Bulletin of the **S**ection of **L**ogic
Volume 53/1 (2024), pp. 105–124
https://doi.org/10.18778/0138-0680.2023.28

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STABILIZERS ON L-ALGEBRAS

Abstract

The main goal of this paper is to introduce the notion of stabilizers in L-algebras and develop stabilizer theory in L-algebras. In this paper, we introduced the notions of left and right stabilizers and investigated some related properties of them. Then, we discussed the relations among stabilizers, ideal and co-annihilators. Also, we obtained that the set of all ideals of a CKL-algebra forms a relative pseudo-complemented lattice. In addition, we proved that right stabilizers in CKL-algebra are ideals. Then by using the right stabilizers we produced a basis for a topology on L-algebra. We showed that the generated topology by this basis is Baire, connected, locally connected and separable and we investigated the other properties of this topology.

Keywords: L-algebra, stabilizer, ideal, co-annihilators, Baire space, topological space.

2020 Mathematical Subject Classification: 06B10, 06B99, 03G25.

1. Introduction

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [8]. Many examples shown that L-algebras are very useful. Yang and Rump [10], characterized pseudo-MV-algebras and

Presented by: Janusz Ciuciura
Received: January 20, 2023
Published online: November 20, 2023

Bosbach's non-commutative bricks as L-algebras. Wu and Yang [13] proved that orthomodular lattices form a special class of L-algebras in different ways. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. [12]. Also, they proved that a basic algebra which satisfies

$$(z \oplus \neg x) \oplus \neg(y \oplus \neg x) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y),$$

can be converted into an L-algebra. Conversely, if an L-algebra with 0 and some conditions such that it is an involutive bounded lattice can be organized into a basic algebra, it must be a lattice-ordered effect algebra. In addition, Aaly in [1], and Ciung in [5] studied the relationship between logical algebraic structures and basic algebras with L-algebras, such as $BCK/$ BCI-algebras, hoop, residuated lattice, equality and EQ-algebras.

A stabilizer is a part of a monoid acting on a set. Specifically, let $\mathbb{X}$ be a monoid operating on a set $\mathbb{X}$ and let $\mathbb{H}$ be a subset of $\mathbb{X}$. The stabilizer of $\mathbb{H}$, sometimes denoted $St(\mathbb{H})$ is the set of elements as a of $\mathbb{X}$ for which $a(\mathbb{H}) \subseteq \mathbb{H}$. The strict stabilizer is the set of $a \in \mathbb{X}$ for which $a(\mathbb{H}) = \mathbb{H}$. In the other words, the stabilizer of $\mathbb{H}$ is the transporter of $\mathbb{H}$ to itself. In recent years, many mathematicians have studied and investigated the characteristics of stabilizers in logical algebraic structures. Also, some of them have used a special type of stabilizers called co-annihilators and have obtained interesting results in this field, and this concept has been investigated on different structures, such as BL-algebra, EQ-algebra, hoop and etc. For more information in this field, we refer the readers to the references [3, 4, 6, 7, 11].

The main goal of this paper is to introduce the notion of stabilizers in L-algebras and develop stabilizer theory in L-algebras. In this paper, we introduce the notions of left and right stabilizers and investigate some related properties of them. Then, we discuss the relations among stabilizers, ideal and co-annihilators. Also, we obtain that the set of all ideals in a CKL-algebra forms a relative pseudo-complemented lattice. In addition, we prove that right stabilizers in CKL-algebra are ideals. Then by using the right stabilizers produce a basis for a topology on L-algebra. We show that the generated topology by this basis is Baire, connected, locally connected and separable and we investigate the other properties of this topology.

2. Preliminaries

In this section, we gather some basic notions relevent to L-algebras which will need in the next sections.

DEFINITION 2.1 ([8]). An *L-algebra* is an algebraic structure $(\mathbb{L}; \to, 1)$ of type $(2, 0)$ satisfying

$(L1)$ $x \to x = x \to 1 = 1$ and $1 \to x = x$,

$(L2)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,

$(L3)$ if $x \to y = y \to x = 1$, then $x = y$,

for any $x, y, z \in \mathbb{L}$. Condition $(L1)$ states that 1 is a logical unit, while $(L2)$ is related to the quantum Yang-Baxter equation. Note that a logical unit is always unique. In addition, we can define the relation

$$x \lesssim y \text{ if and only if } x \to y = 1,$$

on $\mathbb{L}$. By (L_1) and (L_2), clearly this relation is reflexive and transitive, respectively and by $(L3)$, untisymmetric is proved. So, $(\mathbb{L}, \lesssim)$ is a poset. If $\mathbb{L}$ admits a smallest element 0, then it is called a *bounded L-algebra.*

Let $\mathbb{L}$ be bounded. We define a binary operation "$'$" on $\mathbb{L}$ by $x' = x \to 0$, for all $x \in \mathbb{L}$. If for any $x \in \mathbb{L}$, $x'' = x$, then the bounded L-algebra $\mathbb{L}$ is called to have the double negation properties.

PROPOSITION 2.2 ([10]). Let $\mathbb{L}$ be an L-algebra. Then $x \lesssim y$ implies $z \to x \lesssim z \to y$, for any $x, y, z \in \mathbb{L}$.

PROPOSITION 2.3 ([10]). For an L-algebra $\mathbb{L}$, the following are equivalent:

(i) $x \lesssim y \to x$,

(ii) if $x \lesssim z$, then $z \to y \lesssim x \to y$,

(iii) $((x \to y) \to z) \to z \lesssim ((x \to y) \to z) \to ((y \to x) \to z)$,

for any $x, y, z \in \mathbb{L}$.

DEFINITION 2.4 ([9]). An L-algebra $\mathbb{L}$ which satisfies

$$x \to (y \to x) = 1, \qquad (K)$$

for any $x, y \in \mathbb{L}$ is called a *KL-algebra.*

A CKL-*algebra* is an L-algebra which satisfies

$$x \to (y \to z) = y \to (x \to z), \qquad (C)$$

for any $x, y, z \in \mathbb{L}$ (see [9]).

