(\varnothing;\neg(p \wedge -q)) \vdash^{+} p}}{(\varnothing;\neg(p \wedge -q)) \vdash^{-} \neg p} \quad \dfrac{\dfrac{\dfrac{(p;q) \vdash^{-} q}{(p,-q;\varnothing) \vdash^{-} q}}{((p \wedge -q);\varnothing) \vdash^{-} q}}{(\varnothing;\neg(p \wedge -q)) \vdash^{-} q}}{(\varnothing;\neg(p \wedge -q)) \vdash^{-} (\neg p \vee q)} \approx \dfrac{\dfrac{\dfrac{\dfrac{(p;q) \vdash^{+} p}{(\varnothing;\neg p,q) \vdash^{+} p}}{(\varnothing;(\neg p \vee q)) \vdash^{+} p} \quad \dfrac{\dfrac{\dfrac{(p;q) \vdash^{-} q}{(\varnothing;\neg p,q) \vdash^{-} q}}{(\varnothing;(\neg p \vee q)) \vdash^{-} q}}{(\varnothing;(\neg p \vee q)) \vdash^{+} -q}}{(\varnothing;(\neg p \vee q)) \vdash^{+} (p \wedge -q)}}{(\varnothing;(\neg p \vee q)) \vdash^{-} \neg(p \wedge -q)}$$

8.: By 6., 7., and the transitivity of $\approx$. □

Since we have not been able to find pairs of distinct formulas in the language of `2Int` that are synonymous with respect to `SC2Int` in the sense of Definition 4.2, we are led to conjecture that there are no such pairs of formulas.

Conjecture 4.4. There exist no two distinct formulas A, B in the language of `2Int` that are synonymous with respect to `SC2Int` in the sense of Definition 4.2.

If that conjecture is true, then synonymy based on in-identity with respect to `SC2Int` trivializes in the sense that it seems to be an empty concept. However, this is neither really surprising if we reconsider the differences between `2Int` and `N4` nor does it have to be seen as a defect of in-identity or `SC2Int`. While in `N4` there is one negation, which is firstly primitive and secondly serves as a toggle between proofs and refutations, in `2Int` we have two negations, which are mere results from having two *implications*, which in turn are the object language manifestation of having two *derivability relations*. With that in mind it does not seem odd that there are no two (distinct) synonymous formulas interderivable w.r.t. to *both* derivability relations. After all, the interaction rules solely work with the two negations but do not allow a 'toggling' back and forth between proofs and refutations. So, in order to get an interderivability w.r.t. the positively signed derivability relation using the interaction rules, it seems that we will always have to use the $-$ci and the $-$ai rule as the last interaction rules in the derivation. This is not to say that one of them has to be the very last rule applied in the derivation and also not to say that other interaction rules cannot appear within the derivation. As we see in the exemplary derivations above, of course, the very last rule can be a normal operational rule and of course, there can be other interaction rules like the ones for $\neg$. But the last of the interaction rules to occur, must always be $-$ci and $-$ai (the order between those two does not matter). This is just because *if* interaction rules are to be used, then these are the ones getting a formula into the assumptions and switching the derivability relation from $-$ to $+$, which is the result we need for derivations of the form $(A; \varnothing) \vdash^+ B$ and $(B; \varnothing) \vdash^+ A$. The same holds for interderivability w.r.t. the negatively signed derivability relation and the use of the interaction rules $\neg$ai and $\neg$ci. Since applying these rules results in different formulas, though, namely in formulas having $-$, resp. $\neg$ as main operator, it simply does not seem possible to have both interderivabilities for the same pair of formulas.

So, this result can be regarded as an interesting consequence of the basics of SC2Int because what we obtain by having bilateralist concepts also overtly realized in the connectives is an exclusive division between positive and negative synonymy. It highlights the bilateralist principle of verifications (proofs) and falsifications (refutations) being two primitive kinds of derivations in their own right.

5. Conclusion and outlook

By applying the proof methods that [10] use for their calculus G3ip, we were able to show the admissibility of the structural rules of weakening, contraction, and cut in the sequent calculus SC2Int for the bi-intuitionistic logic 2Int. With SC2Int at hand, we could apply the definition of inherited identity of derivations from [18] to define the notion of propositional synonymy of formulas with respect to SC2Int as the combination of two concepts of positive and negative synonymy. We were able to present various pairs of distinct formulas that are either positively or negatively synonymous with respect to SC2Int, and we conjectured that there exist no pairs of distinct formulas that are both positively and negatively synonymous with respect to SC2Int.

An obvious task is to decide Conjecture 4.4. Moreover, as already indicated in [18], it would be interesting to encode derivations in a bilateral sequent calculus that accommodates proofs as well as refutations, such as SC2Int, in a suitable two-sorted typed λ-calculus with terms of one sort denoting proofs and terms of a second sort denoting dual proofs, refutations. This is currently work in progress by one of the authors (cf. [2]). There, it will be pondered what other ways of understanding the concept of identity between proofs and refutations are available and sensible in the light of identifying lambda-term constructions.

Acknowledgements. We would like to thank the two anonymous referees of the *Bulletin of the Section of Logic* for their helpful comments. Heinrich Wansing has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant agreement ERC-2020-ADG, 101018280, ConLog.

6. Appendix

We present a proof of Theorem 3.9 by considering the mentioned case distinction.

Cut with a conclusion of a zero-premise rule as premise

Cut with a conclusion of Rf^+, Rf^-, $\bot L^a$, $\top L^c, \bot R^-$, or $\top R^+$ as premise

If at least one of the premises of cut is a conclusion of one of the zero-premise rules, we distinguish three cases for both cut rules:

-1- Cut^a

-1.1- The left premise $(\Gamma; \Delta) \vdash^+ D$ is a conclusion of a zero-premise-rule. There are four subcases:

(a) The cut formula D is an atom in Γ. Then the conclusion $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is derived from $(\Gamma', D; \Delta') \vdash^* C$ by W^a and W^c.

(b) $\bot$ is a formula in Γ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\bot L^a$.

(c) $\top$ is a formula in Δ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\top L^c$.

(d) $\top =$ D. Then the right premise is $(\Gamma', \top; \Delta') \vdash^* C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ follows by $W^{\top}_{inv}$ as well as W^a and W^c.

-1.2- The right premise $(\Gamma', D; \Delta') \vdash^{+} C$ is a conclusion of a zero-premise rule. There are six subcases:

(a) C is an atom in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of Rf^{+}.

(b) $C = D$. Then the left premise is $(\Gamma; \Delta) \vdash^{+} C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ follows by W^a and W^c.

(c) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of $\bot L^a$.

