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Leonardo Ceragioli

BILATERAL RULES AS COMPLEX RULES

Abstract

Proof-theoretic semantics is an inferentialist theory of meaning originally developed in a unilateral framework. Its extension to bilateral systems opens both opportunities and problems. The problems are caused especially by Coordination Principles (a kind of rule that is not present in unilateral systems) and mismatches between rules for assertion and rules for rejection. In this paper, a solution is proposed for two major issues: the availability of a reduction procedure for *tonk* and the existence of harmonious rules for the paradoxical zero-ary connective $\bullet$. The solution is based on a reinterpretation of bilateral rules as complex rules, that is, rules that introduce or eliminate connectives in a subordinate position. Looking at bilateral rules from this perspective, the problems faced by bilateralism can be seen as special cases of general problems of complex systems, which have been already analyzed in the literature. In the end, a comparison with other proposed solutions underlines the need for further investigation in order to complete the picture of bilateral proof-theoretic semantics.

Keywords: bilateralism, separability, harmony.

1. Introduction

The aim of this paper is to solve some problems faced by a specific flavour of proof-theoretic semantics when applied to bilateral systems. A complete reconstruction of the state of the art of this field of study or its history is far beyond the limits of this contribution, but some of its key aspects have to be reminded. The same holds for bilateralism: we do not intend to give

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the full picture regarding this vast topic, but we will remind some aspects that will be relevant to the problems here at issue.

Proof-theoretic semantics is an approach toward meaning (especially for the logical language), which – as opposed to model-theoretic semantics – assigns meaning without referring to things external to the language and the linguistic practices, such as models or structures. It is a vast and heterogeneous field of study, with different ramifications, tied together by the adoption of proof – as opposed to truth – as the key ingredient of semantics. In its original formulation due to Dummett and Prawitz, proof-theoretic semantics focuses primarily (if not only) on natural deduction systems, and so on systems containing only Operational Rules.[1] For these rules, some *criteria* of acceptability are given:

- For I-rules, something like a complexity condition is usually imposed, with the clause that in all its applications, the conclusion should be more complex than both the premises and the discharged assumptions;[2]

- For E-rules, a criterion called *harmony* guarantees that they are consequences of, and so justified by, the corresponding I-rules.

While there is no consensus about which shape the criterion of *harmony* should take, it is usually agreed that Inversion Principle should be at least one of its ingredients or presuppositions:[3]

> "Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition $[\cdots]$ for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already "contain" a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α."

In practice, a pair of rules for a logical constant suits this principle iff there are some *reduction steps* that: take every derivation in which the

[1] See [28] (general proof theory) and [6].

[2] See [6] p. 258. This criterion has been criticized in [8], where the author proposes a new criterion.

[3] [27], p. 33. For a historical account of the development of this principle, see [24].

major premise of an E-rule is derived using an I-rule as input; return a derivation of the conclusion of the E-rule which is constructed by combining the derivations of the premises of the I-rule and the (eventual) derivations of the minor premises of the E-rule.

To make this intuition more precise, Prawitz introduces the notion of *maximal formulae*:

Definition 1.1 (Maximal Formulae). Given a derivation $\mathfrak{D}$, a maximal formula in it is a formula that is the conclusion of an I-rule and the major premise of an E-rule.

With this definition, we can observe that Inversion Principle asks that for each *maximal formula* generated applying a pair of I and E-rules, there is a reduction step that removes it. As an example, the maximal formula in the derivation on the left is removed in that on the right:

$$\supset\mathrm{E}\,\dfrac{\begin{matrix}\mathfrak{D}_1 \ \vdots \\ A\end{matrix} \qquad \supset\mathrm{I}\,\dfrac{\begin{matrix}[A] \\ \mathfrak{D}_2 \ \vdots \\ B\end{matrix}}{A \supset B}}{B} \quad \rightsquigarrow \quad \begin{matrix} \mathfrak{D}_1 \ \vdots \\ A \\ \mathfrak{D}_2 \ \vdots \\ B\end{matrix}$$

It should be clear that Inversion Principle does not entail that *maximal formulae* can be avoided in general, since a reduction step can generate new *maximal formulae*. As an example, if in the previous example the derivation $\mathfrak{D}_1$ of A ends with an application of an I-rule and the derivation $\mathfrak{D}_2$ of B from A starts with an application of an E-rule of which A is its major premise, the reduction gives birth to a new *maximal formula*, that is A. The generation of new *maximal formulae* poses the problem of circular reductions and in general of the eliminability of *all maximal formulae*.[4] So, defining as in *normal form* a derivation in which there are no *maximal formulae*, there are two properties eligible for harmony:

existence of normal form Given a derivation of C from Γ, there is a derivation in *normal form* of the same conclusion from at most the same assumptions;

[4] As opposed to the eliminability of *each maximal formula*, for which Inversion Principle is enough.

normalization Given a derivation of C from Γ, there is an effective procedure leading from it to a derivation in *normal form* of the same conclusion from at most the same assumptions.

In the most traditional versions of proof-theoretic semantics, when harmony is not equated with Inversion Principle *tout court*,[5] it usually entails both this principle and the request for normalization.[6] In our discussion about Rumfitt's bilateral system, we will follow him and consider normalization as the key ingredient of harmony. Our aim will be to remain as adherent as possible to this traditional conception of proof-theoretic semantics, while endorsing bilateralism and solving the issues pointed out against Rumfitt's system.

Of course, this overview of proof-theoretic semantics is far from complete, and covers just the orthodox developments of this discipline that follow Dummett and Prawitz in favouring single-conclusion natural deduction and harmony *criteria* based on normalization. Admittedly, there are generalizations and different approaches departing from this traditional flavour of proof-theoretic semantics. As an example, there are some attempts to generalize this kind of investigation in the direction of sequent calculus.[7] Nonetheless, Rumfitt's bilateral system is a development of this early tradition, and the author is explicitly skeptical about meaning-theoretical usages of sequent calculus.[8] Moreover, none of the criticisms that we will consider about this system crosses the limits of this traditional approach. Hence, later alternative approaches to proof-theoretic semantics can be overlooked in what follows.

Even though there are some issues with *ex falso quodlibet*, we can consider part of the received wisdom that traditional unilateral proof-theoretic semantics leads to the justification of intuitionistic logic.[9] On the contrary,

[5] As suggested in [16].

[6] To be honest, the situation is far more complex than that. For a recent analysis of the precise relation between normalization and validity in proof-theoretic semantics, see [38].

[7] *Inter alia*, see [14] for inferentialism and proof-theoretic semantics, and [34] for a bilateralist analysis of these calculi.

[8] [36], p. 795.

[9] See [17] for the problems that proof-theoretic semantics has in defining the meaning of $\bot$, and [1] for the problems encountered in trying to prove that *ex falso quodlibet* suits Dummett and Prawitz's definitions of proof-theoretic validity.

$$\wedge\mathrm{I}^{+}\ \frac{+A \qquad +B}{+(A\wedge B)} \qquad \wedge\mathrm{E}^{+}\ \frac{+(A\wedge B)}{+A} \qquad \wedge\mathrm{E}^{+}\ \frac{+(A\wedge B)}{+B}$$

$$\wedge\mathrm{E}^{-}\ \frac{-(A\wedge B) \qquad \begin{array}{c}[-A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[-B]\\ \vdots\\ C\end{array}}{C} \qquad \wedge\mathrm{I}^{-}\ \frac{-A}{-(A\wedge B)} \qquad \wedge\mathrm{I}^{-}\ \frac{-B}{-(A\wedge B)}$$

$$\vee\mathrm{E}^{+}\ \frac{+(A\vee B) \qquad \begin{array}{c}[+A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[+B]\\ \vdots\\ C\end{array}}{C} \qquad \vee\mathrm{I}^{+}\ \frac{+A}{+(A\wedge B)} \qquad \vee\mathrm{I}^{+}\ \frac{+B}{+(A\wedge B)}$$

$$\vee\mathrm{I}^{-}\ \frac{-A \qquad -B}{-(A\vee B)} \qquad \vee\mathrm{E}^{-}\ \frac{-(A\vee B)}{-A} \qquad \vee\mathrm{E}^{-}\ \frac{-(A\vee B)}{-B}$$

$$\supset\mathrm{I}^{+}\ \frac{\begin{array}{c}[+A]\\ \vdots\\ +B\end{array}}{+(A\supset B)} \qquad \supset\mathrm{I}^{-}\ \frac{+A \qquad -B}{-(A\supset B)} \qquad \supset\mathrm{E}^{+}\ \frac{+(A\supset B) \qquad +A}{+B}$$

$$\supset\mathrm{E}^{-}\ \frac{-(A\supset B)}{+A} \qquad \neg\mathrm{I}^{-}\ \frac{+A}{-(\neg A)} \qquad \supset\mathrm{E}^{-}\ \frac{-(A\supset B)}{-B}$$

$$\neg\mathrm{I}^{+}\ \frac{-A}{+(\neg A)} \qquad \neg\mathrm{E}^{-}\ \frac{-(\neg A)}{+A} \qquad \neg\mathrm{E}^{+}\ \frac{+(\neg A)}{-A}$$

Figure 1. Operational rules

while there are some attempts in this direction, there is no clear and uncontended justification of classical logic inside such a unilateral perspective.[10]

In [36], Rumfitt investigates the possibility of justifying classical logic by focusing on a bilateral reformulation of natural deduction. He uses $+A$ to mean that A is asserted and $-A$ to mean that A is rejected, and proposes a system consisting of two kinds of rules:

Operational Rules: rules governing the introduction or elimination of connectives inside propositions prefixed by a sing $+$ or by a sign $-$;

Coordination Principles: principles dealing with propositions prefixed by a sing $+$ or by a sign $-$ regardless of their logical structure.

In Figure 1 the Operational Rules endorsed by Rumfitt are displayed.[11] In Figure 2 two alternative sets of Coordination Principles for a bilateral classical system are displayed: the three on the top (the two rules of Reductio and the rule of Non-Contradiction) or the two on the bottom (the two rules of Smiley).[12] We will work mostly with the system composed of the Operational Rules together with the two Smiley, but sometimes we will consider Reductio and Non-Contradiction as well.

Rumfitt explicitly endorses a *criterion* of harmony based on normalization for the acceptability of the Operational Rules, but has to provide new *criteria* for the Coordination Principles. Indeed, being developed in a unilateral framework, proof-theoretic semantics deals traditionally only with Operational Rules and gives *criteria* only for the acceptability of such rules. Rumfitt proposes different *criteria* for the Coordination Principles, which nonetheless have been proved to be untenable.[13] At the beginning

[10] An anonymous referee suggests that Sandqvist's semantics for classical logic in [37] counts as such an uncontended justification. I thank them for this suggestion. Sandqvist's result is surely thought-provoking for proof-theoretic semantics and uncontested as a formal result, leaving aside some formal issues regarding disjunction and existential quantifier. Anyway, it develops a notion of validity that is very different from the one based on harmony, as remarked also in [26]. What I mean here is that there are no uncontroversial justifications of classical logic inside the specific flavour of proof-theoretic semantics that relies on harmony and normalization, and to which Rumfitt's work belongs, even though there are some attempts in this direction: [22], [32] and [25] *inter alia*.

[11] [36], pp. 800–802.

[12] [36], p. 804.

[13] [9] and [18], p. 635.

of our investigation, it will just be enough to focus on harmony for Operational Rules and leave open the issue of Coordination Principles. Later on, the problem of providing a working *criterion* for Coordination Principles will become central and we will see that a common *criterion* for both Operational Rules and Coordination Principles can be found.