Clearly, every CKL-algebra is a KL-algebra, since for any $x, y \in \mathbb{L}$, we have

$$x \to (y \to x) = y \to (x \to x) = y \to 1 = 1.$$

Proposition 2.5 ([2]). Assume $(\mathbb{L}, \to, 1)$ is a CKL-algebra. Then for any $x, y, z \in \mathbb{L}$, the following properties hold:

(i) if $x \lesssim y$, then $z \to x \lesssim z \to y$,

(ii) $x \to (y \to x) = 1$, i.e., $x \lesssim y \to x$,

(iii) $x \lesssim (x \to y) \to y$,

(iv) $x \lesssim y \to z$ if and only if $y \lesssim x \to z$,

(v) if $x \lesssim y$, then $y \to z \lesssim x \to z$,

(vi) $((x \to y) \to z) \to z \lesssim ((x \to y) \to z) \to ((y \to x) \to z)$,

(vii) $z \to y \lesssim (y \to x) \to (z \to x)$,

$(viii)$ $z \to y \lesssim (x \to z) \to (x \to y)$,

If $\mathbb{L}$ has a least element 0, then

(ix) if $x \lesssim y$, then $y' \lesssim x'$, where $x' = x \to 0$,

(x) $x \lesssim x''$, and $x' = x'''$,

(xi) $x' \lesssim x \to y$,

(xii) $((x \to y) \to y) \to y = x \to y$,

$(xiii)$ If $\mathbb{L}$ has double negation, then $x \to y = y' \to x'$.

Definition 2.6 ([8]). An L-algebra $\mathbb{L}$ is said to be a **semi-regular** if the equation

$$((x \to y) \to z) \to ((y \to x) \to z) = ((x \to y) \to z) \to z,$$

holds in $\mathbb{L}$. Also, $\mathbb{L}$ is called a **regular L-algebra** if in addition, for any pair element $x \lesssim y$ in $\mathbb{L}$, there is an element $z \gtrsim x$ in $\mathbb{L}$ such that $z \to x = y$.

For a bounded L-algebra with negation, we set

$$x \curlywedge y = ((x \to y) \to x')', \qquad x \curlyvee y = (x' \to y') \to x. \tag{2.1}$$

PROPOSITION 2.7 ([10]). Let $\mathbb{L}$ be a semi-regular L-algebra with negation. Then the equations

$$x \to (y \curlywedge z) = (x \to y) \curlywedge (x \to z)$$
$$(x \curlyvee y) \to z = (x \to z) \curlywedge (y \to z)$$

hold for any $x, y, z \in \mathbb{L}$.

DEFINITION 2.8 ([8]). A subset $\mathbb{I}$ of an L-algebra $\mathbb{L}$ is called an *ideal of* $\mathbb{L}$ if it satisfies the following conditions for all $x, y \in \mathbb{L}$,

(I_1) $1 \in \mathbb{I}$,

(I_2) if $x \in \mathbb{I}$ and $x \to y \in \mathbb{I}$, then $y \in \mathbb{I}$,

(I_3) if $x \in \mathbb{I}$, then $(x \to y) \to y \in \mathbb{I}$,

(I_4) if $x \in \mathbb{I}$, then $y \to x \in \mathbb{I}$ and $y \to (x \to y) \in \mathbb{I}$.

The set of all ideals of $\mathbb{L}$ is denoted by $\mathcal{I}d(\mathbb{L})$.

PROPOSITION 2.9 ([2]). Every ideal of $\mathbb{L}$ is upset.

If we consider the ideal of CKL-algebra, the conditions (I_3) and (I_4) can be dropped. In fact, for any $x \in \mathbb{I}$, by (C) and (I_1) we have

$$x \to ((x \to y) \to y) = (x \to y) \to (x \to y) = 1 \in \mathbb{I},$$

for any $y \in \mathbb{L}$. It follows by (I_2) that $(x \to y) \to y \in \mathbb{I}$. Thus (I_3) holds. Furthermore, if $x \in \mathbb{I}$, then for any $y \in \mathbb{L}$, by (K) we have $x \to (y \to x) = 1 \in \mathbb{I}$ and by (I_2), $y \to x \in \mathbb{I}$.

For an L-algebra such as $\mathbb{L}$, a binary relation $\sim$ is a *congruence relation* [8] on $\mathbb{L}$ if it is an equivalence relation such that for any $x, y, z \in \mathbb{L}$,

$$x \sim y \Leftrightarrow \ (z \to x) \sim (z \to y) \ \text{ and } \ (x \to z) \sim (y \to z).$$

THEOREM 2.10 ([8]). *Let* $(\mathbb{L}, \to, 1)$ *be an* L*-algebra. Then every ideal* $\mathbb{I}$ *of* $\mathbb{L}$ *defines a congruence relation on* $\mathbb{L}$*, for any* $x, y \in \mathbb{L}$*, where*

$$x \sim_{\mathbb{I}} y \ \Leftrightarrow \quad x \to y, \ y \to x \in \mathbb{I}.$$

Conversely, every congruence relation $\sim$ defines an ideal

$$\mathbb{I} = \{x \in \mathbb{L} \mid x \sim 1\}.$$

DEFINITION 2.11 ([8]). Let $\mathbb{L}$ and $\mathbb{H}$ be two L-algebras. Then a map $f : \mathbb{L} \to \mathbb{H}$ is called an *L-homomorphism* if for any $x, y \in \mathbb{L}$ we have

$$f(x \to_{\mathbb{L}} y) = f(x) \to_{\mathbb{H}} f(y).$$

Obviously, $f(1_{\mathbb{L}}) = 1_{\mathbb{H}}$.

Note. From now on, we let $(\mathbb{L}, \to, 1)$ or $\mathbb{L}$, for short, be an L-algebra and $\mathbb{X}$ be a non-empty subset of $\mathbb{L}$.

3. Main results

3.1. Stabilizers on L-algebras

In this section, we introduce the notions of left and right stabilizers on L-algebras and investigate some properties of them.