(d) $\bot$ = D. Then the left premise is $(\Gamma; \Delta) \vdash^{+} \bot$ and is either a conclusion of $\bot L^a$ or $\top L^c$ (in which case cf. 1.1 (b) or 1.1 (c)) or it has been derived by a left rule. There are eight cases according to the rule used which can be transformed into derivations with lesser cut-height. We will not show this here, since this is only a special case of the cases 3.1-3.8 below.

(e) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of $\top L^c$.

(f) $\top$ = C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{+} C$ is also a conclusion of $\top R^{+}$.

-1.3- The right premise $(\Gamma', D; \Delta') \vdash^{-} C$ is a conclusion of a zero-premise rule. There are five subcases:

(a) C is an atom in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of Rf^{-}.

(b) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of $\bot L^a$.

(c) $\bot$ = D. Then the left premise is $(\Gamma; \Delta) \vdash^{+} \bot$ and the same as mentioned in 1.2 (d) holds.

(d) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of $\top L^c$.

(e) $\bot$ = C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{-} C$ is also a conclusion of $\bot R^{-}$.

-2- Cut^c

-2.1- The left premise $(\Gamma; \Delta) \vdash^{-} D$ is a conclusion of a zero-premise rule. There are four subcases:

(a) The cut formula D is an atom in Δ. Then the conclusion $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^{*} C$ is derived from $(\Gamma'; \Delta', D) \vdash^{*} C$ by W^a and W^c.

(b) $\bot$ is in Γ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\bot L^a$.

(c) $\top$ is in Δ. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ is also a conclusion of $\top L^c$.

(d) $\bot =$ D. Then the right premise is $(\Gamma'; \Delta', \bot) \vdash^* C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C$ follows by $W^{\bot}_{inv}$ as well as W^a and W^c.

-2.2- The right premise $(\Gamma'; \Delta', D) \vdash^+ C$ is a conclusion of a zero-premise rule. There are five subcases:

(a) C is an atom in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of Rf^+.

(b) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of $\bot L^a$.

(c) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of $\top L^c$.

(d) $\top =$ D. Then the left premise is $(\Gamma; \Delta) \vdash^- \top$ and the same as mentioned in 1.2 (d) holds.

(e) $\top =$ C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ C$ is also a conclusion of $\top R^+$.

-2.3- The right premise $(\Gamma'; \Delta', D) \vdash^- C$ is a conclusion of a zero-premise rule. There are six subcases:

(a) C is an atom in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of Rf^-.

(b) $C = D$. Then the left premise is $(\Gamma; \Delta) \vdash^- C$ and $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ follows by W^a and W^c.

(c) $\bot$ is in Γ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of $\bot L^a$.

(d) $\top$ is in Δ'. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of $\top L^c$.

(e) $\top =$ D. Then the left premise is $(\Gamma; \Delta) \vdash^- \top$ and the same as mentioned in 1.2 (d) holds.

(f) $\bot =$ C. Then $(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- C$ is also a conclusion of $\bot R^-$.

Cut with neither premise a conclusion of a zero-premise rule

We distinguish the cases that a left rule is used to derive the left premise (cf. 3), a right rule is used to derive the left premise (cf. 5), a right or a left rule is used to derive the right premise with the cut formula not being principal there (cf. 4), and that a left rule is used to derive the right premise with the cut formula being principal (cf. 5). These cases can be subsumed in a more compact form as categorized below. We assume, like [10], that in the derivations the topsequents, from left to right, have derivation heights n, m, k,...

-3- Cut not principal in the left premise

If the cut formula D is not principal in the left premise, this means that this premise is derived by a left introduction rule. By permuting the order of the rules for the logical connectives with the cut rules, cut-height can be reduced in each of the following eight cases:

-3.1- $\wedge L^a$ is the last rule used to derive the left premise with $\Gamma = \Gamma'', A \wedge B$. The derivations for Cut^a and Cut^c with cuts of cut-height $n+1+m$ are

$$\dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^+ D}{(\Gamma'', A \wedge B; \Delta) \vdash^+ D}\,\wedge L^a \quad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^- D}{(\Gamma'', A \wedge B; \Delta) \vdash^- D}\,\wedge L^a \quad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^+ D \quad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A, B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^a}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,\wedge L^a \qquad \dfrac{\dfrac{(\Gamma'', A, B; \Delta) \vdash^- D \quad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A, B, \Gamma'; \Delta, \Delta') \vdash^* C}\,Cut^c}{(\Gamma'', A \wedge B, \Gamma'; \Delta, \Delta') \vdash^* C}\,\wedge L^a$$

-3.2- $\wedge L^c$ is the last rule used to derive the left premise with $\Delta = \Delta'', A \wedge B$. The derivations with cuts of cut-height $max(n,m)+1+k$ are

$$\dfrac{\dfrac{(\Gamma; \Delta'', A) \vdash^+ D \quad (\Gamma; \Delta'', B) \vdash^+ D}{(\Gamma; \Delta'', A \wedge B) \vdash^+ D}\,\wedge L^c \quad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \wedge B, \Delta') \vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma; \Delta'', A) \vdash^- D \quad (\Gamma; \Delta'', B) \vdash^- D}{(\Gamma; \Delta'', A \wedge B) \vdash^- D}\,\wedge L^c \quad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \wedge B, \Delta') \vdash^* C}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+k$ and $m+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta'',A)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta'',A,\Delta')\vdash^* C}\,Cut^a \qquad \dfrac{(\Gamma;\Delta'',B)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta'',B,\Delta')\vdash^* C}\,Cut^a}{(\Gamma,\Gamma';\Delta'',A\wedge B,\Delta')\vdash^* C}\,\wedge L^c$$

$$\dfrac{\dfrac{(\Gamma;\Delta'',A)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma,\Gamma';\Delta'',A,\Delta')\vdash^* C}\,Cut^c \qquad \dfrac{(\Gamma;\Delta'',B)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma,\Gamma';\Delta'',B,\Delta')\vdash^* C}\,Cut^c}{(\Gamma,\Gamma';\Delta'',A\wedge B,\Delta')\vdash^* C}\,\wedge L^c$$

-3.3- $\vee L^a$ is the last rule used to derive the left premise with $\Gamma = \Gamma'', A\vee B$. The derivations with cuts of cut-height $max(n,m)+1+k$ are

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^+ D \quad (\Gamma'',B;\Delta)\vdash^+ D}{(\Gamma'',A\vee B;\Delta)\vdash^+ D}\,\vee L^a \qquad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^- D \quad (\Gamma'',B;\Delta)\vdash^- D}{(\Gamma'',A\vee B;\Delta)\vdash^- D}\,\vee L^a \qquad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+k$ and $m+k$, respectively:

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma'',A,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^* C}\,\vee L^a$$

$$\dfrac{\dfrac{(\Gamma'',A;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma'',A,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c}{(\Gamma'',A\vee B,\Gamma';\Delta,\Delta')\vdash^* C}\,\vee L^a$$

-3.4- $\vee L^c$ is the last rule used to derive the left premise with $\Delta = \Delta'', A\vee B$. The derivations with cuts of cut-height $n+1+m$ are

$$\dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^+ D}{(\Gamma;\Delta'',A\vee B)\vdash^+ D}\,\vee L^c \qquad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^- D}{(\Gamma;\Delta'',A\vee B)\vdash^- D}\vee L^c \qquad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^* C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^+ D \qquad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta'',A,B,\Delta')\vdash^* C}\,Cut^a}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^* C}\vee L^c \qquad \dfrac{\dfrac{(\Gamma;\Delta'',A,B)\vdash^- D \qquad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma,\Gamma';\Delta'',A,B,\Delta')\vdash^* C}\,Cut^c}{(\Gamma,\Gamma';\Delta'',A\vee B,\Delta')\vdash^* C}\vee L^c$$

-3.5- $\to L^a$ is the last rule used to derive the left premise with $\Gamma=\Gamma'',A\to B$. The derivations with cuts of cut-height $max(n,m)+1+k$ are

$$\dfrac{\dfrac{(\Gamma'',A\to B;\Delta)\vdash^+ A \quad (\Gamma'',B;\Delta)\vdash^+ D}{(\Gamma'',A\to B;\Delta)\vdash^+ D}\to L^a \qquad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma'',A\to B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma'',A\to B;\Delta)\vdash^+ A \quad (\Gamma'',B;\Delta)\vdash^- D}{(\Gamma'',A\to B;\Delta)\vdash^- D}\to L^a \qquad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma'',A\to B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $m+k$:

$$\dfrac{\dfrac{(\Gamma'',A\to B;\Delta)\vdash^+ A}{(\Gamma'',A\to B,\Gamma';\Delta,\Delta')\vdash^+ A}W^{a/c} \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a}{(\Gamma'',A\to B,\Gamma';\Delta,\Delta')\vdash^* C}\to L^a$$

$$\dfrac{\dfrac{(\Gamma'',A\to B;\Delta)\vdash^+ A}{(\Gamma'',A\to B,\Gamma';\Delta,\Delta')\vdash^+ A}W^{a/c} \qquad \dfrac{(\Gamma'',B;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^* C}{(\Gamma'',B,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c}{(\Gamma'',A\to B,\Gamma';\Delta,\Delta')\vdash^* C}\to L^a$$

-3.6- $\to L^c$ is the last rule used to derive the left premise with $\Delta=\Delta'',A\to B$. The derivations with cuts of cut-height $n+1+m$ are

$$\dfrac{\dfrac{(\Gamma,A;\Delta'',B)\vdash^+ D}{(\Gamma;\Delta'',A\to B)\vdash^+ D}\to L^c \qquad (\Gamma',D;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta'',A\to B,\Delta')\vdash^* C}\,Cut^a$$

$$\dfrac{\dfrac{(\Gamma, A; \Delta'', B) \vdash^- D}{(\Gamma; \Delta'', A \to B) \vdash^- D}\ {\to}L^c \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \to B, \Delta') \vdash^* C}\ Cut^c$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma, A; \Delta'', B) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, A, \Gamma'; \Delta'', B, \Delta') \vdash^* C}\ Cut^a}{(\Gamma, \Gamma'; \Delta'', A \to B, \Delta') \vdash^* C}\ {\to}L^c \qquad \dfrac{\dfrac{(\Gamma, A; \Delta'', B) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, A, \Gamma'; \Delta'', B, \Delta') \vdash^* C}\ Cut^c}{(\Gamma, \Gamma'; \Delta'', A \to B, \Delta') \vdash^* C}\ {\to}L^c$$

-3.7- ${\prec}L^a$ is the last rule used to derive the left premise with $\Gamma = \Gamma'', A{\prec}B$. The derivations with cuts of cut-height $n + 1 + m$ are

$$\dfrac{\dfrac{(\Gamma'', A; \Delta, B) \vdash^+ D}{(\Gamma'', A \prec B; \Delta) \vdash^+ D}\ {\prec}L^a \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^a$$

$$\dfrac{\dfrac{(\Gamma'', A; \Delta, B) \vdash^- D}{(\Gamma'', A \prec B; \Delta) \vdash^- D}\ {\prec}L^a \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ Cut^c$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma'', A; \Delta, B) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma'', A, \Gamma'; \Delta, B, \Delta') \vdash^* C}\ Cut^a}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ {\prec}L^a \qquad \dfrac{\dfrac{(\Gamma'', A; \Delta, B) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma'', A, \Gamma'; \Delta, B, \Delta') \vdash^* C}\ Cut^c}{(\Gamma'', A \prec B, \Gamma'; \Delta, \Delta') \vdash^* C}\ {\prec}L^a$$

-3.8- ${\prec}L^c$ is the last rule used to derive the left premise with $\Delta = \Delta'', A{\prec}B$. The derivations with cuts of cut-height $max(n, m) + 1 + k$ are

$$\dfrac{\dfrac{(\Gamma; \Delta'', A \prec B) \vdash^- B \quad (\Gamma; \Delta'', A) \vdash^+ D}{(\Gamma; \Delta'', A \prec B) \vdash^+ D}\ {\prec}L^c \qquad (\Gamma', D; \Delta') \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^* C}\ Cut^a$$

$$\dfrac{\dfrac{(\Gamma; \Delta'', A \prec B) \vdash^- B \quad (\Gamma; \Delta'', A) \vdash^- D}{(\Gamma; \Delta'', A \prec B) \vdash^- D}\ {\prec}L^c \qquad (\Gamma'; \Delta', D) \vdash^* C}{(\Gamma, \Gamma'; \Delta'', A \prec B, \Delta') \vdash^* C}\ Cut^c$$

These can be transformed into derivations with cuts of cut-height $m + k$:

$$\dfrac{\dfrac{(\Gamma;\Delta'',A\prec B)\vdash^{-} B}{(\Gamma,\Gamma';\Delta'',A\prec B,\Delta')\vdash^{-} B}\,W^{a/c} \qquad \dfrac{(\Gamma;\Delta'',A)\vdash^{+} D \quad (\Gamma',D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A,\Delta')\vdash^{*} C}\,Cut^{a}}{(\Gamma,\Gamma';\Delta'',A\prec B,\Delta')\vdash^{*} C}\,{\prec}L^{c}$$

$$\dfrac{\dfrac{(\Gamma;\Delta'',A\prec B)\vdash^{-} B}{(\Gamma,\Gamma';\Delta'',A\prec B,\Delta')\vdash^{-} B}\,W^{a/c} \qquad \dfrac{(\Gamma;\Delta''A)\vdash^{-} D \quad (\Gamma';\Delta',D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta'',A,\Delta')\vdash^{*} C}\,Cut^{c}}{(\Gamma,\Gamma';\Delta'',A\prec B,\Delta')\vdash^{*} C}\,{\prec}L^{c}$$

As said above, cut-height is reduced in all cases.