Before moving to more technical topics, let us discuss a conceptual difficulty about bilateral systems, since (apart from its intrinsic interest) it will become relevant later on. Kürbis has shown (see the contribution to this volume) that the interpretation of $+$ and $-$ as speech acts is untenable, and has proposed a proof-and-refutation route to bilateralism, as opposed to an assertion-and-rejection one. In a nutshell, his argument is the following: since asserting, denying, and making an assumption are all speech acts, and speech acts cannot be iterated (as an example, in Rumfitt's vocabulary the expression $++p$ is forbidden), then Rumfitt cannot adopt assumptions in his system.[14] While I find his objection well-defended and convincing, in this paper I will focus on what seems to me an orthogonal issue. I will just treat $+$ and $-$ as two modalities, without discussing their nature, and try to address some well-known problems of this system.[15] What will come out are considerations coherent with Kürbis' objection, but independent from it.[16]

The structure of the article is the following. In section 2 we will deal with the first objection regarding bilateral systems in proof-theoretic semantics. In particular, in subsection 2.1 we will display the problem and argue the need for a formal solution, in subsection 2.2 we will propose our

[14] An early exposition of this argument can be found in [19]. An anonymous referee asks whether Hjortland's bilateral sequent system in [15] escapes Kürbis' objection. Even though it is an interesting observation, I have some reservations about such a solution. Indeed, while sequent *calculi* are formalized without assumptions, their inferential interpretation considers *formulae* in the antecedent as open assumptions, and this speaks against the referee's proposal. Moreover, see note 7 for inferentialism and sequent *calculi*.

[15] These modalities are reminiscent of Wittgenstein's reading of negation (see [31], pp. 178–182 and [21] pp. 60–61), even though in Rumfitt's system they are endorsed not in place of negation, but alongside it.

[16] An anonymous referee objected that modalities can be nested, while this is forbidden for $+$ and $-$. I agree that this restriction is quite peculiar. Anyway, when I say that $+$ and $-$ should be treated as modalities, I mean that they should undergo the same scrutiny of the rest of the language and, first of all, be considered for harmony and separability. The restriction on their occurrence only as outermost terms can maybe undermine their reading as simple modalities, but not this point.

$$\text{Reductio}\ \dfrac{\begin{matrix}[+A]\\ \vdots\\ \bot\end{matrix}}{-A} \qquad \text{Non-Contradiction}\ \dfrac{+A \qquad -A}{\bot} \qquad \text{Reductio}\ \dfrac{\begin{matrix}[-A]\\ \vdots\\ \bot\end{matrix}}{+A}$$

$$\text{Smiley}\ \dfrac{\begin{matrix}[+A]\\ \vdots\\ +B\end{matrix} \qquad \begin{matrix}[+A]\\ \vdots\\ -B\end{matrix}}{-A} \qquad \text{Smiley}\ \dfrac{\begin{matrix}[-A]\\ \vdots\\ +B\end{matrix} \qquad \begin{matrix}[-A]\\ \vdots\\ -B\end{matrix}}{+A}$$

Figure 2. Coordination principles

solution, and in subsection 2.3 we will discuss some consequences regarding separability. In section 3 we will deal with the second objection. In particular, in subsection 3.1 we will display the problem and evaluate a formal solution present in the recent literature, and in subsection 3.2 we will extend the proposal developed in subsection 2.2 so to cover this objection as well. In section 4 we will develop a comparison between our proposal and the other alternatives present in the literature. In section 5 we will sum up and conclude.

2. *Tonk* in bilateral systems

2.1. Gabbay's reduction for *tonk*

Prior's connective *tonk*

$$tonk\mathrm{I}\ \dfrac{A}{A\ tonk\ B} \qquad tonk\mathrm{E}\ \dfrac{A\ tonk\ B}{B}$$

is the most famous example of pathological connective in proof-theoretic semantics.[17] It was presented as an objection to an early version of inferentialism proposed by Popper and Kneale, which adopted a completely descriptive approach toward rules: no restrictions were imposed for the rules to attach meaning to the logical constants. One of the features of normative approaches to inferentialism like proof-theoretic semantics, which on

[17] [30].

the countrary impose *criteria* for the acceptability of rules, should be that they exclude pathological connectives like *tonk*. Indeed, it can be observed that unilateral proof-theoretic semantics excludes *tonk*, on the ground that it leads to non-reducible *maximal formulae*: the only derivation of B from A pass through an application of *tonk*I followed immediately by an application of *tonk*E.

The first objection to bilateralism that we will consider focuses precisely on the behavior of *tonk* inside this framework. Michael Gabbay points out that, as opposed to standard unilateral systems, in Rumfitt's bilateral system the rules for *tonk* cannot be excluded by the usual criterion of harmony.[18] Of course, in order to comply with bilateralism, *tonk*-rules have to be modified to work with assertions and rejections, but this step does not pose any problem: we just add $+$ to both the premise and the conclusion of each *tonk*-rule. The problem emerges when we observe that the usual *maximal formula* obtained for *tonk* can now be "reduced" by inserting some applications of Coordination Principles between the conclusion of the I-rule and the assumption of the E-rule.

$$
{\scriptstyle tonk\mathrm{E}}\,\dfrac{{\scriptstyle tonk\mathrm{I}}\,\dfrac{+A}{+(A\,tonk\,B)}}{+B}
\quad\rightsquigarrow\quad
{\scriptstyle \mathrm{Smiley},2}\,\dfrac{{\scriptstyle tonk\mathrm{I}}\,\dfrac{+A}{+(A\,tonk\,B)} \qquad {\scriptstyle \mathrm{Smiley},1}\,\dfrac{{\scriptstyle tonk\mathrm{E}}\,\dfrac{[+(A\,tonk\,B)]^1}{+B} \qquad [-B]^2}{-(A\,tonk\,B)}}{+B}
$$

Even though this "reduction" manages to derive B from A without passing through *maximal formulae*, Francez argues that the second derivation does not qualify as a real reduction of the first, because it does not avoid the need to introduce and then to eliminate a *tonk*-formula, but just spreads it out in the derivation.[19] In this way, the detour is still there, although it is not in plain view.

While I agree with Francez in his evaluation that this should not count as a proper reduction, I believe that he misinterpreted Gabbay's intentions. Indeed Gabbay never claims that what he proposes is a valid reduction, but just points out that there is no formal *criterion* that detects a *maximal formula* in the derivation on the right. In other words, since according to Francez the derivation on the right does not qualify as in normal form, it should qualify as containing a *maximal formula*. Nonetheless, the standard

[18] [13].

[19] [12], section 5.

definition of such a notion is useless for this purpose, and the observation that the detour on the left is just "spread out" in the derivation on the right is just an intuitive observation, which cannot do the work of a formal *criterion*. As a consequence, in the bilateral systems we should take care also of these hidden detours (or fake normal derivations) by providing a formal *criterion* for them. When this is done, the reduction procedure can be evaluated and, if necessary, updated. I am not sure whether this is the original objection planned by Gabbay or a reformulation of it, but what is important is that there seems to be no answer to it in Francez's paper.

It can be seen that something similar happens when we have disjunction in a natural deduction system and we are forced to consider *maximal sequences* in addition to *maximal formulae* in defining *normal form*. If we do not define *maximal sequences*, then we could eventually use $\vee$-rules to "reduce" some *maximal formulae*, by moving an application of $\vee$E between the I and the E-rule in the following way:[20]

$$\vee\mathrm{E}\,\dfrac{A \vee B \qquad \oplus\mathrm{E}\,\dfrac{\oplus\mathrm{I}\,\dfrac{\begin{array}{c}[A]\\ \vdots\\ C\end{array}}{D}}{E} \qquad \oplus\mathrm{E}\,\dfrac{\oplus\mathrm{I}\,\dfrac{\begin{array}{c}[B]\\ \vdots\\ C\end{array}}{D}}{E}}{E} \quad \rightsquigarrow \quad \oplus\mathrm{E}\,\dfrac{\vee\mathrm{E}\,\dfrac{A \vee B \qquad \oplus\mathrm{I}\,\dfrac{\begin{array}{c}[A]\\ \vdots\\ C\end{array}}{D} \qquad \oplus\mathrm{I}\,\dfrac{\begin{array}{c}[B]\\ \vdots\\ C\end{array}}{D}}{D}}{E}$$

This move is available both in "normal" unilateral systems and in bilateral ones, before the definition of *maximal sequences*, and mirrors Gabbay's objection that *tonk-formulae* are reducible in bilateral systems.

Admittedly, this fake reduction procedure cannot be applied to every *maximal formula*, since it asks for a very specific position of the *maximal formula* in relation to an application of $\vee$E. As a consequence, it cannot be used to argue for the harmony of the *tonk*-rules, or in general of rules that generate irreducible *maximal formulae*. Indeed, if in general a pair of rules gives rise to non-reducible *maximal formulae*, even accepting this fake reduction, the vast majority of *maximal formulae* would remain without reduction. Nonetheless, when we have rules with particularly restrictive

[20] We will use $\oplus$ and $\ominus$ for generic logical constant.

side conditions, this "reduction" could be sufficient to argue for their harmony. As an example, let us consider what happens if we add to *tonk*I the following conditions of applicability:

- the conclusion of *tonk*I or the conclusion of the rule that is applied immediately after *tonk*I must be one of the minor premises of ∨E;
- an identical application of *tonk*I and, eventually, of the other rule occurring immediately after must conclude also the sub-derivation of the other premise of ∨E.[21]

These odd clauses entail that the only acceptable applications of *tonk*-rules that generate *maximal formulae* have the form

$$\vee\mathrm{E}\;\dfrac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ \mathit{tonk}\mathrm{E}\;\dfrac{\mathit{tonk}\mathrm{I}\;\dfrac{C}{C\ \mathit{tonk}\ D}}{D}\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ \mathit{tonk}\mathrm{E}\;\dfrac{\mathit{tonk}\mathrm{I}\;\dfrac{C}{C\ \mathit{tonk}\ D}}{D}\end{array}}{D}$$

But this is precisely the maximality that *can* be reduced if we do not consider *maximal sequences* alongside *maximal formulae*. So, the exclusion of this weakened version of *tonk* requires *maximal sequences*.

The situation here clearly resembles Gabbay's objection because, without a generalization of maximality that includes *maximal sequences*, we are forced to conclude that all *maximal formulae* generated by these weakened *tonk*-rules can be reduced according to the pattern that we already saw: by moving ∨E between *tonk*I and *tonk*E. Hence, without the definition of *maximal sequences* it seems that we need to accept this weakened reformulation of the *tonk*-rules. However, some of these "reducible" derivations prove blatantly unacceptable consequences, such as:

[21] The extra clause that the major premise of ∨E is an assumption (open or closed) may be added in order to prevent problems for the reduction of *maximal formulae* of disjunctive form.

$$\vee\text{E}\ \dfrac{A \vee A \qquad \textit{tonk}\text{E}\ \dfrac{\textit{tonk}\text{I}\ \dfrac{[A]}{A\, \textit{tonk}\, B}}{B} \qquad \textit{tonk}\text{E}\ \dfrac{\textit{tonk}\text{I}\ \dfrac{[A]}{A\, \textit{tonk}\, B}}{B}}{B}$$

Of course, the solution here is just to extend the characterization of maximality so as to cover *maximal sequences* as well. This leads to rejecting the alleged reduction, because the derivation on the right is not in normal form and it even contains a *maximal sequence* longer than the one contained in the derivation on the left.[22] In the same way, the rejection of Gabbay's proposed reduction should be grounded on an extension of *maximality*. Unfortunately, such an explicit generalization is missing in Francez's paper, so we can not see his answer as satisfactory. In the following sections, we will search for a generalization of maximality that deals with Gabbay's objection.

2.2. Complex maximality

My proposal is to apply to Gabbay's provocatory reduction a generalization of maximality developed by Milne in order to justify classical logic in traditional unilateral proof-theoretic semantics.