DEFINITION 3.1. A **left stabilizer and a right stabilizer of** $\mathbb{X}$ are defined as follows:

$$\mathcal{S}_r(\mathbb{X}) = \{a \in \mathbb{L} \mid \text{for any } x \in \mathbb{X},\ a \to x = x\}.$$
$$\mathcal{S}_l(\mathbb{X}) = \{a \in \mathbb{L} \mid \text{for any } x \in \mathbb{X},\ x \to a = a\}.$$

Example 3.2. (*i*) Assume $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a chain where $a < b < c < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	1	1
b	c	1	1	1
c	b	c	1	1
1	a	b	c	1

Clearly, $\mathcal{S}_r(\{b\}) = \mathcal{S}_l(\{b\}) = \{1\}$.
(*ii*) Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a chain where $a < b < c < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\rightarrow$	a	b	c	1
a	1	1	1	1
b	b	1	1	1
c	a	b	1	1
1	a	b	c	1

If $\mathbb{X}_1 = \{c\}$ and $\mathbb{X}_2 = \{a, b, 1\}$, then $\mathcal{S}_l(\mathbb{X}_1) = \{a, b, 1\}$ and $\mathcal{S}_r(\mathbb{X}_2) = \{c, 1\}$.

Note. $\mathcal{S}_r(\mathcal{S}_l(\mathbb{X}))$ is called a **right-left stabilizer of** $\mathbb{X}$ and we denote it by $(\mathcal{S}_l(\mathbb{X}))_r$, for short. Similarly, $(\mathcal{S}_r(\mathbb{X}))_l$ is a **left-right stabilizer of** $\mathbb{X}$.

PROPOSITION 3.3. For all $x, y \in \mathbb{L}$ and $\emptyset \neq \mathbb{X}, \mathbb{Y} \subseteq \mathbb{L}$, the following statements hold:

(i) $1 \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

(ii) If $\mathbb{X} \subseteq \mathbb{Y}$, then $\mathcal{S}_r(\mathbb{Y}) \subseteq \mathcal{S}_r(\mathbb{X})$ and $\mathcal{S}_l(\mathbb{Y}) \subseteq \mathcal{S}_l(\mathbb{X})$.

(iii) $\mathbb{X} \subseteq (\mathcal{S}_r(\mathbb{X}))_l \cap (\mathcal{S}_l(\mathbb{X}))_r$.

(iv) $\mathcal{S}_r(\mathbb{X}) = (((\mathcal{S}_r(\mathbb{X}))_l)_r$ and $\mathcal{S}_l(\mathbb{X}) = (((\mathcal{S}_l(\mathbb{X}))_r)_l$.

(v) If $\{\mathbb{X}_i\}_{i \in I}$ is a family of non-empty subsets of $\mathbb{L}$, then $\mathcal{S}_r(\bigcup\limits_{i \in I} \mathbb{X}_i) = \bigcap\limits_{i \in I} \mathcal{S}_r(\mathbb{X}_i)$ and $\mathcal{S}_l(\bigcup\limits_{i \in I} \mathbb{X}_i) = \bigcap\limits_{i \in I} \mathcal{S}_l(\mathbb{X}_i)$.

(vi) $\mathcal{S}_r(\mathbb{L}) = \mathcal{S}_l(\mathbb{L}) = \{1\}$.

(vii) $\mathcal{S}_r(\{1\}) = \mathcal{S}_l(\{1\}) = \mathbb{L}$.

$(viii)$ If $x \in \mathcal{S}_r(\{x\}) \cap \mathcal{S}_l(\{x\})$, then $x = 1$.

(ix) If $h : \mathbb{L} \to \mathbb{L}$ is a homomorphism and $x \in \mathbb{L}$, then $h(\mathcal{S}_r(\{x\})) \subseteq \mathcal{S}_r(\{h(x)\})$ and $h(\mathcal{S}_l(\{x\})) \subseteq \mathcal{S}_l(\{h(x)\})$.

(x) If $\mathbb{L}$ is a bounded L-algebra with DNP, then $\mathcal{S}_r(\{0\}) = \{1\}$.

(xi) If $\mathbb{L}$ is a bounded L-algebra, then $\mathcal{S}_r(\{0\}) = \{1\}$ if and only if for any $x, y \in \mathbb{L}$, $x \to y, y \to x \in \mathcal{S}_r(\{0\})$ implies $x = y$.

PROOF: (i) Clearly, by $(L1)$, since for any $x \in \mathbb{X}$, $1 \to x = x$, we get $1 \in \mathcal{S}_r(\mathbb{X})$. In addition, by $(L1)$, $x \to 1 = 1$, and so $1 \in \mathcal{S}_l(\mathbb{X})$. Hence, $1 \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

(ii) Assume $a \in \mathcal{S}_r(\mathbb{Y})$. Then for any $y \in \mathbb{Y}$, $a \to y = y$. Since $\mathbb{X} \subseteq \mathbb{Y}$, clearly, for any $x \in \mathbb{X}$, $a \to x = x$ and so $a \in \mathcal{S}_r(\mathbb{X})$. Hence, $\mathcal{S}_r(\mathbb{Y}) \subseteq \mathcal{S}_r(\mathbb{X})$. The proof of the other case is similar.

(iii) Suppose $a \in \mathbb{X}$. Then for any $y \in \mathcal{S}_r(\mathbb{X})$, $y \to a = a$, and so $a \in (\mathcal{S}_r(\mathbb{X}))_l$. In addition, for any $y \in \mathcal{S}_l(\mathbb{X})$, $a \to y = y$ and so $a \in (\mathcal{S}_l(\mathbb{X}))_r$. Hence, $a \in (\mathcal{S}_r(\mathbb{X}))_l \cap (\mathcal{S}_l(\mathbb{X}))_r$. Therefore, $\mathbb{X} \subseteq (\mathcal{S}_r(\mathbb{X}))_l \cap (\mathcal{S}_l(\mathbb{X}))_r$.