-4- Cut formula D principal in the left premise only

The cases distinguished here concern the way the right premise is derived. We can distinguish 16 cases and show for each case that the derivation of the right premise can be transformed into one containing only occurrences of cut with a reduced cut-height.

-4.1- $\wedge L^a$ is the last rule used to derive the right premise with $\Gamma' = \Gamma'', A \wedge B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma'',A,B,D;\Delta')\vdash^{*} C}{(\Gamma'',A\wedge B,D;\Delta')\vdash^{*} C}\,\wedge L^{a}}{(\Gamma,\Gamma'',A\wedge B;\Delta,\Delta')\vdash^{*} C}\,Cut^{a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma'',A,B;\Delta',D)\vdash^{*} C}{(\Gamma'',A\wedge B;\Delta',D)\vdash^{*} C}\,\wedge L^{a}}{(\Gamma,\Gamma'',A\wedge B;\Delta,\Delta')\vdash^{*} C}\,Cut^{c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma'',A,B,D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma'',A,B;\Delta,\Delta')\vdash^{*} C}\,Cut^{a}}{(\Gamma,\Gamma'',A\wedge B;\Delta,\Delta')\vdash^{*} C}\,\wedge L^{a} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma'',A,B;\Delta',D)\vdash^{*} C}{(\Gamma,\Gamma'',A,B;\Delta,\Delta')\vdash^{*} C}\,Cut^{c}}{(\Gamma,\Gamma'',A\wedge B;\Delta,\Delta')\vdash^{*} C}\,\wedge L^{a}$$

-4.2- $\wedge L^c$ is the last rule used to derive the right premise with $\Delta' = \Delta'', A \wedge B$. The derivations with cuts of cut-height $n + max(m, k) + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma',D;\Delta'',A)\vdash^{*} C \quad (\Gamma',D;\Delta'',B)\vdash^{*} C}{(\Gamma',D;\Delta'',A\wedge B)\vdash^{*} C}\,\wedge L^{c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\wedge B)\vdash^{*} C}\,Cut^{a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \quad \dfrac{(\Gamma';\Delta'',A,D)\vdash^* C \quad (\Gamma';\Delta'',B,D)\vdash^* C}{(\Gamma';\Delta'',A\wedge B,D)\vdash^* C}\,{\scriptstyle \wedge L^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\wedge B)\vdash^* C}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta'',A)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',A)\vdash^* C}\,{\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta'',B)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',B)\vdash^* C}\,{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta,\Delta'',A\wedge B)\vdash^* C}\,{\scriptstyle \wedge L^c}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta'',A,D)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',A)\vdash^* C}\,{\scriptstyle Cut^c} \quad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta'',B,D)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',B)\vdash^* C}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\wedge B)\vdash^* C}\,{\scriptstyle \wedge L^c}$$

-4.3- $\vee L^a$ is the last rule used to derive the right premise with $\Gamma' = \Gamma'', A\vee B$. The derivations with cuts of cut-height $n+max(m,k)+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \quad \dfrac{(\Gamma'',A,D;\Delta')\vdash^* C \quad (\Gamma'',B,D;\Delta')\vdash^* C}{(\Gamma'',A\vee B,D;\Delta')\vdash^* C}\,{\scriptstyle \vee L^a}}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,{\scriptstyle Cut^a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \quad \dfrac{(\Gamma'',A;\Delta',D)\vdash^* C \quad (\Gamma'',B;\Delta',D)\vdash^* C}{(\Gamma'',A\vee B;\Delta',D)\vdash^* C}\,{\scriptstyle \vee L^a}}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma'',A,D;\Delta')\vdash^* C}{(\Gamma,\Gamma'',A;\Delta,\Delta')\vdash^* C}\,{\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma'',B,D;\Delta')\vdash^* C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^* C}\,{\scriptstyle Cut^a}}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,{\scriptstyle \vee L^a}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma'',A;\Delta',D)\vdash^* C}{(\Gamma,\Gamma'',A;\Delta,\Delta')\vdash^* C}\,{\scriptstyle Cut^c} \quad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma'',B;\Delta',D)\vdash^* C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^* C}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma'',A\vee B;\Delta,\Delta')\vdash^* C}\,{\scriptstyle \vee L^a}$$

-4.4- $\vee L^c$ is the last rule used to derive the right premise with $\Delta' = \Delta'', A\vee B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \quad \dfrac{(\Gamma',D;\Delta'',A,B)\vdash^* C}{(\Gamma',D;\Delta'',A\vee B)\vdash^* C}\,{\scriptstyle \vee L^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\vee B)\vdash^* C}\,{\scriptstyle Cut^a}$$

$$\frac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma';\Delta'',A,B,D)\vdash^{*} C}{(\Gamma';\Delta'',A\vee B,D)\vdash^{*} C}\,{\vee L^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\vee B)\vdash^{*} C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma',D;\Delta'',A,B)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta'',A,B)\vdash^{*} C}\,Cut^a}{(\Gamma,\Gamma';\Delta,\Delta''A\vee B)\vdash^{*} C}\,{\vee L^c} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma';\Delta'',A,B,D)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta'',A,B)\vdash^{*} C}\,Cut^c}{(\Gamma,\Gamma';\Delta,\Delta'',A\vee B)\vdash^{*} C}\,{\vee L^c}$$