Dummett defined the following notions regarding the structure of an I-rule:[23]

Purity Only one logical constant figures in each rule;

Simplicity Every logical constant which occurs in a rule, occurs as principal operator;

Directness Discharged assumptions are completely general, rules do not specify some connectives that must occur in them.

The I-rules of Gentzen's system **NJ** suit all these properties, apart from $\neg$I, which is not pure since $\bot$ occurs in it. Moreover, Dummett proposes a pure, even though oblique (that is, non-direct), rule for negation, so that

[22] Of course, this is not the whole story about *maximal sequences*, which must be considered to prove normalization, regardless of these fake reductions. See [27].

[23] [6], p. 257.

at least purity and simplicity can be obtained for the complete unilateral system for intuitionistic logic.[24]

Nevertheless, while all these properties are clearly *desiderata* for an I-rule, not least for feasibility reasons, it is far from clear that they should be required as necessary conditions. Indeed, as we have just seen, in each system for intuitionistic logic at least one of them fails and even Dummett pointed out that they together constituted an "exorbitant" demand.[25] Despise this early and authoritative declaration, there are quite few attempts to generalize proof-theoretic semantics by allowing for impure and complex (non-simple) rules.[26] The first real attempt of including complex rules in proof-theoretic semantics is [22], in which the author gives a harmonious and, to some extent, separable unilateral system for classical logic.[27]

Milne proposes the following impure and complex rules for classical conditional and classical negation:[28]

$$\supset\mathrm{I}_{Mln}\ \frac{\begin{array}{c}[A]\\ \vdots\\ B\{\vee D\}\end{array}}{(A\supset B)\{\vee D\}} \qquad \neg\mathrm{I}_{Mln}\ \frac{\begin{array}{c}[A]\\ \vdots\\ D\end{array}}{\neg A\vee D}$$

Here, the meaning of curly brackets is that the formula between them may either be or not be present, the rule remaining valid anyway. In other words, Milne's rules can introduce $\supset$ both as the principal connective of a formula and inside a disjunction, depending on the premise.

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[24]The rule (displayed in [7], p. 89) is $\frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}\quad\begin{array}{c}[A]\\ \vdots\\ \neg B\end{array}}{\neg A}$. Read shows why obliquity, in this case, is not a problem by pointing out that all derivations containing applications of this rule can be modified so as to ensure that, for each of these applications, the discharged hypotheses are always less complex than the conclusion (and so the rule follows Dummett's complexity condition). See [33] for a complete analysis.

[25][6], p. 257.

[26]Having worked on this subject, I strongly suspect that the main reasons are not ideological, but rather practical: trying to prove something about or in a system with complex rules can be very frustrating!

[27]I have pushed further some of Milne's intuitions in [2], which nonetheless lacks much of the elegance of Milne's work.

[28][22], p. 514.

The adoption of these complex rules comes with the need to revise the definition of *maximal formula*. Indeed, when $\supset$ and $\neg$ are introduced inside a disjunction, they cannot be directly eliminated. There need to be an application of $\vee$E that enables such an elimination. As a consequence, in the following derivation $(A \supset B) \vee C$ counts as a *maximal formula*:

$$\dfrac{\dfrac{\begin{matrix}[A]^1\\ \vdots \\ \mathfrak{D}_1 \quad B \vee C\end{matrix}}{(A \supset B) \vee C}\,{\supset}\mathrm{I},\ 1 \qquad \begin{matrix}\dfrac{[A \supset B]^2 \qquad A}{B}\,{\supset}\mathrm{E}\\ \vdots \\ \mathfrak{D}_2 \quad D\end{matrix} \qquad \begin{matrix}[C]^2\\ \vdots \\ \mathfrak{D}_3 \quad D\end{matrix}}{D}\,{\vee}\mathrm{E},\ 2$$

The reduction of such a derivation should remove both I and E-rule for $\supset$, possibly maintaining $\vee$E. Milne proposes the following reduction step:

$$\dfrac{\begin{matrix}A\\ \vdots \\ \mathfrak{D}_1 \quad B \vee C\end{matrix} \qquad \begin{matrix}[B]^2\\ \vdots \\ \mathfrak{D}_2 \quad D\end{matrix} \qquad \begin{matrix}[C]^2\\ \vdots \\ \mathfrak{D}_3 \quad D\end{matrix}}{D}\,{\vee}\mathrm{E},\ 2$$

The need for such a revision of maximality can be shown by considering the following weakened rule for *tonk*.

$$tonk\mathrm{I}\ \dfrac{A \vee C}{(A\, tonk\, B) \vee C} \qquad tonk\mathrm{E}\ \dfrac{A\, tonk\, B}{B}$$

Since there are no curly brackets, as opposed to Milne's rule for $\supset$, the introduction of *tonk* inside a disjunction is not optional but explicitly required by the rule. In other words, the premise of *tonk*I *must* be a disjunction and *tonk* cannot be introduced as the principal connective of the conclusion, but only inside the disjunction itself.

The complex structure of *tonk*I does not allow for the construction of a traditional *maximal formula*, since it cannot be paired with an immediate application of *tonk*E. Moreover, *tonk*I should not count as an I-rule for disjunction, so that an immediate application of $\vee$E to its conclusion does not generate any maximality by itself.[29] As a consequence, in order to reject this complex reformulation of *tonk* we need an extension in

[29] For some objections to this conclusion, see [39] and [40], p. 345.

the definition of *maximal formula* which singles out the maximality in the following derivation of B from $A \vee B$.

$$\dfrac{\dfrac{A \vee B}{(A\, tonk\, B) \vee B}\, tonk\mathrm{I} \qquad \dfrac{[A\, tonk\, B]^1}{B}\, tonk\mathrm{E} \qquad [B]^1}{B}\, \vee\mathrm{E},\, 1$$

There seems to be some obvious similarities between this derivation and the alleged reduction proposed by Gabbay for the bilateral version of *tonk*. Indeed, in both cases there is an application of *tonk*I that is followed (indirectly) by an application of *tonk*E, with some applications of extra rules between them. These extra rules are in both cases rules for the more external logical constants in the conclusions of the I-rules for *tonk*: in Milne's unilateral case, they are $\vee$-rules; in Gabbay's bilateral case, they are Coordination Principles, that is rules for $+$ and $-$. Moreover, also the meaning-theoretical justification of the application of such extra rules could rely on the same basis in both cases: the dependence of meaning of classical conditional and negation (and of the complex reformulation of *tonk*, of course) upon disjunction in Milne's system, and the dependence of meaning of all the connectives (*tonk* included) upon $+$ and $-$ in Rumfitt's bilateral system. Of course, this is just an intuitive analysis of the analogy between Milne's complex maximality and Gabbay's reduction for *tonk*. To check whether complex maximality can solve Gabbay's objection, we need to take into consideration the formal developments of Milne's ideas.

Unfortunately, Milne does not propose any formal *criterion* for his generalization of maximality, even though he discusses informally the cases with negation and implication and provides an interesting semantic analysis for them. However, in a previous work, I have provided a general definition of *maximal formulae* in unilateral systems with complex rules, and I have proved that there are harmonious complex systems for both intuitionistic and classical logic.[30] In the rest of this section, we will see how this definition can be adapted to bilateral systems, and what follows from its application in this framework.

[30]See [2]. To solve an issue about circularity of meaning, in this previous paper I have worked with single-assumption (and single-conclusion) systems, but I do not want to pose the same restriction here. Also because circularity of meaning seems to be ineliminable for bilateral systems, as concluded in section 2.3.

First of all, the formal definition of dependence of meaning is the expected one:[31]

Definition 2.1 (Dependence of Meaning). For every pair of logical terms $\ominus$ and $\oplus$, the meaning of $\oplus$ depends on the meaning of $\ominus$ ($\ominus \prec \oplus$) iff there is a sequence of logical terms $\circ_1, \dots, \circ_n$ such that $\circ_1 = \ominus$, $\circ_n = \oplus$ and for every $1 \leq i < n$, $\circ_i$ occurs in the premisses or in the discharged assumptions of an I-rule for $\circ_{i+1}$.

For example, since $\perp$ occurs in $\neg$I in **NJ** the meaning of $\neg$ depends on that of $\perp$, and since $\vee$ occurs in the premise of $\supset\mathrm{I}_{Mln}$ in Milne's system the meaning of $\supset$ depends on that of $\vee$. As for Rumfitt's bilateral system, the meaning of the connectives depends on that of $+$ and $-$, since these terms occur in their I-rules. Moreover, the Coordination Principles characterize the meaning of $+$ and $-$, pointing out that each of them depends circularly on the other one.[32] Indeed, we will treat each Coordination Principle as contemporarily an I-rule for the modality in the conclusion and an E-rule for the modalities in the premises or in the discharged assumptions.[33] In particular, Reductio introduces one of the modalities in the conclusion, eliminating the other one, and for this reason it counts as an element of $\oplus\mathrm{I}^+$ or $\oplus\mathrm{I}^-$, depending on the modality of its conclusion, and an element of $\oplus\mathrm{E}^+$ or $\oplus\mathrm{E}^-$, depending on the modality of its discharged assumption. Moreover, since Non-Contradiction eliminates both modalities, it belongs to both $\oplus\mathrm{E}^+$ and $\oplus\mathrm{E}^-$. Hence, since meaning dependence is transitive by definition, each connective depends on both $+$ and $-$.

As for Smiley, it surely works as an E-rule for the modality that is not in the conclusion, and as an I-rule for the modality that is in it. However, the occurrence of the introduced modality in one of the premises of Smiley makes inaccurate to characterize it as just an I-rule for it: it rather seems both an I and an E-rule for the modality in the conclusion. For this reason, it seems less confusing to use Reductio and Non-Contradiction when dealing with separability and harmony.

[31] Notice that meaning-dependence between logical terms is defined on the base of their occurrence in the rules, not in their applications (that is, in inferences).

[32] For now, let us take for granted that this is not a problem. See end of section 2.3.

[33] This interpretation of the logical terms in the discharged assumptions as eliminated is not uncontroversial. As an example, [23] treats them as introduced connectives, and so Peirce's rule and Classical *reductio ad absurdum* as I-rules.

To be more precise, the use of Reductio and Non-Contradiction, in place of Smiley, seems needed to prove that $+$ and $-$ can be characterized with harmonious rules, that is, to prove that Coordination Principles are in harmony with each other. However, given the circular dependence between $+$ and $-$ and the fact that each Coordination Principle eliminates at least one of these modalities, the choice between Reductio and Non-Contradiction, or Smiley is irrelevant to prove harmony and separability of the Operational Rules, regardless of complex maximality. These aspects will become clearer after the display of the formal *criteria* for harmony and separability, in this and in the following section. In particular, in the proof of Theorem 2.6, we will use Smiley until we take into account *maximal formulae* that are the conclusion of Reductio.

The definition of *maximal formulae* for complex systems rests on the notions of elimination path and active logical term in an inference:

Definition 2.2 (Elimination Path (E-path)). Given a derivation $\mathfrak{D}$, a list of formulae $A_1, \ldots, A_n$ is an E-path iff for every m such that $1 \leq m \leq n$, A_m is:

1. the *major* premise of an E-rule, A_{m+1} is one of its discharged assumption, and A_m does not depend on A_{m+1} before the discharge;[34] or
2. the *major* premise of an E-rule that does not discharge assumptions, and A_{m+1} is its conclusion.[35]

Definition 2.3 (Active Logical Term in an Inference). An occurrence of a logical term in a formula A is active in an inference iff the inference is an application of a rule in which, in the formula exemplified by A, the term already has the same occurrence.

The first definition is quite simple. An E-path is a list of *major* premises of E-rules such that, the next element after such a premise is one of the discharged assumptions of the rule, if there is one, or its conclusion otherwise. Sometimes, for brevity we will speak of E-rules of an E-path, to refer

[34]The last clause about the dependence of A_m on A_{m+1} excludes E-paths that go from the major premise of an E-rule A_m to the discharged assumption that is above A_m. This clause is not needed in [2], because the systems there presented do not contain E-rules that discharge open assumptions above their major premises.