(iv) By (iii), we have $\mathbb{X} \subseteq (\mathcal{S}_r(\mathbb{X}))_l$ and by (ii), we get $((\mathcal{S}_r(\mathbb{X}))_l)_r \subseteq \mathcal{S}_r(\mathbb{X})$. Also, by (iii), $\mathbb{Y} \subseteq (\mathcal{S}_l(\mathbb{Y}))_r$. Consider $\mathbb{Y} = \mathcal{S}_r(\mathbb{X})$. Then $\mathcal{S}_r(\mathbb{X}) \subseteq ((\mathcal{S}_r(\mathbb{X}))_l)_r$. Hence, $\mathcal{S}_r(\mathbb{X}) = (((\mathcal{S}_r(\mathbb{X}))_l)_r$. The proof of the other case is similar.

(v) Since $\mathbb{X}_i \subseteq \bigcup\limits_{i\in I}\mathbb{X}_i$, by (ii), $\mathcal{S}_r(\bigcup\limits_{i\in I}\mathbb{X}_i) \subseteq \mathcal{S}_r(\mathbb{X}_i)$, and so $\mathcal{S}_r(\bigcup\limits_{i\in I}\mathbb{X}_i) \subseteq \bigcap\limits_{i\in I}\mathcal{S}_r(\mathbb{X}_i)$. Conversely, assume $a \in \bigcap\limits_{i\in I}\mathcal{S}_r(\mathbb{X}_i)$, then for any $i \in I$, $a \in \mathcal{S}_r(\mathbb{X}_i)$, and so for any $x_i \in \mathbb{X}_i$, $a \to x_i = x_i$. Thus for any $x \in \bigcup\limits_{i\in I}\mathbb{X}_i$, there exists $i \in I$ such that $x \in \mathbb{X}_i$, and so $a \to x = x$. So, $a \in \mathcal{S}_r(\bigcup\limits_{i\in I}\mathbb{X}_i)$. Therefore, $\mathcal{S}_r(\bigcup\limits_{i\in I}\mathbb{X}_i) = \bigcap\limits_{i\in I}\mathcal{S}_r(\mathbb{X}_i)$.

(vi) Clearly, by (i), $\{1\} \subseteq \mathcal{S}_r(\mathbb{L})$. Assume $1 \neq a \in \mathcal{S}_r(\mathbb{L})$. Then for any $x \in \mathbb{L}$, $a \to x = x$. Let $x = a$. Then $1 = a \to a = a$, and so $a = 1$, which is a contradiction. Hence, $\mathcal{S}_r(\mathbb{L}) = \{1\}$. The proof of the other case is similar.

(vii) Obviously, $\mathcal{S}_r(\{1\}), \mathcal{S}_l(\{1\}) \subseteq \mathbb{L}$. Suppose $a \in \mathbb{L}$. Then by $(L1)$, $a \to 1 = 1$ and $1 \to a = a$. Thus $a \in \mathcal{S}_r(\{1\}) \cap \mathcal{S}_l(\{1\})$. Hence $\mathcal{S}_r(\{1\}) = \mathcal{S}_l(\{1\}) = \mathbb{L}$.

$(viii)$ Straightforward.

(ix) Assume $y \in h(\mathcal{S}_l(\{x\}))$. Then there exists $a \in \mathcal{S}_l(\{x\})$ such that $y = h(a)$. Since $x \to a = a$ and h is a homomorphism on $\mathbb{L}$, we have

$$y = h(a) = h(x \to a) = h(x) \to h(a) = h(x) \to y.$$

Thus $y \in \mathcal{S}_l(\{h(x)\})$. Hence, $h(\mathcal{S}_l(\{x\})) \subseteq \mathcal{S}_l(h(x))$. The proof of the other case is similar.

(x) Assume $a \in \mathcal{S}_r(\{0\})$. Then $a \to 0 = 0$, and so $a' = 0$. By hypothesis, $a'' = 0' = 1$ and so $a = 1$. Thus $\mathcal{S}_r(\{0\}) = \{1\}$.

(xi) If $\mathcal{S}_r(\{0\}) = \{1\}$ and $x \to y, y \to x \in \mathcal{S}_r(\{0\})$, then $x \to y = y \to x = 1$, and by $(L3)$, we have $x = y$. Conversely, by (i), $\{1\} \subseteq \mathcal{S}_r(\{0\})$. Consider

$a \in \mathcal{S}_r(\{0\})$. Since $1 \to a = a \in \mathcal{S}_r(\{0\})$ and $a \to 1 = 1 \in \mathcal{S}_r(\{0\})$, we get $1 \to a$, $a \to 1 \in \mathcal{S}_r(\{0\})$. So, by assumption, we have $a = 1$. Hence, $\mathcal{S}_r(\{0\}) = \{1\}$. □

In the following example we show that for any non-empty subset $\mathbb{X}$ of $\mathbb{L}$, $\mathcal{S}_r(\mathbb{X})$ and $\mathcal{S}_l(\mathbb{X})$ are not ideals of $\mathbb{L}$, in general.

Example 3.4. (*i*) According to Example 3.2(i), $\mathcal{S}_r(\{b\}) = \mathcal{S}_l(\{b\}) = \{1\}$. So, both are ideals of $\mathbb{L}$.

(*ii*) According to Example 3.2(ii), $\mathcal{S}_r(\{a, b, 1\}) = \{c, 1\}$ is an ideal of $\mathbb{L}$ but $\mathcal{S}_l(\{c\}) = \{a, b, 1\}$ is not an ideal of $\mathbb{L}$ since $b \to c = 1 \in \{a, b, 1\}$ and $b \in \{a, b, 1\}$ but $c \notin \{a, b, 1\}$.

(*iii*) Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a poset where $a, c < b < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	b	1	1	1
1	a	b	c	1

If $\mathbb{X} = \{b\}$, then $\mathcal{S}_l(\mathbb{X}) = \{a, c, 1\}$ is not an ideal of $\mathbb{L}$, because $a \to b = 1 \in \mathcal{S}_l(\mathbb{X})$ and $a \in \mathcal{S}_l(\mathbb{X})$, but $b \notin \mathcal{S}_l(\mathbb{X})$.

(*iv*) Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a poset where $a < b, c < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	1	1
b	c	1	c	1
c	b	b	1	1
1	a	b	c	1

Assume $\mathbb{X} = \{b\}$. Then $\mathcal{S}_r(\mathbb{X}) = \mathcal{S}_l(\mathbb{X}) = \{c, 1\}$ are ideals of $\mathbb{L}$.