-4.5- $\to L^a$ is the last rule used to derive the right premise with $\Gamma' = \Gamma'', A\to B$. The derivations with cuts of cut-height $n+max(m,k)+1$ are

$$\frac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma'',A\to B,D;\Delta')\vdash^{+} A \quad (\Gamma'',B,D;\Delta')\vdash^{*} C}{(\Gamma'',A\to B,D;\Delta')\vdash^{*} C}\,{\to L^a}}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^{*} C}\,Cut^a$$

$$\frac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma'',A\to B;\Delta',D)\vdash^{+} A \quad (\Gamma'',B;\Delta',D)\vdash^{*} C}{(\Gamma'',A\to B;\Delta',D)\vdash^{*} C}\,{\to L^a}}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^{*} C}\,Cut^c$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\frac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma'',A\to B,D;\Delta')\vdash^{+} A}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^{+} A}\,Cut^a \qquad \dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma'',B,D;\Delta')\vdash^{*} C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^{*} C}\,Cut^a}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^{*} C}\,{\to L^a}$$

$$\frac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \quad (\Gamma'',A\to B;\Delta',D)\vdash^{+} A}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^{+} A}\,Cut^c \qquad \dfrac{(\Gamma;\Delta)\vdash^{-} D \quad (\Gamma'',B;\Delta',D)\vdash^{*} C}{(\Gamma,\Gamma'',B;\Delta,\Delta')\vdash^{*} C}\,Cut^c}{(\Gamma,\Gamma'',A\to B;\Delta,\Delta')\vdash^{*} C}\,{\to L^a}$$

-4.6- $\to L^c$ is the last rule used to derive the right premise with $\Delta' = \Delta'', A\to B$. The derivations with cuts of cut-height $n+m+1$ are

$$\frac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma',A,D;\Delta'',B)\vdash^{*} C}{(\Gamma',D;\Delta'',A\to B)\vdash^{*} C}\,{\to L^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\to B)\vdash^{*} C}\,Cut^a$$

$$\frac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma',A;\Delta'',B,D)\vdash^{*} C}{(\Gamma';\Delta'',A\to B,D)\vdash^{*} C}\,{\to L^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\to B)\vdash^{*} C}\,Cut^c$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma', A, D; \Delta'', B)\vdash^{*} C}{(\Gamma, \Gamma', A; \Delta, \Delta'', B)\vdash^{*} C}\,Cut^{a}}{(\Gamma, \Gamma'; \Delta, \Delta'', A\rightarrow B)\vdash^{*} C}\,{\rightarrow}L^{c} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma', A; \Delta'', B, D)\vdash^{*} C}{(\Gamma, \Gamma', A; \Delta, \Delta'', B)\vdash^{*} C}\,Cut^{c}}{(\Gamma, \Gamma'; \Delta, \Delta'', A\rightarrow B)\vdash^{*} C}\,{\rightarrow}L^{c}$$

-4.7- $\prec L^{a}$ is the last rule used to derive the right premise with $\Gamma' = \Gamma'', A \prec B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma'', A, D; \Delta', B)\vdash^{*} C}{(\Gamma'', A\prec B, D; \Delta')\vdash^{*} C}\,{\prec}L^{a}}{(\Gamma, \Gamma'', A\prec B; \Delta, \Delta')\vdash^{*} C}\,Cut^{a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma'', A; \Delta', B, D)\vdash^{*} C}{(\Gamma'', A\prec B; \Delta', D)\vdash^{*} C}\,{\prec}L^{a}}{(\Gamma, \Gamma'', A\prec B; \Delta, \Delta')\vdash^{*} C}\,Cut^{c}$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad (\Gamma'', A, D; \Delta', B)\vdash^{*} C}{(\Gamma, \Gamma'', A; \Delta, \Delta', B)\vdash^{*} C}\,Cut^{a}}{(\Gamma, \Gamma'', A\prec B; \Delta, \Delta')\vdash^{*} C}\,{\prec}L^{a} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad (\Gamma'', A; \Delta', B, D)\vdash^{*} C}{(\Gamma, \Gamma'', A; \Delta, \Delta', B)\vdash^{*} C}\,Cut^{c}}{(\Gamma, \Gamma'', A\prec B; \Delta, \Delta')\vdash^{*} C}\,{\prec}L^{a}$$

-4.8- $\prec L^{c}$ is the last rule used to derive the right premise with $\Delta' = \Delta'', A \prec B$. The derivations with cuts of cut-height $n + max(m,k) + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^{+} D \qquad \dfrac{(\Gamma', D; \Delta'', A\prec B)\vdash^{-} B \quad (\Gamma', D; \Delta'', A)\vdash^{*} C}{(\Gamma', D; \Delta'', A\prec B)\vdash^{*} C}\,{\prec}L^{c}}{(\Gamma, \Gamma'; \Delta, \Delta'', A\prec B)\vdash^{*} C}\,Cut^{a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^{-} D \qquad \dfrac{(\Gamma'; \Delta'', A\prec B, D)\vdash^{-} B \quad (\Gamma'; \Delta'', A, D)\vdash^{*} C}{(\Gamma'; \Delta'', A\prec B, D)\vdash^{*} C}\,{\prec}L^{c}}{(\Gamma, \Gamma'; \Delta, \Delta'', A\prec B)\vdash^{*} C}\,Cut^{c}$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma', D; \Delta'', A\prec B)\vdash^{-} B}{(\Gamma, \Gamma'; \Delta, \Delta'', A\prec B)\vdash^{-} B}\,Cut^{a} \qquad \dfrac{(\Gamma;\Delta)\vdash^{+} D \quad (\Gamma', D; \Delta'', A)\vdash^{*} C}{(\Gamma, \Gamma'; \Delta, \Delta'', A)\vdash^{*} C}\,Cut^{a}}{(\Gamma, \Gamma'; \Delta, \Delta'', A\prec B)\vdash^{*} C}\,{\prec}L^{c}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta'',A\prec B,D)\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^- B}{\scriptstyle Cut^c} \quad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta'',A,D)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta'',A)\vdash^* C}{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta'',A\prec B)\vdash^* C}{\scriptstyle \prec L^c}$$

-4.9- $\wedge R^+$ is the last rule used to derive the right premise with $C = A\wedge B$. The derivations with cuts of cut-height $n + max(m,k) + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \quad \dfrac{(\Gamma',D;\Delta')\vdash^+ A \quad (\Gamma',D;\Delta')\vdash^+ B}{(\Gamma',D;\Delta')\vdash^+ A\wedge B}{\scriptstyle \wedge R^+}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\wedge B}{\scriptstyle Cut^a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \quad \dfrac{(\Gamma';\Delta',D)\vdash^+ A \quad (\Gamma';\Delta',D)\vdash^+ B}{(\Gamma';\Delta',D)\vdash^+ A\wedge B}{\scriptstyle \wedge R^+}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\wedge B}{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n + m$ and $n + k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}{\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^+ B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ B}{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\wedge B}{\scriptstyle \wedge R^+}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}{\scriptstyle Cut^c} \quad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^+ B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ B}{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\wedge B}{\scriptstyle \wedge R^+}$$