[35]We will apply this second clause only for Non-Contradiction, as we will see in the proof of Theorem 2.6.

to the E-rules that have formulae of the E-path as major premises. The second notion is a little tricky, but an example will clarify its definition. In the derivation

$$\dfrac{\supset\mathrm{I}^+\ \dfrac{[+B]^1}{+(A \wedge C) \supset B} \qquad -(A \wedge C) \supset B}{-B}\ \text{Smiley, 1}$$

$\supset$ and $+$ in $+(A \wedge C) \supset B$ are active in the instantiation of $\supset\mathrm{I}^+$, since their occurrence are already present in the schema of $\supset\mathrm{I}^+$. On the contrary, the occurrence of $\wedge$ in the same formula is not active, since conjunction does not occur in the schema of $\supset\mathrm{I}^+$.

With the previous notions at hand, we can give the following definition:

Definition 2.4 (Maximal Formulae). Given a derivation $\mathfrak{D}$, a formula A that occurs in it is a *maximal formula* iff it is the conclusion of an application of $\oplus\mathrm{I}^+$ ($\oplus\mathrm{I}^-$) and the first formula of an E-path such that:

1. the last rule of the E-path is $\oplus\mathrm{E}^+$ ($\oplus\mathrm{E}^-$), and the last formula of the E-path is identical to A;[36]
2. each rule in the E-path eliminates occurrences of logical terms that are active in the conclusion of the application of $\oplus\mathrm{I}$, or logical terms on which those depend.

If the E-path contains only one formula, we say that the *maximal formula* is *simple*. Otherwise, we say that it is *complex*. Notice that, since Coordination Principles are considered as I and/or E-rules for the modalities, a formula that is conclusion of Reductio (which counts as $\oplus\mathrm{I}^+$ or $\oplus\mathrm{I}^-$) and premise of Non-Contradiction (which counts as $\oplus\mathrm{E}^+$ or $\oplus\mathrm{E}^-$) is maximal according to this definition. Moreover, notice that Reductio can introduce only *simple maximal formulae*, not complex ones, since only one logical term occurs actively in its conclusion, and the E-path can go on only with an application of Non-Contradiction, which ends the E-path.

Regarding Operational Rules, in each application of an I-rule of Rumfitt's bilateral system, the only logical terms active in the conclusion are the connective introduced and the modality in which it is introduced (+

[36] In the unilateral systems presented in [2], the restriction on the form of the last formula of the E-path is not needed because of the specific form of its E-rules.

or $-$). Moreover, the only dependences of meaning are the circular one between $+$ and $-$, and the dependence of every connective on at least one of the modalities (and so indirectly on both of them, for transitivity). Apart from this, there is no other occurrence of connectives in the I-rules that establish dependences of meaning. As a consequence, complex *maximal formulae* can only be composed of an I-rule for a connective $\oplus$ inside a modality, followed by an E-path of Coordination Principles that ends with $\oplus$E for the same modality. Simple *maximal formulae* remain the standard ones individuated by Rumfitt in which an I-rule is immediately followed by an E-rule for the same connective and the same modality. Let us see some examples of *maximal formulae* and some of their properties.

First of all, it is quite clear that according to Definition 2.4 we have a *simple maximal formula* for *tonk* in

$$\dfrac{\dfrac{+A}{+(A\,tonk\,B)}\,tonk\mathrm{I}}{+B}\,tonk\mathrm{E}$$

and a *complex maximal formula* in

$$\dfrac{\dfrac{+A}{+(A\,tonk\,B)}\,tonk\mathrm{I} \qquad \dfrac{\dfrac{[+(A\,tonk\,B)]^1}{+B}\,tonk\mathrm{E} \qquad [-B]^2}{-(A\,tonk\,B)}\,\text{Smiley},\,1}{+B}\,\text{Smiley},\,2$$

Indeed, the conclusion of *tonk*I starts an E-path, since it is the major premise of a Smiley. The E-path continues with the occurrence of $-B$ discharged by this Smiley (see the first clause of the Definition 2.2), and then with the occurrence of $+(A\,tonk\,B)$ discharged by the Smiley that has $-B$ as premise. This occurrence of $+(A\,tonk\,B)$ then ends the E-path, being the premise of *tonk*E.

So, our formal *criterion* of maximality confirms the previous intuition about the non-normality of Gabbay's reduction. There is however something more to say about complex maximality in bilateral systems. What is peculiar in Rumfitt's system is that there are complex *maximal formulae* for every pair of I and E-rules, such as:

$$\text{Smiley, 2}\ \dfrac{\wedge\mathrm{I}^+\ \dfrac{+A \qquad +B}{+A \wedge B} \qquad \text{Smiley, 1}\ \dfrac{\wedge\mathrm{E}^+\ \dfrac{[+A \wedge B]^1}{+A} \qquad [-A]^2}{-A \wedge B}}{+A}$$

Moreover, they are all quite easily reducible to traditional simple *maximal formulae*. In this case, to:

$$\wedge\mathrm{E}^+\ \dfrac{\wedge\mathrm{I}^+\ \dfrac{+A \qquad +B}{+A \wedge B}}{+A}$$

It should not come as a surprise that equivalent *maximal formulae* can be individuated when Reductio and Non-Contradiction are used in place of Smiley, such as:

$$\text{Reductio, 2}\ \dfrac{\text{Non-Contradiction}\ \dfrac{\wedge\mathrm{I}^+\ \dfrac{+A \qquad +B}{+A \wedge B} \qquad \text{Reductio, 1}\ \dfrac{\text{Non-Contradiction}\ \dfrac{\wedge\mathrm{E}^+\ \dfrac{[+A \wedge B]^1}{+A} \qquad [-A]^2}{\bot}}{-A \wedge B}}{\bot}}{+A}$$

Here, the E-path originated with the conclusion of $\wedge\mathrm{I}^+$ continues with the $\bot$ that is concluded by Non-Contradiction (an E-rule that does not discharge assumptions, see the second clause of Definition 2.2), then with $-A$ that is discharged by Reductio, then $\bot$ and $+A \wedge B$, which ends the E-path. Moreover, also this complex *maximal formula* reduces to its simple counterpart.

This general reducibility of the complex maximality to the simple one is particularly interesting. Indeed, in Rumfitt's system the generalization of the definition of *maximal formula* to include complex cases seems ineffective – even though justified from a meaning-theoretic point of view –, and poses just minor problems for normalizability, as we will see. Nonetheless, it is the key ingredient to reject Gabbay's alleged reduction for *tonk*. On the contrary, in Milne's system *complex maximality* properly extends *simple maximality*.[37] Indeed, *maximal formulae* obtained using his complex version of I-rules for negation and implication are *not* reducible to *maximal formulae* obtained using their simple counterparts, and this is the reason why they must be addressed independently, as Milne himself does.

In Rumfitt's system, the reducibility of complex *maximal formulae* to simple *maximal formulae* holds for *tonk*-rules as well. The difference is that, while with well-behaving connectives the reduction can then go on and lead to a normal derivation, simple *maximal formulae* for *tonk* are not reducible, so the reduction has to stop there. Hence, Gabbay is wrong not only because his alleged reduction for *tonk* is not in normal form, but also because the reduction procedure goes in the opposite direction of what he claims: what he is proposing is not a reduction to a normal form, but a step backward from a non-normal derivation with a simple *maximal formula* to one with a complex *maximal formula*.

Let us now substantiate the previous intuitive analysis with a formal treatment of normalization for Rumfitt's system that deals with complex maximality. First of all, we will need a formal definition of *maximal sequence*:

Definition 2.5 (Maximal Sequences). Given a derivation $\mathfrak{D}$, a list of identical formulae that occur in it $A_1, \ldots, A_n$ is a *maximal sequence* iff A_1 is the conclusion of an application of $\oplus\mathrm{I}^+$ ($\oplus\mathrm{I}^-$) and the first formula of an E-path such that:

1. each rule in the E-path eliminates occurrences of logical terms that are active in the conclusion of the application of $\oplus\mathrm{I}$, or logical terms on which those depend;

2. if the E-path contains more than one formula, then the last formula of the E-path is the second formula of *the maximal sequence* A_2, and, for each $1 < i < n$, A_i is a minor premise of an application of $\vee\mathrm{E}^+$ or $\wedge\mathrm{E}^-$;

[37] This holds for the systems in [2] as well.

3. if the E-path contains just one formula, then it is the first formula of *the maximal sequence* A_1, and, for each $1 \leq i < n$, A_i is a minor premise of an application of $\vee E^+$ or $\wedge E^-$;

4. A_n is the major premise of $\oplus E^+$ ($\oplus E^-$).

In summary, the structure of a *maximal sequence* is the following. It starts with the conclusion of an I-rule $\oplus$I. If it is the major premise of the E-rule $\oplus$E or the minor premise of $\vee E^+$ or $\wedge E^-$, we have a simple *maximal sequence* like the usual ones considered by Prawitz. Otherwise, the conclusion of $\oplus$I is the first formula of an E-path. In this case, the last formula of the E-path is the second formula of the *maximal sequence* and can be a minor premise of $\vee E^+$ or $\wedge E^-$, or the major premise of the E-rule $\oplus$E. In the last case, we have a complex *maximal formula*, which is a specific case of complex *maximal sequence.* As usual, a derivation in which there are no *maximal sequences* is called in *normal form.*

Finally, let us prove normalization for this complex reformulation of maximality:

Theorem 2.6 (Normalization). *For every derivation $\mathfrak{D}$ of the system consisting of Rumfitt's Operational Rules in table 1 together with both the rules of Smiley in table 2, or together with both the rules of Reductio and the rule of Non-Contradiction in the same table, there is a reduction procedure that leads from $\mathfrak{D}$ to a derivation $\mathfrak{D}'$ in normal form with the same conclusion of $\mathfrak{D}$ and the same or less open assumptions.*

Proof: The structure of the proof is the following. In the first part, we will prove the result for the system constructed with the two Smiley as the only Coordination Principles. Then, we will extend the result to the system with Reductio and Non-Contradiction in place of Smiley.

First of all, let us see the reduction steps for the complex *maximal sequences.* For reasons of space, we will not show the reduction steps for all of them. In particular, we will focus on the case of positive sequences and we will assume that the Smiley that starts the E-path discharges positive formulae. Nonetheless, all the other cases are trivial variations of those here displayed.