Proposition 3.5. If $\mathbb{L}$ is a KL-algebra and for any $x, y \in \mathbb{L}$, $(x \to y) \to y = (y \to x) \to x$, then for any $\mathbb{X} \subseteq \mathbb{L}$, $\mathcal{S}_r(\mathbb{X}) = \mathcal{S}_l(\mathbb{X})$.

Proof: Let $a \in \mathcal{S}_r(\mathbb{X})$. Then for any $x \in \mathbb{X}$, $a \to x = x$. Since $\mathbb{L}$ is a KL-algebra, by Proposition 2.3, $a \lesssim x \to a$. By assumption,

$$(x \to a) \to a = (a \to x) \to x = x \to x = 1.$$

Thus, $x \to a \lesssim a$, and so by $(L3)$, we have $x \to a = a$. Hence, $a \in \mathcal{S}_l(\mathbb{X})$ and so $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}_l(\mathbb{X})$. The proof of the other case is similar. $\square$

Example 3.6. Assume $\mathbb{L}$ is an L-algebra as in Example 3.4(iv). This example demonstrates Proposition 3.5.

In the following example we show that the condition $(x \to y) \to y = (y \to x) \to x$ in Proposition 3.5 is necessary.

Example 3.7. Let $\mathbb{L}$ be an L-algebra as in Example 3.2(ii). Clearly, $\mathbb{L}$ is a KL-algebra but

$$(a \to c) \to c = 1 \to c = c \neq 1 = a \to a = (c \to a) \to a.$$

As we see in this example, if $\mathbb{X} = \{c\}$, then $\mathcal{S}_r(\mathbb{X}) = \{1\} \neq \{a, b, 1\} = \mathcal{S}_l(\mathbb{X})$.

THEOREM 3.8. *If $\mathbb{L}$ is a CKL-algebra, $\mathcal{S}_r(\mathbb{X})$ is an ideal of $\mathbb{L}$, for any non-empty subset $\mathbb{X}$ of $\mathbb{L}$.*

PROOF: By Proposition 3.3(i), $1 \in \mathcal{S}_r(\mathbb{X})$. Assume $a, a \to b \in \mathcal{S}_r(\mathbb{X})$, for any $a, b \in \mathbb{L}$. Then for any $x \in \mathbb{X}$, $a \to x = x$ and $(a \to b) \to x = x$. Thus,

$$\begin{aligned} b \to x &\lesssim (a \to b) \to (a \to x) \qquad \text{by Proposition 2.5(viii)} \\ &= a \to ((a \to b) \to x) \qquad \text{by (C)} \\ &= a \to x \qquad \text{since } a \to b \in \mathcal{S}_r(\mathbb{X}) \\ &= x. \qquad \text{since } a \in \mathcal{S}_r(\mathbb{X}) \end{aligned}$$

Thus, $b \to x \lesssim x$. By Proposition 2.5(ii), $x \lesssim b \to x$. Hence, $b \to x = x$, and so $b \in \mathcal{S}_r(\mathbb{X})$. Therefore, $\mathcal{S}_r(\mathbb{X})$ is an ideal of $\mathbb{L}$. $\square$

The next example shows that the condition CKL-algebra in Theorem 3.8 is necessary.

Example 3.9. Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a poset where $a, c < b < 1$. Then $(\mathbb{L}, \to, 1)$ is an L-algebra such that

$\to$	a	b	c	1
a	1	1	a	1
b	c	1	a	1
c	a	1	1	1
1	a	b	c	1

But $\mathbb{L}$ is not a CKL-algebra, since

$$a \to (b \to c) = a \to a = 1 \neq c = b \to a = b \to (a \to c).$$

If $\mathbb{X} = \{a\}$, then $\mathcal{S}_r(\mathbb{X}) = \{c, 1\}$, which is not an ideal of $\mathbb{L}$, because since $c \in \mathcal{S}_r(\mathbb{X})$ by (I_3), we have to have $b \to c \in \mathcal{S}_r(\mathbb{X})$, but $b \to c = a \notin \mathcal{S}_r(\mathbb{X})$.

By the following example we show that in any CKL-algebra, $\mathcal{S}_l(\mathbb{X})$ is not an ideal of $\mathbb{L}$.

Example 3.10. Suppose $(\mathbb{L} = \{a, b, c, 1\}, \lesssim)$ is a chain where $a < b < c < 1$. Then $(\mathbb{L}, \to, 1)$ is a CKL-algebra such that

$\to$	a	b	c	1
a	1	1	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

Assume $\mathbb{X} = \{b\}$. Then $\mathcal{S}_l(\mathbb{X}) = \{a, 1\}$ which is not an ideal of $\mathbb{L}$, since $a \to b = 1 \in \mathcal{S}_l(\mathbb{X})$ and $a \in \mathcal{S}_l(\mathbb{X})$ but $b \notin \mathcal{S}_l(\mathbb{X})$.

Remark 3.11. If $\mathbb{L}$ is a CKL-algebra with DNP, then by Proposition 2.5(xiii) and (2.1), we have

$$x \curlyvee y = (x' \to y') \to x = (y \to x) \to x.$$

Proposition 3.12. Assume $\mathbb{L}$ is a CKL-algebra with DNP and $a \in \mathbb{L}$. If $a \curlyvee x = 1$, then for any $x \in \mathbb{X}$, $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

Proof: Since $\mathbb{L}$ is a CKL-algebra, by Proposition 2.5(ii), $a \lesssim x \to a$, for any $x \in \mathbb{X}$. By assumption and Remark 3.11, we have

$$1 = a \curlyvee x = (a' \to x') \to a = (x \to a) \to a.$$

So, $x \to a \lesssim a$. Thus by Proposition 2.5(ii), we get $x \to a = a$, and so $a \in \mathcal{S}_l(\mathbb{X})$. By similar discussion, we can prove $a \in \mathcal{S}_r(\mathbb{X})$. Hence, $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. □