-4.10.1- $\wedge R_1^-$ is the last rule used to derive the right premise with $C = A\wedge B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \quad \dfrac{(\Gamma',D;\Delta')\vdash^- A}{(\Gamma',D;\Delta')\vdash^- A\wedge B}{\scriptstyle \wedge R_1^-}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\wedge B}{\scriptstyle Cut^a} \quad \dfrac{(\Gamma;\Delta)\vdash^- D \quad \dfrac{(\Gamma';\Delta',D)\vdash^- A}{(\Gamma';\Delta',D)\vdash^- A\wedge B}{\scriptstyle \wedge R_1^-}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\wedge B}{\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^- A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A}{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\wedge B}{\scriptstyle \wedge R_1^-} \qquad \dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^- A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A}{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\wedge B}{\scriptstyle \wedge R_1^-}$$

-4.10.2- $\wedge R_2^-$ is the last rule used to derive the right premise with $C = A \wedge B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^- B}{(\Gamma', D; \Delta') \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', D) \vdash^- B}{(\Gamma'; \Delta', D) \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^- B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- B}\ {\scriptstyle Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^- B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- B}\ {\scriptstyle Cut^c}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \wedge B}\ {\scriptstyle \wedge R_2^-}$$

-4.11.1- $\vee R_1^+$ is the last rule used to derive the right premise with $C = A \vee B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^+ A}{(\Gamma', D; \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', D) \vdash^+ A}{(\Gamma'; \Delta', D) \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^+ A}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A}\ {\scriptstyle Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^+ A}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A}\ {\scriptstyle Cut^c}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_1^+}$$

-4.11.2- $\vee R_2^+$ is the last rule used to derive the right premise with $C = A \vee B$. The derivations with cuts of cut-height $n+m+1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta') \vdash^+ B}{(\Gamma', D; \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', D) \vdash^+ B}{(\Gamma'; \Delta', D) \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n+m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta') \vdash^+ B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ B}\ {\scriptstyle Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', D) \vdash^+ B}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ B}\ {\scriptstyle Cut^c}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^+ A \vee B}\ {\scriptstyle \vee R_2^+}$$

-4.12- $\vee R^-$ is the last rule used to derive the right premise with $C = A \vee B$. The derivations with cuts of cut-height $n + max(m, k) + 1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D;\Delta') \vdash^- A \quad (\Gamma', D;\Delta') \vdash^- B}{(\Gamma', D;\Delta') \vdash^- A \vee B}\,{\scriptstyle \vee R^-}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- A \vee B}\,{\scriptstyle Cut^a}$$

$$\dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma';\Delta', D) \vdash^- A \quad (\Gamma';\Delta', D) \vdash^- B}{(\Gamma';\Delta', D) \vdash^- A \vee B}\,{\scriptstyle \vee R^-}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- A \vee B}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n + m$ and $n + k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \quad (\Gamma', D;\Delta') \vdash^- A}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- A}\,{\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta) \vdash^+ D \quad (\Gamma', D;\Delta') \vdash^- B}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- B}\,{\scriptstyle Cut^a}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- A \vee B}\,{\scriptstyle \vee R^-}$$

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \quad (\Gamma';\Delta', D) \vdash^- A}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- A}\,{\scriptstyle Cut^c} \qquad \dfrac{(\Gamma;\Delta) \vdash^- D \quad (\Gamma';\Delta', D) \vdash^- B}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- B}\,{\scriptstyle Cut^c}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- A \vee B}\,{\scriptstyle \vee R^-}$$

-4.13- $\rightarrow R^+$ is the last rule used to derive the right premise with $C = A \rightarrow B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', A, D;\Delta') \vdash^+ B}{(\Gamma', D;\Delta') \vdash^+ A \rightarrow B}\,{\scriptstyle \rightarrow R^+}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^+ A \rightarrow B}\,{\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma', A;\Delta', D) \vdash^+ B}{(\Gamma';\Delta', D) \vdash^+ A \rightarrow B}\,{\scriptstyle \rightarrow R^+}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^+ A \rightarrow B}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', A, D;\Delta') \vdash^+ B}{(\Gamma, \Gamma', A;\Delta, \Delta') \vdash^+ B}\,{\scriptstyle Cut^a}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^+ A \rightarrow B}\,{\scriptstyle \rightarrow R^+} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma', A;\Delta', D) \vdash^+ B}{(\Gamma, \Gamma', A;\Delta, \Delta') \vdash^+ B}\,{\scriptstyle Cut^c}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^+ A \rightarrow B}\,{\scriptstyle \rightarrow R^+}$$

-4.14- $\rightarrow R^-$ is the last rule used to derive the right premise with $C = A \rightarrow B$. The derivations with cuts of cut-height $n + max(m, k) + 1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D;\Delta') \vdash^+ A \quad (\Gamma', D;\Delta') \vdash^- B}{(\Gamma', D;\Delta') \vdash^- A \rightarrow B}\,{\scriptstyle \rightarrow R^-}}{(\Gamma, \Gamma';\Delta, \Delta') \vdash^- A \rightarrow B}\,{\scriptstyle Cut^a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma';\Delta',D)\vdash^+ A \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma';\Delta',D)\vdash^- A\to B}\,{\scriptstyle \to R^-}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\to B}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,{\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\to B}\,{\scriptstyle \to R^-}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,{\scriptstyle Cut^c} \qquad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- A\to B}\,{\scriptstyle \to R^-}$$

-4.15- $\prec R^+$ is the last rule used to derive the right premise with $C = A\prec B$. The derivations with cuts of cut-height $n+max(m,k)+1$ are

$$\dfrac{(\Gamma;\Delta)\vdash^+ D \qquad \dfrac{(\Gamma',D;\Delta')\vdash^+ A \quad (\Gamma',D;\Delta')\vdash^- B}{(\Gamma',D;\Delta')\vdash^+ A\prec B}\,{\scriptstyle \prec R^+}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,{\scriptstyle Cut^a}$$

$$\dfrac{(\Gamma;\Delta)\vdash^- D \qquad \dfrac{(\Gamma';\Delta',D)\vdash^+ A \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma';\Delta',D)\vdash^+ A\prec B}\,{\scriptstyle \prec R^+}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,{\scriptstyle Cut^c}$$

These can be transformed into derivations each with two cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,{\scriptstyle Cut^a} \qquad \dfrac{(\Gamma;\Delta)\vdash^+ D \quad (\Gamma',D;\Delta')\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,{\scriptstyle Cut^a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,{\scriptstyle \prec R^+}$$