Let us consider the complex *maximal sequence*:

$$
\begin{prooftree}
\AxiomC{$[+A]^1$}
\noLine
\UnaryInfC{$\vdots\ \mathfrak{D}_1$}
\noLine
\UnaryInfC{$+B$}
\LeftLabel{$\oplus\mathrm{I}^+$}
\UnaryInfC{$+C$}
\AxiomC{$[+A]^1$}
\AxiomC{$+C \vee F$}
\AxiomC{$[+C]^n$}
\AxiomC{$[+F]^{n+1}$}
\noLine
\UnaryInfC{$\vdots\ \mathfrak{D}_2$}
\noLine
\UnaryInfC{$+C$}
\LeftLabel{$\vee\mathrm{E}^+$, n+1}
\TrinaryInfC{$+C$}
\noLine
\UnaryInfC{$\vee\mathrm{E}^+$ or $\wedge\mathrm{E}^-$, n+2,...,m $\quad \vdots\ \mathfrak{D}_3$}
\noLine
\UnaryInfC{$+C$}
\LeftLabel{$\oplus\mathrm{E}^+$}
\UnaryInfC{$+E$}
\noLine
\UnaryInfC{Smiley $\times\ n-2$, 3,...,n $\quad \vdots\ \mathfrak{D}_4$}
\noLine
\UnaryInfC{$-A$}
\RightLabel{Smiley, 2}
\BinaryInfC{$-D$}
\noLine
\UnaryInfC{$\vdots\ \mathfrak{D}_5$}
\noLine
\UnaryInfC{$-C$}
\RightLabel{Smiley, 1}
\BinaryInfC{$-A$}
\end{prooftree}
$$

It can be reduced to a complex *maximal formula* in just one step:

$$
\begin{prooftree}
\AxiomC{$[+A]^1$}
\noLine
\UnaryInfC{$\vdots\ \mathfrak{D}_1$}
\noLine
\UnaryInfC{$+B$}
\LeftLabel{$\oplus\mathrm{I}^+$}
\UnaryInfC{$+C$}
\AxiomC{$[+A]^1$}
\AxiomC{$[+C]^n$}
\LeftLabel{$\oplus\mathrm{E}^+$}
\UnaryInfC{$+E$}
\noLine
\UnaryInfC{Smiley $\times\ n-2$, 3,...,n $\quad \vdots\ \mathfrak{D}_4$}
\noLine
\UnaryInfC{$-A$}
\LeftLabel{Smiley, 2}
\BinaryInfC{$-D$}
\noLine
\UnaryInfC{$\vdots\ \mathfrak{D}_5$}
\noLine
\UnaryInfC{$-C$}
\LeftLabel{Smiley, 1}
\BinaryInfC{$-A$}
\end{prooftree}
$$

This in turn can be reduced to a simple *maximal formula* in the following way:

$$
\text{Smiley, 1}\ \dfrac{[+A]^1 \qquad \begin{array}{c} [+A]^1 \\ \vdots\ \mathfrak{D}_1 \\ \dfrac{\dfrac{+B}{+C}\,\oplus\mathrm{I}^+}{+E}\,\oplus\mathrm{E}^+ \\ \text{Smiley} \times n-2,\ 3,\ldots,\text{n} \\ \vdots\ \mathfrak{D}_4 \\ -A \end{array}}{-A}
$$

Notice that no new *maximal sequences* can be generated. For simple ones this is obvious. For complex ones, $\mathfrak{D}_1$ is composed *above* $\mathfrak{D}_2$, so from the last point of Definition 2.2 it follows that no new E-path for complex *maximal sequences* is generated. When A is not an open assumption of the derivation $\mathfrak{D}_1$, we can drop the last application of Smiley, as done in the examples already shown for *maximal formulae* introduced by $\wedge\mathrm{I}^+$. Of course, the simple *maximal formulae* and *maximal sequences* are reducible, as already noticed by Rumfitt.[38] Hence, for *each maximal sequence* a reduction step is available. Let us now prove normalization *stricto sensu*, that is that reduction steps can be composed to reduce *all maximal sequences*, and so lead to a derivation in normal form.

Our proof of normalization is a development of Prawitz's normalization for NJ.[39] The definitions of degree and length of a *maximal sequence* are as usual. The proof is by induction on $\langle d, l, e\rangle$, where d is the highest degree of a *maximal sequence* in $\mathfrak{D}$, l is the sum of the lengths of *maximal sequences* in $\mathfrak{D}$ of degree d, and e is the sum of the lengths of the E-paths of complex *maximal sequences* in $\mathfrak{D}$ of degree d. We assume that $\langle d', l', e'\rangle < \langle d, l, e\rangle$ iff $d' < d$, or $d' = d$ and $l' < l$, or $d' = d$, $l' = l$ and $e' < e$.

We prove that, given a derivation $\mathfrak{D}$ not in normal form with induction value $v = \langle d, l, e\rangle$, we can find a derivation $\mathfrak{D}'$ with an induction value less than v of the same conclusion, from the same or fewer assumptions. Let us choose a *maximal sequence* α of degree d with the following properties:

[38] In general, the availability of reduction steps for simple *maximal sequences* in Rumfitt's bilateral system is a well-established starting point of the discussion about bilateralism. The problem is how to extend this result to obtain at least coherence, and how to treat Coordination Principles.

[39] [27], pp. 50–51.

1. there are no *maximal sequences* of degree d above α; and

2. if α is a simple *maximal sequence*, then there are no simple *maximal sequences* of degree d that are above or contain the other premises of the inference of which the last formula of α is premise.[40]

If the *maximal sequence* is simple, we apply the standard reduction steps for Rumfitt's bilateral system. They clearly reduce d or l, if it is a simple *maximal formula*, and l, if it is a simple *maximal sequence* longer than one. In both cases, we obtain a derivation $\mathfrak{D}'$ with a value of $\langle d, l, e\rangle$ that is less than v. The reduction steps do not generate any new *maximal sequence* of degree d. This is well known for simple *maximal sequences* and easy to see for complex *maximal sequences* as well. As an example, let us focus on the case of *permutative conversions* that reduce the length of simple *maximal sequences*. If $+B \vee C$ (or $-B \wedge C$) is major premise of $\vee\mathrm{E}^+$ (respectively $\wedge\mathrm{E}^-$), it does not begin an E-path, sice $\vee$ (respectively $\wedge$) is active only in applications of $\vee\mathrm{I}^+$ (respectively $\wedge\mathrm{I}^-$). Hence, even though there is a change in the derivation of the minor premises of $\vee\mathrm{E}^+$ ($\wedge\mathrm{E}^-$), it cannot generate any new complex *maximal sequence*.

If the *maximal sequence* is complex, we apply the reduction steps displayed at the beginning of this proof. If the *maximal sequence* is longer than two, then the reduction step reduces the value of l. Otherwise, the reduction step reduces the value of e. In both cases, the value of $\langle d, l, e\rangle$ is decreased. As we have already argued in the first part of the proof, no new *maximal sequence* is generated by these reduction steps. Hence, the result follows by induction.

So far, we have considered a system with Smiley as Coordination Principle. Let us show that the use of Reductio and Non-Contradiction in its place does not constitute any problem. First of all, Smiley is clearly derivable from the other two rules: just derive $\perp$ from $+B$ and $-B$ using Non-Contradiction, and then discharge the open assumption $+A$ (or $-A$) to derive $-A$ (or $+A$) using Reductio. Moreover, as we have seen, the cyclic dependence of meaning between $+$ and $-$ entails that we can use

[40] This second clause is required by Prawitz to make *permutative conversions* effective in reducing the inductive value of the derivation. Since the reduction procedure for complex *maximal sequences* is very different and does not use *permutative conversions*, this clause can be dropped for them.

both Reductio and Non-Contradiction in the E-path of a *maximal sequence*. Hence, the occurrences of Smiley in the reductions can be substituted with occurrences of Reductio and Non-Contradiction without affecting the normalization process. This is the reason why we claimed that for the harmony of the Operational Rules (and for separability as well), the adoption of Reductio and Non-Contradiction, or Smiley makes no difference.

However, while using Smiley it is not clear how we should address simple maximality regarding Coordination Principles, that is regarding $+$ and $-$, with Reductio and Non-Contradiction we clearly have *maximal formulae*, when the conclusion of an application of Reductio is premise of an application of Non-Contradiction. Fortunately, in these cases we can easily find a reduction, such as:

$$\text{Non-Contradiction}\ \dfrac{+A \qquad \text{Reductio, 1}\ \dfrac{\begin{matrix}[+A]^1\\ \vdots\\ \bot\end{matrix}}{-A}}{\bot} \qquad \rightsquigarrow \qquad \begin{matrix}+A\\ \vdots\\ \bot\end{matrix}$$

Moreover, given a derivation $\mathfrak{D}$ and chosen a *maximal sequence* according to the instruction seen previously, this reduction step clearly reduces the value of d or l.

The conclusion of an application of Reductio can also be the first formula of a simple *maximal sequence* that ends with an application of Non-Contradiction. In this case, a *permutative conversion* can be provided. As an example, the derivation

$$\text{Non-Contradiction}\ \dfrac{\vee\text{E}^+,2\ \dfrac{+B \vee C \qquad \text{Reductio, 1}\ \dfrac{\begin{matrix}[+A]^1 \quad [+B]^2\\ \vdots\ \mathfrak{D}_1\\ \bot\end{matrix}}{-A} \qquad \begin{matrix}[+C]^2\\ \vdots\ \mathfrak{D}_2\\ -A\end{matrix}}{-A} \qquad +A}{\bot}$$

can be reduced to the derivation

$$\dfrac{+B \vee C \qquad \text{Non-Con}\ \dfrac{\text{Reductio, 1}\ \dfrac{\begin{array}{c}[+A]^1 \quad [+B]^2\\ \vdots\\ \mathfrak{D}_1\\ \bot\end{array}}{-A} \qquad +A}{\bot} \qquad \text{Non-Con}\ \dfrac{\begin{array}{c}[+C]^2\\ \vdots\\ \mathfrak{D}_2\\ -A\end{array} \qquad +A}{\bot}}{\bot}\ \vee\text{E}^+, 2$$

Of course, a similar reduction can be provided also when the *maximal sequence* applies $\wedge\text{E}^-$ instead of $\vee\text{E}^+$. In this case as well, given a derivation $\mathfrak{D}$ and chosen a *maximal sequence* according to the instructions seen previously, the reduction step reduces the value of l.[41]

As for complex *maximal sequences*, let me remind the reader that the conclusion of Reductio cannot be the first formula of an E-path, since there is only one active logical term in its conclusion. Hence, complex maximality is not an issue in this case. In conclusion, normalization holds also for the system composed with Reductio and Non-Contradiction in place of Smiley. □

Given Theorem 2.6, it follows that the generalization of the notion of *maximal formula* does not exclude the well-behaved system proposed by Rumfitt. In other words, the complex *maximal formulae* and *maximal sequences* constructed using Rumfitt's Operational Rules and Coordination Principles are all reducible. Moreover, in [2] I have shown that also in unilateral systems this generalization does not exclude well-behaved systems, but on the contrary extends the class of acceptable systems. As we have claimed in the introduction, normalization is usually the key ingredient of any proof-theoretic *criterion* of harmony that decides the acceptability for a set of rules, at least in the flavour of proof-theoretic semantics that is based on Dummett's and Prawitz's works – and to which Rumfitt's bilateralism belongs. Hence, since the adoption of complex maximality results in a proof of normalization for a classical bilateral system, it is compatible with the standard approach to validity endorsed in proof-theoretic semantics, and at most extends the class of its valid systems. The only systems ruled

[41] Notice that the second clause imposed for the selection of the *maximal sequence* to be reduced is relevant here, since if there is a *maximal sequence* of degree d in the derivation of the right minor premise of $\vee\text{E}^+$ the value of l remains unchanged at the end of the reduction.

out by this extension of maximality are pathological systems that use complex rules to introduce paradoxical connectives avoiding simple *maximal formulae* (such as the introduction of *tonk* in the scope of the disjunction), or to fake a reduction process (such as in Gabbay's case).

2.3. Weak separability

Complex rules, like all impure rules, impose a dependence of meaning between logical constants. As an example, it is part of the received wisdom that the meaning of intuitionistic negation depends upon that of $\perp$. This is the semantic counterpart of the occurrence of this constant in the I-rule for negation, that is of the impurity of $\neg$I. In the same way, the occurrence of disjunction in Milne's rule for the introduction of classical negation and conditional entails that the meaning of both $\supset$ and $\neg$ depends upon that of $\vee$.