Theorem 3.13. *Let $\mathbb{L}$ be a semi-regular L-algebra with negation. If $x \curlyvee y \in \mathcal{S}_r(\mathbb{X})$, then $x \in \mathcal{S}_r(\mathbb{X})$ or $y \in \mathcal{S}_r(\mathbb{X})$.*

Proof: Suppose $x \curlyvee y \in \mathcal{S}_r(\mathbb{X})$ such that $x \notin \mathcal{S}_r(\mathbb{X})$ and $y \notin \mathcal{S}_r(\mathbb{X})$. Then for any $a \in \mathbb{X}$, $(x \curlyvee y) \to a = a$, $x \to a \neq a$ and $y \to a \neq a$. By Proposition 2.3, $a < x \to a$ and $a < y \to a$. Thus by Proposition 2.7 we have

$$a < (x \to a) \curlywedge (y \to a) = (x \curlyvee y) \to a = a,$$

which is a contradiction. Hence, $x \in \mathcal{S}_r(\mathbb{X})$ or $y \in \mathcal{S}_r(\mathbb{X})$. $\square$

Theorem 3.14. *Let $\mathbb{L}$ be a semi-regular L-algebra with negation. If $\mathbb{I}_1, \mathbb{I}_2 \in \mathcal{I}d(\mathbb{X})$ such that $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1 \cap \mathbb{I}_2$, then $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1$ or $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_2$.*

Proof: By the assumption, $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1 \cap \mathbb{I}_2$. So, clearly, $\mathcal{S}_r(\mathbb{X}) \subseteq \mathbb{I}_1$ and $\mathcal{S}_r(\mathbb{X}) \subseteq \mathbb{I}_2$. Now, suppose $\mathbb{I}_1, \mathbb{I}_2 \nsubseteq \mathcal{S}_r(\mathbb{X})$. Then there exist $a \in \mathbb{I}_1 \setminus \mathcal{S}_r(\mathbb{X})$ and $b \in \mathbb{I}_2 \setminus \mathcal{S}_r(\mathbb{X})$. Since $\mathbb{I}_1, \mathbb{I}_2 \in \mathcal{I}d(\mathbb{X})$, by Proposition 2.9, $\mathbb{I}_1$ and $\mathbb{I}_2$ are upset. So, $a \lesssim a \curlyvee b$ and $b \lesssim a \curlyvee b$, we get $a \curlyvee b \in \mathbb{I}_1 \cap \mathbb{I}_2$. Thus $a \curlyvee b \in \mathcal{S}_r(\mathbb{X})$. By Theorem 3.13, we obtain $a \in \mathcal{S}_r(\mathbb{X})$ or $b \in \mathcal{S}_r(\mathbb{X})$, which is a contradiction. Hence, $\mathbb{I}_1 \subseteq \mathcal{S}_r(\mathbb{X})$ or $\mathbb{I}_2 \subseteq \mathcal{S}_r(\mathbb{X})$. Therefore, $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_1$ or $\mathcal{S}_r(\mathbb{X}) = \mathbb{I}_2$. $\square$

Note. The set ${}^{\perp}\mathbb{X} = \{a \in \mathbb{L} \mid a \curlyvee x = 1, \text{ for all } x \in \mathbb{X}\}$, if $x \curlyvee a$ exists, is called a *co-annihilator* of $\mathbb{X}$. In the following theorem we investigate the condition showing ${}^{\perp}\mathbb{X} = \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.

Theorem 3.15. *Consider $\mathbb{L}$ be a CKL-algebra with DNP. Then ${}^{\perp}\mathbb{X} = \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$.*

Proof: Assume $a \in^{\perp} \mathbb{X}$. Then for all $x \in \mathbb{X}$, $a \curlyvee x = 1$, and by Remark 3.11 and since $x \curlyvee a = a \curlyvee x$ we have

$$1 = a \curlyvee x = (a \to x) \to x = (x \to a) \to a.$$

Thus, $(a \to x) \to x = 1$ and $(x \to a) \to a = 1$. Hence, $a \to x \lesssim x$ and $x \to a \lesssim a$. Also, by Proposition 2.5(ii), we have $x \lesssim a \to x$ and $a \lesssim x \to a$. Then $a \to x = x$ and $x \to a = a$. Therefore, $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. Conversely, suppose $a \in \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. Then $a \in \mathcal{S}_r(\mathbb{X})$ and $a \in \mathcal{S}_l(\mathbb{X})$. Thus, for any $x \in \mathbb{X}$, $a \to x = x$ and $x \to a = a$. So $(a \to x) \to x = 1$ and $(x \to a) \to a = 1$. By Remark 3.11, we have

$$(a \to x) \to x = (x \to a) \to a = a \curlyvee x = 1.$$

Hence, $a \in^{\perp} \mathbb{X}$. Therefore, ${}^{\perp}\mathbb{X} = \mathcal{S}_r(\mathbb{X}) \cap \mathcal{S}_l(\mathbb{X})$. $\square$

Stabilizer topology

In this section, we use of the right and left stabilizers of an L-algebra and produce a basis for a topology on it. Then we show that the generated topology by this basis is Baire, connected, locally connected and separable and investigate the other properties of this topology.

DEFINITION 3.16. A map $C : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L})$ is a **closure operator** if for any $\mathbb{X}, \mathbb{Y} \in \mathcal{P}(\mathbb{L})$ we have

(C_1) $\mathbb{X} \subseteq C(\mathbb{X})$,

(C_2) If $\mathbb{X} \subseteq \mathbb{Y}$, then $C(\mathbb{X}) \subseteq C(\mathbb{Y})$,

(C_3) $C(C(\mathbb{X})) = C(\mathbb{X})$.

THEOREM 3.17. *Define* $\omega : \mathcal{P}(\mathbb{L}) \to \mathcal{P}(\mathbb{L})$ *such that* $\omega(\mathbb{X}) = (\mathcal{S}_l(\mathbb{X}))_r$, *for all* $\mathbb{X} \in \mathcal{P}(\mathbb{L})$. *Then*

(i) ω *is a closure map.*

(ii) $\mathbb{X} \subseteq \omega(\mathbb{Y})$ *if and only if* $\omega(\mathbb{X}) \subseteq \omega(\mathbb{Y})$, *for all* $\mathbb{Y} \subseteq \mathbb{L}$.