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^+ A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A}\,{\scriptstyle Cut^c} \qquad \dfrac{(\Gamma;\Delta)\vdash^- D \quad (\Gamma';\Delta',D)\vdash^- B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^- B}\,{\scriptstyle Cut^c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^+ A\prec B}\,{\scriptstyle \prec R^+}$$

-4.16- $\prec R^-$ is the last rule used to derive the right premise with $C = A \prec B$. The derivations with cuts of cut-height $n + m + 1$ are

$$\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad \dfrac{(\Gamma', D; \Delta', B) \vdash^- A}{(\Gamma', D; \Delta') \vdash^- A \prec B}{\prec R^-}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}{Cut^a} \qquad \dfrac{(\Gamma;\Delta) \vdash^- D \qquad \dfrac{(\Gamma'; \Delta', B, D) \vdash^- A}{(\Gamma'; \Delta', D) \vdash^- A \prec B}{\prec R^-}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}{Cut^c}$$

These can be transformed into derivations with cuts of cut-height $n + m$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ D \qquad (\Gamma', D; \Delta', B) \vdash^- A}{(\Gamma, \Gamma'; \Delta, \Delta', B) \vdash^- A}{Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}{\prec R^-} \qquad \dfrac{\dfrac{(\Gamma;\Delta) \vdash^- D \qquad (\Gamma'; \Delta', B, D) \vdash^- A}{(\Gamma, \Gamma'; \Delta, \Delta', B) \vdash^- A}{Cut^c}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^- A \prec B}{\prec R^-}$$

It is shown that cut-height is reduced in all cases.

-5- Cut formula D principal in both premises

For each cut rule four cases can be distinguished. Here, it can be shown for each case that the derivations can be transformed into ones in which the occurrences of cut have a reduced cut-height or the cut formula has a lower weight (or both).

-5.1- $D = A \wedge B$. The derivation for Cut^a with a cut of cut-height $max(n, m) + 1 + k + 1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ A \quad (\Gamma;\Delta) \vdash^+ B}{(\Gamma;\Delta) \vdash^+ A \wedge B}{\wedge R^+} \qquad \dfrac{(\Gamma', A, B; \Delta') \vdash^* C}{(\Gamma', A \wedge B; \Delta') \vdash^* C}{\wedge L^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}{Cut^a}$$

and can be transformed into a derivation with two cuts of cut-height (from top to bottom) $n + k$ and $m + max(n, k) + 1$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ B \qquad \dfrac{(\Gamma;\Delta) \vdash^+ A \qquad (\Gamma', A, B; \Delta') \vdash^* C}{(\Gamma, \Gamma', B; \Delta, \Delta') \vdash^* C}{Cut^a}}{(\Gamma, \Gamma, \Gamma'; \Delta, \Delta, \Delta') \vdash^* C}{Cut^a}}{(\Gamma, \Gamma'; \Delta, \Delta') \vdash^* C}{C_{a/c}}$$

Note that in both cases the weight of the cut formula is reduced. The upper cut is also reduced in height, while with the lower cut we have a case where cut-height is not necessarily reduced.

The possible derivations for Cut^c with a cut of cut-height $n+1+max(m,k)+1$ are

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- A}{(\Gamma;\Delta)\vdash^- A\wedge B}\wedge R_1^- \qquad \dfrac{(\Gamma';\Delta',A)\vdash^* C \quad (\Gamma';\Delta',B)\vdash^* C}{(\Gamma';\Delta',A\wedge B)\vdash^* C}\wedge L^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c$$

or

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- B}{(\Gamma;\Delta)\vdash^- A\wedge B}\wedge R_2^- \qquad \dfrac{(\Gamma';\Delta',A)\vdash^* C \quad (\Gamma';\Delta',B)\vdash^* C}{(\Gamma';\Delta',A\wedge B)\vdash^* C}\wedge L^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c$$

and those can be transformed into derivations with cuts of cut-height $n+m$ or $n+k$, respectively:

$$\dfrac{(\Gamma;\Delta)\vdash^- A \qquad (\Gamma';\Delta',A)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c \quad \dfrac{(\Gamma;\Delta)\vdash^- B \qquad (\Gamma';\Delta',B)\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c$$

Here, both cut-height and weight of the cut formulas are reduced.

-5.2- $D = A\vee B$. The possible derivations for Cut^a with a cut of cut-height $n+1+max(m,k)+1$ are

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ A}{(\Gamma;\Delta)\vdash^+ A\vee B}\vee R_1^+ \qquad \dfrac{(\Gamma',A;\Delta')\vdash^* C \quad (\Gamma',B;\Delta')\vdash^* C}{(\Gamma',A\vee B;\Delta')\vdash^* C}\vee L^a}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a$$

or

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^+ B}{(\Gamma;\Delta)\vdash^+ A\vee B}\vee R_2^+ \qquad \dfrac{(\Gamma',A;\Delta')\vdash^* C \quad (\Gamma',B;\Delta')\vdash^* C}{(\Gamma',A\vee B;\Delta')\vdash^* C}\vee L^a}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a$$

and those can be transformed into derivations with cuts of cut-height $n+m$ and $n+k$, respectively:

$$\dfrac{(\Gamma;\Delta)\vdash^+ A \qquad (\Gamma',A;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a \quad \dfrac{(\Gamma;\Delta)\vdash^+ B \qquad (\Gamma',B;\Delta')\vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^a$$

Again, both cut-height and weight of the cut formulas are reduced.

The derivation for Cut^c with a cut of cut-height $max(n,m)+1+k+1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^- A \quad (\Gamma;\Delta)\vdash^- B}{(\Gamma;\Delta)\vdash^- A\vee B}\vee R^- \qquad \dfrac{(\Gamma';\Delta',A,B)\vdash^* C}{(\Gamma';\Delta',A\vee B)\vdash^* C}\vee L^c}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^* C}\,Cut^c$$

and can be transformed into a derivation with two cuts of cut-height $n+k$ and $m+max(n,k)+1$:

$$\dfrac{(\Gamma;\Delta)\vdash^{-} B \qquad \dfrac{(\Gamma;\Delta)\vdash^{-} A \qquad (\Gamma';\Delta',A,B)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta',B)\vdash^{*} C}\,Cut^{c}}{\dfrac{(\Gamma,\Gamma,\Gamma';\Delta,\Delta,\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,C^{a/c}}\,Cut^{c}$$

Note that again, in the case of the lower cut, although the cut-height might increase, the weight of the cut formula is reduced. For the upper cut both cut-height and weight of the cut formula is reduced.