For separability, the fact that the meaning of $\neg$ depends upon that of $\perp$ means that in order to prove that C follows logically from Γ the $\perp$-rules (that is *ex falso quodlibet*) could be needed together with those for $\neg$, even if $\neg$ occurs in $\Gamma \cup \{C\}$ but $\perp$ does not.[42] A clear example of this phenomenon is that any derivation of $A \wedge \neg A \vdash C$, with C fully general, requires *ex falso*. Traditionally, instead of considering the meaning of intuitionistic negation as depending on that of absurdity, $\neg A$ has been sometimes considered just a shortening for $A \supset \perp$. With complex rules, this solution is not viable and meaning-dependence becomes an incontestable phenomenon of the system.[43] This leads to revising the traditional definition of separability in the following way:

Definition 2.7 (Weak Separability). To prove a logical consequence $\Gamma \vdash C$ we only need to use the rules for the logical constants that occur in Γ or C, together with the rules for the constants on which those depend. That is, in order to prove a logical consequence $\Gamma \vdash C$, it is enough to use the rules for the constants $\circ_1, \dots, \circ_n$ such that for every $1 \leq i \leq n$:

- $\circ_i$ occurs in Γ or C; or
- for some $j \neq i$ such that $1 \leq j \leq n$, $\circ_j$ occurs in Γ or C and $\circ_i \prec \circ_j$.

[42] We will use $\Gamma \vdash C$ to indicate that C is a logical consequence of Γ.

[43] Cozzo investigated something similar, even though for non-logical terms: see [3], pp. 246–250, [4], pp. 32–34 and [5], p. 305. Also Prawitz developed a similar idea for logical terms, apparently independently of Milne: see [29].

To see a clear example of why such a weakening of separability is needed, let us just consider the following proof of the purely implicational classical theorem usually called *Peirce's law*:[44]

$$\dfrac{\dfrac{\dfrac{[A]^1}{A \vee B}\vee\mathrm{I}}{A \vee (A \supset B)}\supset\mathrm{I},1 \qquad \dfrac{[(A \supset B) \supset A]^4 \qquad [A \supset B]^2}{A}\supset\mathrm{E} \qquad [A]^3}{\dfrac{A}{((A \supset B) \supset A) \supset A}\supset\mathrm{I},4}\vee\mathrm{E},2{,}3$$

Since disjunction does not occur in the conclusion, Peirce's law is not intuitionistically valid and the only difference between Milne's classical $\supset$I and intuitionistic $\supset$I is the possibility of introducing $\supset$ inside a disjunction, the usage of $\vee$-rules is obviously needed in the previous derivation. Hence, Milne's system does not suit separability, even though it can be shown to suit *weak* separability.

Rumfitt claims that his system is separable, but he considers only Operational Rules to show this result. It is far from obvious that separability holds when we consider $+$ and $-$ as well, asking for example that in order to prove purely assertive consequences (that is, consequences that have only $+$-*formulae* both between the assumptions and as the conclusion) only rules for assertion are needed. On the contrary, any derivation of the purely assertive consequence $+\neg\neg A \vdash +A$ seems to require an application of rules for the rejection of $\neg$-*formulae*, such as:

$$\neg\mathrm{E}^{-}\,\dfrac{\neg\mathrm{E}^{+}\,\dfrac{+(\neg\neg A)}{-(\neg A)}}{+A}$$

The generalization of maximality seen in the previous paragraph gives ground for considering $+$ and $-$ too for separability, since they contribute to the meaning of the connectives and Coordination Principles can be used to construct complex maximality. Moreover, Kürbis' observations about the problems of interpreting these signs as standing for speech acts suggests that they should be treated more like modalities and so, arguably, considered for separability as well.[45] Other circumstantial pieces of evidence that

[44] [22], p. 527.

[45] We have seen briefly Kürbis' observations at the end of section 1.

$+$ and $-$ should be considered for separability can be provided: changing Coordination Principles entails a change in the logic,[46] some *criteria* are needed to balance $+$ and $-$ rules for the same connective.[47] Nonetheless, considering these signs for separability also means adopting a weakened version of this requirement. Indeed, assertion and rejection are used to give meaning to all the connectives, and if the Coordination Principles define the meaning of $+$ and $-$, it seems obvious that there can only be a cyclic dependence of meaning of each of them upon the other. Hence, weak separability for Rumfitt's bilateral system asks that all logical consequences are provable using only rules for the connectives that occur in the premises or in the conclusion of the consequence, signed with the signs that occur in the consequence, together with Coordination Principles.[48] As an example, if both premises and conclusion of a consequence are signed $+$ and $\oplus$ occurs in its premises or in its conclusion, we can use $\oplus^+$-rules and Coordination Principles, while we should not use $\oplus^-$-rules. With this clarification, we obtain:

Theorem 2.8 (Weak Separability). *Weak separability holds for the system consisting of Rumfitt's Operational Rules in table 1 together with both the rules of Smiley in table 2, or together with both the rules of Reductio and the rule of Non-Contradiction in the same table.*

Proof: The part about the connectives is given by Rumfitt: in order to prove $\Gamma \vdash C$ we need to use only rules for connectives that occur in $\Gamma \cup \{C\}$.[49] About $+$ and $-$, showing that for every connective $\oplus$, $\oplus^+$-rules ($\oplus^-$-rules) are derivable from $\oplus^-$-rules ($\oplus^+$-rules) together with Coordination Principles is sufficient to establish the result. Indeed, from this derivability it follows that any application of a $\oplus^-$-rule can be substituted with an application of the $\oplus^+$-rules and of Coordination Principles. Hence, it cannot be necessary to use $\oplus^-$-rules to prove that C follows from Γ if $-$ does not occur neither in the premises nor in the conclusion. The same

[46] [18]

[47] We will deal with this issue in section 3.1.

[48] In his [20], the author asks for the normalization of Reductio followed by E-rules or Non-Contradiction. In concrete, this means considering Coordination Principles (and sometimes even Operational Rules, as we will see) as meaning conferring rules for $+$ and $-$.

[49] See [36].

argument proves that $\oplus^+$-rules are not needed to derive consequences that regard only rejections.

We will not give a complete proof of the derivability of all the rules of assertion (rejection) for a connective from the rules of rejection (assertion) for the same connective together with Coordination Principles, since they are quite easy. The example of implication will be sufficient to illustrate the procedure:

- The rule $\supset^+$I is derived by

$$\text{Smiley, 1}\ \dfrac{\begin{array}{c}\supset\mathrm{E}^-\ \dfrac{[-A\supset B]^1}{+A}\\ \vdots\\ +B\end{array} \qquad \supset\mathrm{E}^-\ \dfrac{[-A\supset B]^1}{-B}}{+A\supset B}$$

- The rule $\supset^+$E is derived by

$$\text{Smiley, 1}\ \dfrac{+A\supset B \qquad \supset\mathrm{I}^-\ \dfrac{+A \qquad [-B]^1}{-A\supset B}}{+B}$$

- The rule $\supset^-$I is derived by

$$\text{Smiley, 1}\ \dfrac{\supset\mathrm{E}^+\ \dfrac{+A \qquad [+A\supset B]^1}{+B} \qquad -B}{-A\supset B}$$

- The rule $\supset^{-}\mathrm{E}_1$ is derived by

$$\dfrac{-A \supset B \qquad \dfrac{\dfrac{[-A]^2 \qquad [+A]^1}{+B}\,\text{Smiley}}{+A \supset B}\,\supset\mathrm{I}^{+}, 1}{+A}\,\text{Smiley}, 2$$

- The rule $\supset^{-}\mathrm{E}_2$ is derived by

$$\dfrac{-A \supset B \qquad \dfrac{[+B]^1}{+A \supset B}\,\supset\mathrm{I}^{+}}{-B}\,\text{Smiley}, 1$$

$\square$

Whether cyclic dependencies of meaning like the one between $+$ and $-$ are acceptable in logic is at least a controversial issue.[50] Here I will not discuss this issue, which would need a further article in itself, but I will be satisfied with having pointed at what seems to me the deepest problem of the bilateral systems.

That bilateral systems have problems with *complexity criterion* and non-circularity of meaning-dependence should not come as a surprise. Indeed, Milne already considered the possibility of reading

$$\dfrac{\begin{matrix}[\neg A]\\ \vdots \\ \bot\end{matrix}}{A}\,\text{Classical Reductio}$$

as an I-rule for the atomic formula in the conclusion, and of course the main obstacle for such a reading is the *complexity criterion*.[51] The same *criterion* is still violated if we substitute $-$ to $\neg$ and add $+$ to the conclusion, so obtaining the bilateral rule of Reductio. Moreover, it can be observed that in the bilateral framework the *complexity condition* is violated less heavily, since the conclusion of Reductio is not less complex but has the same complexity of its discharged assumption, but only at the cost of a violation of the circularity of meaning-dependence between $+$ and $-$.

[50] Dummett clearly rejects such a possibility in [6], p. 257. Nonetheless, the same criticism that has been raised against his complexity *criterion* could maybe be used against this non-cyclic requirement.

[51] [21], p. 59.

3. Bullet and the balance between assertion and rejection

3.1. Gabbay's reappraisal of Read's •

In section 2 we have seen one of the objections raised by Gabbay against bilateral systems and we have evaluated a possible solution that rests on a generalization of *maximality*. Here we will consider another objection raised in the same paper and evaluate whether a common solution to both these problems can be found.

While the first objection was a reinterpretation of *tonk* inside the bilateral framework, the second one is a reinterpretation in the same framework of Read's •, an inferentialist version of liar's paradox.[52] Gabbay presents the following set of rules for this zero-ary connective:

$$\bullet\mathrm{I}^{+}\ \frac{+A \qquad -A}{+\bullet} \qquad\qquad \bullet\mathrm{I}^{-}\ \frac{+A \qquad -A}{-\bullet}$$

$$\bullet\mathrm{E}^{+}\ \frac{+\bullet}{+A} \qquad \bullet\mathrm{E}^{+}\ \frac{+\bullet}{-A} \qquad \bullet\mathrm{E}^{-}\ \frac{-\bullet}{+A} \qquad \bullet\mathrm{E}^{-}\ \frac{-\bullet}{-A}$$

It can be shown that they are harmonious in bilateral systems, since any *maximal formula* obtained by pairing an I-rule (for assertion or for rejection) with the corresponding E-rule can be reduced. They are nonetheless unacceptable, since they lead any system that is equipped with the standard Coordination Principles to trivialism:

$$\text{Smiley, 1}\ \frac{\bullet\mathrm{E}^{+}\ \dfrac{[+\bullet]^1}{+B} \qquad \bullet\mathrm{E}^{+}\ \dfrac{[+\bullet]^1}{-B}}{\bullet\mathrm{E}^{-}\ \dfrac{-\bullet}{+A}} \qquad\qquad \text{Smiley, 1}\ \frac{\bullet\mathrm{E}^{+}\ \dfrac{[+\bullet]^1}{+B} \qquad \bullet\mathrm{E}^{+}\ \dfrac{[+\bullet]^1}{-B}}{\bullet\mathrm{E}^{-}\ \dfrac{-\bullet}{-A}}$$

Francez proposes a diagnosis of what is wrong with these rules and a formal *criterion* to exclude them.[53] He claims that the problem is not a disharmony between I and E-rules, but a lack of balance between rules for assertion and rules for rejection. According to Francez, bilateralism should

[52] [32], p. 141. Prawitz's inferentialist interpretation of Russell's paradox displays some similarities with Read's rules; see [27], p. 95.

[53] His answer to Gabbay is in [12], but the principle he employs in his reply was already formulated in his [11].

endorse a *principle of coherence* that prohibits the assertion and the rejection of the same formula. Technically, in order to assure this principle, he asks that the rules of rejection be a function of the rules of assertion, labeling *Horizontal Balance* this condition. The formal details of this functionality are very complex, and there is no need to go into the details here. What is important is that, technically speaking, his principle works fine for this objection (even though not for the one regarding *tonk*, still raised by Gabbay).[54]

There are, however, some more conceptual perplexities that could be raised. Francez's requirement of coherence is explicitly inspired by a similar requirement imposed by Restall, who nonetheless works in a completely different framework. Restall proposes a meaning-theoretical and inferentialist reading of sequent calculus, and his principle of coherence is just a reading of the uncontroversial axiom $A \Rightarrow A$, which works as starting point of every sequent calculus. On the contrary, in Rumfitt's bilateralism, to endorse coherence means both to exclude the rules of Incoherence

$$\text{Incoherence}\ \frac{+A}{-A} \qquad \text{Incoherence}\ \frac{-A}{+A}$$

from the set of Coordination Principles, and to ask that they are not derivable rules of the system. So in this case there is a positive restriction to be imposed, which moreover raises both conceptual and formal issues.