(iii) $\gamma_\omega = \{\mathbb{X} \in \mathcal{P}(\mathbb{L}) \mid \ \omega(\mathbb{X}) = \mathbb{X}\}$ *is a basis for a topology on* $\mathbb{L}$.

PROOF: (i) By Proposition 3.3(ii), (iii) and (iv) the proof is clear.

(ii) By (i) is clear.

(iii) Let $\gamma_\omega = \{\mathbb{X} \in \mathcal{P}(\mathbb{L}) \mid \ \omega(\mathbb{X}) = \mathbb{X}\}$. Obviously, $\emptyset \in \gamma_\omega$. Also, by Proposition 3.3(vi) and (vii), $\omega(\mathbb{L}) = (\mathcal{S}_l(\mathbb{L}))_r = \mathcal{S}_r(\{1\}) = \mathbb{L}$. Thus, $\omega(\mathbb{L}) = \mathbb{L}$, and so $\mathbb{L} \in \gamma_\omega$. Now, suppose $\mathbb{X}, \mathbb{Y} \in \gamma_\omega$. Then $\omega(\mathbb{X}) = \mathbb{X}$ and $\omega(\mathbb{Y}) = \mathbb{Y}$. We show $\mathbb{X} \cap \mathbb{Y} \in \gamma_\omega$. Since $\mathbb{X} \cap \mathbb{Y} \subseteq \mathbb{X}, \mathbb{Y}$, by (i), $\omega(\mathbb{X} \cap \mathbb{Y}) \subseteq \omega(\mathbb{X})$ and $\omega(\mathbb{Y})$. Thus, $\omega(\mathbb{X} \cap \mathbb{Y}) \subseteq \omega(\mathbb{X}) \cap \omega(\mathbb{Y})$. In addition, from $\mathbb{X}, \mathbb{Y} \in \gamma_\omega$, we have $\omega(\mathbb{X} \cap \mathbb{Y}) \subseteq \mathbb{X} \cap \mathbb{Y}$. Moreover, by Proposition 3.3(iii), $\mathbb{X} \cap \mathbb{Y} \subseteq \omega(\mathbb{X} \cap \mathbb{Y})$. Then $\omega(\mathbb{X} \cap \mathbb{Y}) = \mathbb{X} \cap \mathbb{Y}$, and so $\mathbb{X} \cap \mathbb{Y} \in \gamma_\omega$. Therefore, γ_ω is a basis. □

Note. (i) According to the definition γ_ω, clearly, $(St_l(\mathbb{L}))_r = \mathbb{L}$ and $(St_l(\emptyset))_r = \emptyset$, so $\emptyset, \mathbb{L} \in \gamma_\omega$ and by Proposition 3.3(i), for any $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$, $1 \in (St_l(\mathbb{X}))_r$, so for any $\mathbb{X} \in \gamma_\omega$, $1 \in \mathbb{X}$. We have to notice that in general form, $\mathbb{X} \in \gamma_\omega$ is not an ideal of $\mathbb{L}$.
(ii) Since $(St_l(\emptyset))_r = \emptyset$, by Proposition 3.3$(vi)$ and (vii), $\{\emptyset, \{1\}, \mathbb{L}\} \subseteq \gamma_\omega$.

DEFINITION 3.18. According to Theorem 3.17, the topological space, $(\mathbb{L}, \tau_\omega)$ is called a *stabilizer topology*.

Note. Since in any CKL-algebra, $\mathcal{S}_r(\mathbb{X}) \in \mathcal{I}d(\mathbb{L})$, for any $\mathbb{X} \subseteq \mathbb{L}$, every element of γ_ω is an ideal of $\mathbb{L}$.

Example 3.19.

(*i*) In Example 3.2(i), $(\mathbb{L}, \to, 1)$ is an L-algebra. By Proposition 3.3(i) and (vii), $\{1\} \in \omega(\mathbb{X})$, for all $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$. So, if $1 \notin \mathbb{X}$, then $\mathbb{X} \notin \gamma_\omega$. By some manipulations, we get $\gamma_\omega = \{\emptyset, \mathbb{L}, \{1\}\}$. Thus, $\tau_\omega = \{\emptyset, \mathbb{L}, \{1\}\}$. In addition, $\{1, b\} \notin \gamma_\omega$, because $\mathcal{S}_l(\{1, b\}) = \{1\}$ and by Proposition 3.3(vii), $\mathcal{S}_r(\{1\}) = \mathbb{L}$, then $\omega(\{1, b\}) = \mathbb{L}$, and so $\omega(\{1, b\}) \neq \{1, b\}$.

(*ii*) Assume $\mathbb{L}$ is an L-algebra as in Examples 3.2(ii) and 3.10. Then $\gamma_\omega = \{\emptyset, \{1\}, \{c, 1\}, \mathbb{L}\}$.

(*iii*) Consider an L-algebra as in Example 3.4(iii). Then $\gamma_\omega = \{\emptyset, \{1\}, \{b, 1\}, \mathbb{L}\}$.

(*iv*) According to Example 3.4(iv), $\gamma_\omega = \{\emptyset, \{1\}, \{b, 1\}, \{c, 1\}, \mathbb{L}\}$.

THEOREM 3.20. *The stabilizer topology* $(\mathbb{L}, \tau_\omega)$ *is*

(*i*) *connected.*

(*ii*) *locally connected.*

(*iii*) *Hausdorff space if and only if* $\mathbb{L} = \{1\}$.

THEOREM 3.21. *Let* $(\mathbb{L}, \tau_\omega)$ *be a stabilizer topology. If* $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$ *such that* $1 \in \mathbb{X}$, *then* $\overline{\mathbb{X}} = \mathbb{L}$.