-5.3- $D = A \to B$. The derivation for Cut^a with a cut of cut-height $n+1+max(m,k)+1$ is

$$\dfrac{\dfrac{(\Gamma,A;\Delta)\vdash^{+} B}{(\Gamma;\Delta)\vdash^{+} A\to B}\,{\to}R^{+} \qquad \dfrac{(\Gamma',A\to B;\Delta')\vdash^{+} A \quad (\Gamma',B;\Delta')\vdash^{*} C}{(\Gamma',A\to B;\Delta')\vdash^{*} C}\,{\to}L^{a}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,Cut^{a}$$

and this can be transformed into a derivation with three cuts of cut-height (from left to right and from top to bottom) $n+1+m$, $n+k$, and $max(n+1,m)+1+max(n,k)+1$ respectively:

$$\dfrac{\dfrac{\dfrac{(\Gamma,A;\Delta)\vdash^{+} B}{(\Gamma;\Delta)\vdash^{+} A\to B}\,{\to}R^{+} \quad (\Gamma',A\to B;\Delta')\vdash^{+} A}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{+} A}\,Cut^{a} \qquad \dfrac{(\Gamma,A;\Delta)\vdash^{+} B \quad (\Gamma',B;\Delta')\vdash^{*} C}{(\Gamma,A,\Gamma';\Delta,\Delta')\vdash^{*} C}\,Cut^{a}}{\dfrac{(\Gamma,\Gamma,\Gamma',\Gamma';\Delta,\Delta,\Delta',\Delta')\vdash^{*} C}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,C^{a/c}}\,Cut^{a}$$

In the first case cut-height is reduced, in the second case cut-height and weight of the cut formula is reduced and in the third case weight of the cut formula is reduced.

The derivation for Cut^c with a cut of cut-height $max(n,m)+1+k+1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta)\vdash^{+} A \quad (\Gamma;\Delta)\vdash^{-} B}{(\Gamma;\Delta)\vdash^{-} A\to B}\,{\to}R^{-} \qquad \dfrac{(\Gamma',A;\Delta',B)\vdash^{*} C}{(\Gamma';\Delta',A\to B)\vdash^{*} C}\,{\to}L^{c}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}\,Cut^{c}$$

This can be transformed into a derivation with two cuts of cut-height $n+k$ and $m+max(n,k)+1$:

$$\dfrac{\dfrac{(\Gamma; \Delta, A) \vdash^{+} A \qquad \dfrac{}{}}{}}{}$$

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^- B \qquad \dfrac{(\Gamma;\Delta) \vdash^+ A \qquad (\Gamma', A;\Delta', B) \vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta', B) \vdash^* C}\,Cut^a}{(\Gamma,\Gamma,\Gamma';\Delta,\Delta,\Delta') \vdash^* C}\,Cut^c}{(\Gamma,\Gamma';\Delta,\Delta') \vdash^* C}\,C^{a/c}$$

In the first case cut-height and weight of the cut formula is reduced, while in the second case the weight of the cut formula is reduced. Here we can observe a result specific for this calculus due to the mixture of derivability relations $\vdash^+$ and $\vdash^-$ in $\to R^-$ and the position of the active formulas in the assumptions *and* in the counterassumptions in $\to L^c$: Derivations containing instances of Cut^c are not necessarily transformed into derivations with a lesser cut-height or a reduced weight of the cut formula of another instance of Cut^c but it can also happen that Cut^c is replaced by Cut^a.

-5.4- $D = A \prec B$. The derivation for Cut^a with a cut of cut-height $max(n, m) + 1 + k + 1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^+ A \quad (\Gamma;\Delta) \vdash^- B}{(\Gamma;\Delta) \vdash^+ A \prec B}\,{\prec}R^+ \qquad \dfrac{(\Gamma', A;\Delta', B) \vdash^* C}{(\Gamma', A \prec B;\Delta') \vdash^* C}\,{\prec}L^a}{(\Gamma,\Gamma';\Delta,\Delta') \vdash^* C}\,Cut^a$$

This can be transformed into a derivation with two cuts of cut-height $n + k$ and $m + max(n, k) + 1$:

$$\dfrac{\dfrac{(\Gamma;\Delta) \vdash^- B \qquad \dfrac{(\Gamma;\Delta) \vdash^+ A \qquad (\Gamma', A;\Delta', B) \vdash^* C}{(\Gamma,\Gamma';\Delta,\Delta', B) \vdash^* C}\,Cut^a}{(\Gamma,\Gamma,\Gamma';\Delta,\Delta,\Delta') \vdash^* C}\,Cut^c}{(\Gamma,\Gamma';\Delta,\Delta') \vdash^* C}\,C^{a/c}$$

Again, due to the mixture of derivability relations $\vdash^+$ and $\vdash^-$ in ${\prec}R^+$ and the presence of the active formulas both in assumptions and counterassumptions in ${\prec}L^a$, in this case Cut^a can be replaced by instances of Cut^c with a reduced weight of the cut formula. In the upper cut we have a reduction of both cut-height and weight of the cut formula.

The derivation for Cut^c with a cut of cut-height $n+1+max(m, k)+1$ is

$$\dfrac{\dfrac{(\Gamma;\Delta, B) \vdash^- A}{(\Gamma;\Delta) \vdash^- A \prec B}\,{\prec}R^- \qquad \dfrac{(\Gamma';\Delta', A \prec B) \vdash^- B \quad (\Gamma';\Delta', A) \vdash^* C}{(\Gamma';\Delta', A \prec B) \vdash^* C}\,{\prec}L^c}{(\Gamma,\Gamma';\Delta,\Delta') \vdash^* C}\,Cut^c$$

and this can be transformed into a derivation with three cuts of cut-height (from left to right and from top to bottom) $n+1+m$, $n+k$, and $max(n+1,m)+1+max(n,k)+1$ respectively:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\Gamma;\Delta,B)\vdash^{-} A}{(\Gamma;\Delta)\vdash^{-} A\prec B}{\scriptstyle\prec R^{-}} \quad (\Gamma';\Delta',A\prec B)\vdash^{-} B}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{-} B}{\scriptstyle Cut^{c}} \quad \dfrac{(\Gamma;\Delta,B)\vdash^{-} A \quad (\Gamma';\Delta',A)\vdash^{*} C}{(\Gamma,\Gamma';\Delta,B,\Delta')\vdash^{*} C}{\scriptstyle Cut^{c}}}{(\Gamma,\Gamma,\Gamma',\Gamma';\Delta,\Delta,\Delta',\Delta')\vdash^{*} C}{\scriptstyle Cut^{c}}}{(\Gamma,\Gamma';\Delta,\Delta')\vdash^{*} C}{\scriptstyle C^{a/c}}$$

In the first case cut-height is reduced, in the second case cut-height and weight of the cut formula and in the third case weight of the cut formula. □

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