The first formal problem is that Horizontal Balance applies only to Operational Rules and leads to Coherence *only if* Coordination Principles behave well. Hence, since Francez poses no restriction on Coordination Principles, we can only be sure that the rules of Incoherence are not derivable using Operational Rules, as would be the case with Gabbay's $\bullet$-rules, but we cannot prevent their adoption as Coordination Principles or exclude that they are derivable because of the Coordination Principles. The reason why Francez seems not to consider this as an issue is that he works with a predetermined set of Coordination Principles, but this solution seems *ad hoc*: it would be clearly preferable to have a *criterion* for Coordination Principles as we have for Operational Rules.

The second formal problem is that rejecting the rules of Incoherence by decree seems to be unjustified, since these rules have nothing wrong *per se*. Indeed they lead to trivialism only together with standard Coordination Principles, while by themselves they "just" establish the interderivability

[54] The formal details are developed in [10] and in section 4.4.1.7 of [11].

of $+A$ and $-A$, leading to maybe unpalatable consequences but neither to trivialism nor to proof of $\bot$. Arguably, this outcome prevents endorsing Incoherence if we want to preserve the reading of $+$ and $-$ as assertion and rejection, but the acceptability of a rule in itself, due to inferentialist reasons, should not be mixed with its adequacy to its standard interpretation. As an example, the issue of inferentialist acceptability of a set of rules for classical conditional should not be confused with the issue of whether the conditional we use in everyday arguments has classical properties or not.

In summary, from a formal point of view both issues suggest that, without a criterion that deals with Coordination Principles as well as Operational Rules, the problems raised by $\bullet$ can only be moved, not solved. Moreover, also from a purely conceptual point of view the adoption of a principle of coherence is far from obvious from the perspective of proof-theoretic semantics. Indeed, Restall's inferentialism, which inspires Francez's solution, explicitly departs from the standard Dummettian antirealist theory of meaning, which is at the core of proof-theoretic semantics, and relies on a theory of meaning based on Brandom's works.[55]

3.2. The search for a solution

Having dismissed Francez's analysis of Gabbay's rules for $\bullet$, we are in need of a solution that explains what is wrong with them. It seems to me that, following the interpretation of bilateralism outlined in section 2 (and especially in subsection 2.2), we can retort a criticism pointed out by Gabbay himself against Read's $\bullet$ to Gabbay's bilateral reformulation of this zero-ary connective.

Read proposed the following rules for $\bullet$ in a standard unilateral framework

$$\bullet\mathrm{I}\ \frac{\neg\bullet}{\bullet} \qquad\qquad \bullet\mathrm{E}\ \frac{\bullet \qquad \begin{matrix}[\neg\bullet]\\ \vdots \\ C\end{matrix}}{C}$$

arguing that they are in harmony with each other, since *each maximal formula* obtained from an application of $\bullet$I immediately followed by $\bullet$E

[55] See [34] and [35] *inter alia*. For a comparison between these two approaches to inferentialism, see [42] and [41].

can be removed. Nonetheless, together with standard rules for negation they supply a closed proof of $\bot$:

$$\neg\mathrm{E}\ \dfrac{\neg\mathrm{I},2\ \dfrac{\neg\mathrm{E}\ \dfrac{[\bullet]^2 \qquad \bullet\mathrm{E},1\ \dfrac{[\bullet]^2 \qquad [\neg\bullet]^1}{\neg\bullet}}{\bot}}{\neg\bullet} \qquad \bullet\mathrm{I}\ \dfrac{\neg\mathrm{I},2\ \dfrac{\neg\mathrm{E}\ \dfrac{[\bullet]^2 \qquad \bullet\mathrm{E},1\ \dfrac{[\bullet]^2 \qquad [\neg\bullet]^1}{\neg\bullet}}{\bot}}{\neg\bullet}}{\bullet}}{\bot}$$

A first standard objection to the acceptability of these rules is that they are not really harmonious, but just suit Inversion Principle. Indeed, even though *each maximal formula* can be removed, not for every derivation this procedure leads to a derivation without *maximal formulae*, that is to a normalization of the derivation. As an example, the closed derivation of $\bot$ just seen cannot be given in normal form.[56] In the original formulation of proof-theoretic semantics, this objection leads to rejecting the set of rules that causes the non-normalizability. This is indeed the choice made by Prawitz when the same issue arises regarding his inferentialist version of Russell's paradox.[57] Nonetheless, Read rejects this objection, accepting Inversion Principle as the only criterion for harmony. A discussion of Read's position regarding harmony and Inversion Principle exceeds the scope of this article. Nonetheless, it should be noticed that the lack of normalizability cannot be used as an objection against Gabbay's bilateral version of $\bullet$. Indeed, first of all, only the E-rules of bilateral $\bullet$ are used in the proof of $\bot$ that we have just displayed. Moreover, the reduction step for a *maximal formula* obtained via introduction and elimination rules for $\bullet$ completely erases the occurrence of $\bullet$ itself, as opposed to the reduction step for the unilateral version of these rules proposed by Read. So, even taking for granted Prawitz's requirement of normalizability, it can at most exclude Read's $\bullet$, but not Gabbay's.

There is anyway another objection to Read's $\bullet$ that, taken together with the interpretation of $+$ and $-$ presented in section 2, could be extended to cover Gabbay's bilateral version as well. Ironically enough, this objection is raised by Gabbay himself in the same paper in which he proposes his

[56] Tranchini shows that the reduction procedure goes in circle: [43], p. 413.

[57] [27], p. 95.

bilateral $\bullet$. Gabbay observes that, even though Read considers it only as an introduction rule, $\bullet$I is also an elimination rule for $\neg$.[58] As a consequence, $\bullet$I should suit Inversion Principle also *qua* E-rule for negation. But this is clearly not the case, since there is no reduction procedure for the *maximal formula* $\neg\bullet$ in the derivation:[59]

$$\dfrac{\dfrac{\begin{array}{c}[\bullet] \\ \vdots \\ \bot\end{array}}{\neg\bullet}\,\neg\mathrm{I}}{\bullet}\,\bullet\mathrm{I}$$

Of course, since Inversion Principle is at least a necessary requirement for harmony, lacking it $\bullet$-rules cannot be harmonious.

In the previous section, we have seen that to solve Gabbay's puzzle regarding the reduction of *tonk*-formulae we have to consider bilateral Operational Rules as if they were complex rules introducing or eliminating their connectives in the scope of $+$ and $-$, and to extend the notion of *maximal formulae* accordingly. We have also seen that according to this interpretation the meaning of the logical constants depends on those of $+$ and $-$, and that this dependence of meaning requires both a weakening of separability and considering Coordination Principles as meaning-determining rules for $+$ and $-$.

Given this reinterpretation of what goes on in bilateralism, Gabbay's objection against Read's $\bullet$-rules can be retorted against his own $\bullet$-rules. Indeed, following his argument, $\bullet\mathrm{I}^+$ and $\bullet\mathrm{E}^-$ count also as E-rules for $-$, $\bullet\mathrm{I}^-$ and $\bullet\mathrm{E}^+$ also as E-rules for $+$, $\bullet\mathrm{E}^+$ also as an I-rule for $-$, and $\bullet\mathrm{E}^-$ also as an I-rule for $+$. This entails that Gabbay's rules are not really harmonious if considered together with standard Coordination Principles. Indeed, for example, the derivation

$$\dfrac{\dfrac{+\bullet}{-A}\,\bullet\mathrm{E}^+ \qquad +A}{\bot}\,-\mathrm{E}$$

cannot be reduced, as $\bullet\mathrm{E}^+$ intended *qua* I-rule for $-$ would on the contrary require.

[58] [13], p. S113.

[59] Peter Milne claimed that $\bullet$-rules do not suit Inversion Principle already in his [23], but gives only an indirect argument for such a conclusion.

This line of reasoning makes the unacceptability of $\bullet$ rely on the availability of the standard Coordination Principles. Indeed, their disharmony holds only in relation to this or other sets of Coordination Principles, and not in itself. Far from being a problem, it seems to me that, since Coordination Principles are needed to obtain $\perp$ from the rules for $\bullet$, it is plausible that they should be excluded by our *criterion* only when these Principles are present. Moreover, this feature could work as a solution to the problems that we saw at the end of subsection 3.1 regarding the rules of Incoherence and Francez's proposal of a Horizontal Balance between rules for assertion and rules for rejection. Our main reservations against Francez's *criterion* and the exclusion of the rules of Incoherence were that:

1. the rules of Incoherence lead to triviality only if endorsed together with some sets of Coordination Principles;

2. Horizontal Balance cannot be applied to Coordination Principles;

3. when applied to Operational Rules, it does not take into consideration the Coordination Principles, which are nonetheless needed to obtain triviality of the system.

Let us now check how our proposal could deal with them, and whether the involvement of Coordination Principles in our solution could be of some use.

About the first point, such as bilateral rules for $\bullet$ are evaluated in combination with Coordination Principles, the same can be done for the rules of Incoherence. Also in this case, an example of irreducible maximality can be displayed. Indeed, in the derivation

$$\dfrac{\dfrac{+A}{-A}\,\text{Incoherence} \qquad +A}{\perp}\,\text{Non-Contradiction}$$

Incoherence works as an I-rule for $-$ and Non-Contradiction as an E-rule for the same term, and so $-A$ is a *maximal formula* for which no reduction is available. As a consequence, Incoherence is excluded by our *criterion*, but not for itself; only on the background assumption of the standard Coordination Principles.[60] Since Incoherence is a Coordination Principle and it is not excluded by *ad hoc* decisions but because of a general *criterion*, this observation answers both point 1 and point 2. As for the third point, it clearly does not hold for our *criterion*, since the evaluation of the Operational Rules for $\bullet$ leads to a rejection explicitly because of the Coordination Principles, so our proposal seems to solve all the open issues seen in the previous section.

Nevertheless, someone could object that we are posing a too strong restriction on the Operational Rules by asking that they cohere also with Coordination Principles when considered as I or E-rules for $+$ and $-$. Indeed, if $\neg\mathrm{I}^+$ is also an I-rule for $+$ and $\supset\mathrm{E}^-$ is also an E-rule for $+$, they form together the following maximality, which is irreducible:

$$\dfrac{\dfrac{-A}{+(\neg A)}\,{\scriptstyle +\mathrm{I}} \qquad -B}{-(\neg A \supset B)}\,{\scriptstyle +\mathrm{E}}$$

So, we are in danger of proving that Rumfitt's system is not in harmony and throwing away the baby with the dirty water.

Nonetheless, we should not be too hasty in abandoning our alleged solution. Indeed, it should be taken into account that Gabbay's rules for $\bullet$ extend the logical consequences regarding only $+$ and $-$, since they make provable $+A \dashv\vdash -A$, while Rumfitt's rules for negation and implication do not. Of course they extend the provable results regarding $\neg$, $\supset$, $+$ and $-$, but not the consequences regarding only these last two terms. The rules for negation and those for implication constitute, in more formal terms, a conservative extension of the system composed of only the Coordination Principles. As a consequence, there is ground to claim that while Gabbay's rules for $\bullet$ work also as I and E-rules for $+$ and $-$, this is not true for the rules of negation and implication. The intuitive principle that we are

[60] An anonymous referee asks whether there are some *formal* reasons to drop Incoherence instead of Non-Contradiction. It seems to me that the only reasons for this choice regard the intended meaning of $+$ and $-$, and that there are no formal reasons to dismiss Incoherence in a context in which the other Coordination Principles are absent. The situation here resembles the one evaluated by Dummett about harmonious rules that are nonetheless unacceptable, if proposed to capture the meaning of counterfactual conditional. In other words, harmony can be enough for the formal acceptability of a rule, but not for its adequacy with respect to actual usage. See [6], p. 206. [33] as well proposes harmony as just a precondition for validity.

applying here is the same that we implicitly relied on when we, *contra* Steinberger, rejected to Milne's I-rules for classical negation and conditional the *status* of I-rules for disjunction.[61] In conclusion, even though some technicalities are needed, our *criterion* can exclude bilateral $\bullet$, bilateral *tonk* and the rule of Incoherence, without at the same time excluding the well-behaving part of Rumfitt's system.[62]

4. Comparison with other proposals

We have already seen that our proposal of treating bilateral rules as complex has some advantages over Francez's proposal of a principle of Horizontal Balance. Indeed, using this approach we have provided both a solution to Gabbay's puzzle of the 'reduction' for *tonk*-rules, and a *criterion* for Coordination Principles. Nonetheless, it must be admitted that Francez's criterion is much more elegant than mine, since the adoption of complex rules entails a great complication both in constructing the proofs inside the system, and in proving metatheorems about the system itself.

There is nonetheless another, more recent proposal that we have considered only in passing, and that we should compare with our analysis. In a recent paper, Kürbis has proposed a normalization procedure for a variation of Rumfitt's system.[63] What is peculiar about his work is that he has taken into consideration Coordination Principles as well, asking for the reduction of *maximal formulae* obtained: with only Operational Rules, with only Coordination Principles, and with both Operational Rules and Coordination Principles.

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[61] See section 2.2. The same principle is applied also in my work about complex rules in unilateral systems; see [2], p. 1043.

[62] A referee wonders whether $\neg\mathrm{I}^+$ could be considered an I-rule for $\neg$ and $+$ together, instead of an I-rule for both of them taken separately. I thank them for this suggestion. I share their feeling about this interpretation of $\neg\mathrm{I}^+$. Anyway, there are some difficulties, and this reinterpretation cannot be seen as a general solution to the apparent maximality between Operation Rules and Coordination Principles. Indeed, in order to keep our rejection of Gabbay's $\bullet$, we need to interpret $\bullet\mathrm{E}^+$ as an I-rule for $-$. Moreover, even rejecting to $\neg\mathrm{I}^+$ its status of I-rule for $+$, on the basis that it introduces positive formulae only if their most external connective is $\neg$, it is hard to do the same for $\supset\mathrm{E}^-$. Indeed, just like for $\bullet\mathrm{E}^+$, the only logical term occurring in the conclusion of $\supset\mathrm{E}^-$ is $+$. So, in order to reject Gabbay's $\bullet$ but keep the ordinary rules for the logical connectives, a *criterion* relying on *conservative extension* seems needed anyway.

[63] [20]; a strengthening of Kürbis' result has been proposed by Pedro del Valle-Inclan in his contribution to this conference.

While there are some similarities between Kürbis' approach and mine, they should not be confused with each other. Confronting the requirements of these two *criteria* in general would take too long, so we will instead analyze how they behave in some relevant situations, focusing on a variation of Gabbay's rules for $\bullet$ presented by Kürbis himself: two binary connectives that he calls *conk* and *honk* and that, like $\bullet$-rules, are harmonious but leads to triviality.[64] Of course, Kürbis' revision of *harmony* is expressly designed to exclude these connectives, so the issue is whether our *criterion* can do the same.

Let us start by considering the rules for *honk*:

$$\mathit{honk}\mathrm{I}^{+}\ \frac{-A \qquad +B}{+A\,\mathit{honk}\,B} \qquad \mathit{honk}\mathrm{E}_1^{+}\ \frac{+A\,\mathit{honk}\,B}{-A} \qquad \mathit{honk}\mathrm{E}_2^{+}\ \frac{+A\,\mathit{honk}\,B}{+B}$$

$$\mathit{honk}\mathrm{I}^{-}\ \frac{+A \qquad -B}{-A\,\mathit{honk}\,B} \qquad \mathit{honk}\mathrm{E}_1^{-}\ \frac{-A\,\mathit{honk}\,B}{+A} \qquad \mathit{honk}\mathrm{E}_2^{-}\ \frac{-A\,\mathit{honk}\,B}{-B}$$

Our *criterion* excludes *honk* in the same way in which it excludes $\bullet$, that is $\mathit{honk}\mathrm{E}_1^{+}$ can be read also as an I-rule for $-$ and $\mathit{honk}\mathrm{E}_1^{-}$ also as an I-rule for $+$. Interpreted in this way, they do not suit harmony with respect to the other Coordination Principles. The other *honk*-rules are harmonious and so we could propose an amended version of *honk* composed of solely *honk*I, $\mathit{honk}\mathrm{E}_2^{+}$ and $\mathit{honk}\mathrm{E}_2^{-}$, which indeed does not lead to triviality. So, at least for this first connective, our *criterion* can be used in place of Kürbis' one.

The rules for *conk* are similar to those for *honk*, but with a relevant difference: all rules have the same modality both in the premise and in the conclusion.

$$\mathit{conk}\mathrm{I}^{+}\ \frac{+A \qquad +B}{+A\,\mathit{conk}\,B} \qquad \mathit{conk}\mathrm{E}^{+}\ \frac{+A\,\mathit{conk}\,B}{+A} \qquad \mathit{conk}\mathrm{E}^{+}\ \frac{+A\,\mathit{conk}\,B}{+B}$$

$$\mathit{conk}\mathrm{I}^{-}\ \frac{-A \qquad -B}{-A\,\mathit{conk}\,B} \qquad \mathit{conk}\mathrm{E}^{-}\ \frac{-A\,\mathit{conk}\,B}{-A} \qquad \mathit{conk}\mathrm{E}^{-}\ \frac{-A\,\mathit{conk}\,B}{-B}$$

This being the case, it is plain that we cannot exclude *conk* by applying the same strategy seen for $\bullet$. Indeed, none of *conk*-rules can be interpreted as introducing or eliminating $+$ or $-$. On the contrary, Kürbis' *criterion* excludes this connective as well, and so we have to admit the incompleteness of our *criterion*.

[64] [20], pp. 537–538.

Moreover, even though Kürbis never explicitly states this, his *criterion* excludes also Gabbay's alleged reduction for *tonk*, which we have seen in section 2. Indeed, in Gabbay's reduction of a *tonk*-maximality

$$
\text{Reductio}_2\;\dfrac{\text{Non-Contradiction}\;\dfrac{\text{tonkI}\;\dfrac{+A}{+(A\,tonk\,B)} \qquad \text{Reductio}_1\;\dfrac{\text{Non-Contradiction}\;\dfrac{\text{tonkE}\;\dfrac{[+(A\,tonk\,B)]^1}{+B} \qquad [-B]^2}{\bot}}{-(A\,tonk\,B)}}{\bot}}{+B}
$$

the occurrence of $-AtonkB$ immediately after $\bot$ is a *maximal formula* according to the definition given by Kürbis, since it is the conclusion of an application of Reductio and a premise of an application of Non-Contradiction.[65] Hence, Kürbis' extension of *maximality* can be used in place of our proposal in order to solve this puzzle.

We can nonetheless strike a blow for our *criterion*. Indeed, even though our proposal has problems to exclude *conk*, it seems an open issue whether Kürbis' *criterion* can exclude Gabbay's rules for $\bullet$, which are on the contrary excluded by our *criterion*.[66] Surely, it is possible to proof both $+\bullet$ and $-\bullet$ using derivations that are normal according to Kürbis definition, as displayed in:

$$
\text{Red}_1\;\dfrac{\text{Non-Contr}\;\dfrac{\bullet\text{E}^+\;\dfrac{[+\bullet]^1}{+B} \qquad \bullet\text{E}^+\;\dfrac{[+\bullet]^1}{-B}}{\bot}}{-\bullet}
\qquad\qquad
\text{Red}_1\;\dfrac{\text{Non-Contr}\;\dfrac{\bullet\text{E}^+\;\dfrac{[-\bullet]^1}{+B} \qquad \bullet\text{E}^+\;\dfrac{[-\bullet]^1}{-B}}{\bot}}{+\bullet}
$$

Admittedly, in order to go a step further and obtain triviality, we need an application of an E-rule for $\bullet$, which would lead to maximality according to Kürbis' definition. Nonetheless, the availability of normal closed proofs for both $+\bullet$ and $-\bullet$ suggests at least that some more careful reflection is needed regarding these requirements.

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[65] The occurrence of $+(AtonkB)$ on the left branch should not be *maximal*, since the other premise of Non-Contradiction is not a conclusion of an I-rule. Moreover, notice that according to our Definition 2.4, $-(A\,tonk\,B)$ is a simple *maximal formula* like for Kürbis, and $+(AtonkB)$ is a complex *maximal formula*.

[66] Kürbis seems to be aware of this lack, see [20], p. 539 note 5.

In conclusion, both *criteria* seem unable to exclude at least one clearly pathological set of rules, with no clear solution on the horizon: my proposal being unable to exclude *conk* and Kürbis' proposal having at least some trouble to exclude bilateral $\bullet$. Even though the objections that could be raised against my *criterion* are more serious than the ones raised against Kürbis', there seems to be enough ground to argue that neither of them is the full story. Maybe a more in-depth comparison between the proposals could lead to a deeper understanding of what is still lacking in the picture, but such a comparison would require a further paper of its own.

5. Conclusion

We have opened this article with a brief introduction about the development of bilateral systems in proof-theoretic semantics, focusing on the difficulty of individuating a clear *criterion* of acceptability for Coordination Principles. Then we have moved to two major objections raised by Gabbay against bilateral systems: an alleged reduction procedure for *tonk* and the availability of paradoxical but harmonious rules for a bilateral reformulation of Read's $\bullet$.

First of all, in Section 2 we have focused on *tonk*, arguing that an extension in the definition of *maximal formulae* is needed in order to reject Gabbay's reduction. We have found such an extension by applying to bilateralism some ideas taken from Milne's work on complex rules. Nonetheless, this solution has forced us to consider $+$ and $-$ as well for separability and so to skip to a weakened version of this notion, a change that is in line with Milne's work. In passing, we have observed that the main problems regarding bilateralism seem to remain: the circular interdependence of meaning of $+$ and $-$ and the violation of the *complexity condition* by Coordination Principles.

Then, in Section 3 we have moved to $\bullet$, analyzing the solution proposed by Francez and finding it conceptually unsatisfactory (even though formally unquestionable), the main problem being the lack of a *criterion* for Coordination Principles. Hence, we have claimed that some Operational Rules should be considered as I and E-rules for $+$ and $-$, together with Coordination Principles. This gives ground for a common *criterion* for both Coordination Principles and Operational Rules, which suffices to exclude $\bullet$ and accept the standard Coordination Principles. The worry that this *criterion* excludes well-behaving Operational Rules as well is dealt with through a solution that is still in line with Milne's work.

In the end, in Section 4 we displayed a comparison with other solutions to the problems of bilateralism present in the literature. In particular, the comparison with Kürbis' *criterion* seems to show that both proposals are to some extent incomplete, even though Kürbis' one in a less serious way than mine, and so that further investigations seem desirable.

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Leonardo Ceragioli
University of Pisa
Department of Civilizations and Forms of Knowledge
Via Pasquale Paoli, #15
56126 Pisa, Italy
e-mail: leo.crg.uni@gmail.com