PROOF: Suppose $\emptyset \neq \mathbb{X} \subseteq \mathbb{L}$ such that $1 \in \mathbb{X}$. Consider $x \in \mathbb{L}$. If $x = 1$, then $x \in \mathbb{X}$. Hence, $\overline{\mathbb{X}} = \mathbb{L}$. Now, suppose $1 \neq x \in \mathbb{L}$. Then there exists an open subset $\mathcal{U} \in \gamma_\omega$ such that $x \in \mathcal{U}$. Since $1 \in \mathcal{U}$, we have $\mathcal{U} \cap (\mathbb{X} \setminus \{x\}) \neq \emptyset$. Hence, $x \in \overline{\mathbb{X}}$, and so $\overline{\mathbb{X}} = \mathbb{L}$. □

Note. A topological space is called *separable* if it contains a countable dense subset.

COROLLARY 3.22. $(\mathbb{L}, \tau_\omega)$ is separable.

PROOF: Since $\{1\} \in \gamma_\omega$, by Theorem 3.21, $\overline{\{1\}} = \mathbb{L}$. Hence, $(\mathbb{L}, \tau_\omega)$ is separable. □

THEOREM 3.23. *$(\mathbb{L}, \tau_\omega)$ is Baire space, where $\mathbb{L}$ is a CKL-algebra.*

PROOF: Let $\mathcal{U} \in \tau_\omega$. Since $\mathbb{L}$ is a CKL-algebra, by Theorem 3.8, $\mathcal{U} \in \mathcal{I}d(\mathbb{L})$ and so $1 \in \mathcal{U}$. Since $1 \in \mathcal{U}$ by Theorem 3.21, $\overline{\mathcal{U}} = \mathbb{L}$. Thus, every open set of $(\mathbb{L}, \tau_\omega)$ is dense. On the other side, for each collection of open set $\mathcal{U}_n$, $1 \in \bigcap\limits_{n\in\mathbb{N}} \mathcal{U}_n$. Thus, by Theorem 3.21, $\overline{\bigcap\limits_{n\in\mathbb{N}} \mathcal{U}_n} = \mathbb{L}$, and so $\bigcap\limits_{n\in\mathbb{N}} \mathcal{U}_n$ is dense. Therefore, $(\mathbb{L}, \tau_\omega)$ is Baire space. $\square$

In the following example, we show that $(\mathbb{L}, \tau_\omega)$ is not a T_0-space or T_1-space.

Example 3.24. In Example 3.2(i), $\gamma_\omega = \{\emptyset, \mathbb{L}, \{1\}\}$. Since $b \neq c$, for $b, c \in \mathbb{L}$, there is not $\mathcal{U} \in \gamma_\omega$ such that $b \in \mathcal{U}$ and $c \notin \mathcal{U}$. Therefore, $(\mathbb{L}, \tau_\omega)$ is not a T_0-space. Obviously, $(\mathbb{L}, \tau_\omega)$ is not a T_1-space.

THEOREM 3.25. *Let $\mathbb{L}$ be a bounded CKL-algebra. If $\mathbb{L}$ has a cover of $\mathcal{U}_i \in \gamma_\omega$, for $i \in I$, then there exists $i \in I$ such that $\mathcal{U}_i = \mathbb{L}$. Particularly, $\mathbb{L}$ is compact.*

PROOF: Let $\mathbb{L}$ be bounded and $\{\mathcal{U}_i\}_{i\in I}$ be a cover of $\mathbb{L}$ such that, for all $i \in I$, $\mathcal{U}_i \in \gamma_\omega$ and $\mathbb{L} \subseteq \bigcup\limits_{i\in I} \mathcal{U}_i$. Since, for all $i \in I$, $\mathcal{U}_i \in \gamma_\omega$, by Theorem 3.8, we have $\mathcal{U}_i \in \mathcal{I}d(\mathbb{L})$. On the other side, $\mathbb{L}$ is bounded, then $0 \in \mathbb{L}$, and so $0 \in \bigcup\limits_{i\in I} \mathcal{U}_i$. Thus, there exists $i \in I$ such that $0 \in \mathcal{U}_i$. Since $\mathcal{U}_i \in \mathcal{I}d(\mathbb{L})$ and $0 \in \mathcal{U}_i$, by Proposition 2.9, $\mathcal{U}_i = \mathbb{L}$. Hence, there exists a finite family of $\{\mathcal{U}_i\}_{i\in I}$ such that $\mathbb{L} \subseteq \bigcup_{i=1}^{n} \mathcal{U}_i$. $\square$

3.2. Generalization of stabilizers on L-algebras

In this section, we introduce the generalization of stabilizers on L-algebra and investigate their properties and relation of them with stabilizers.

DEFINITION 3.26. Let $\mathbb{X}, \mathbb{Y}$ be two non-empty subsets of $\mathbb{L}$. Then a right(left) stabilizer of $\mathbb{X}$ with respect to $\mathbb{Y}$ are defined by

$$\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \{a \in \mathbb{L} \mid \text{for all } x \in \mathbb{X}, (a \to x) \to x \in \mathbb{Y}\},$$
$$\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) = \{a \in \mathbb{L} \mid \text{for all } x \in \mathbb{X}, (x \to a) \to a \in \mathbb{Y}\}.$$

Example 3.27. According to Example 3.2(i), let $\mathbb{X} = \{a, b\}$ and $\mathbb{Y} = \{b, c, 1\}$. Then $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \{b, c, 1\}$ and $\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) = \{b, c, 1\}$.

Proposition 3.28. Let $\mathbb{X}, \mathbb{Y}, \mathbb{X}_i, \mathbb{Y}_i$ be non-empty subsets of $\mathbb{L}$ and $\mathbb{I} \in \mathcal{I}d(\mathbb{L})$. Then the following statements hold:

(i) If $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \mathbb{L}$ or $\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) = \mathbb{L}$, then $\mathbb{X} \subseteq \mathbb{Y}$.

(ii) If $\mathbb{L}$ is a CKL-algebra and $\mathbb{I} \subseteq \mathbb{Y}$, then $\mathcal{S}t_r(\mathbb{I}, \mathbb{Y}) = \mathbb{L}$ and $\mathcal{S}t_l(\mathbb{I}, \mathbb{Y}) = \mathbb{L}$.

(iii) If $\mathbb{L}$ is a CKL-algebra, then $\mathcal{S}t_r(\mathbb{I}, \mathbb{I}) = \mathbb{L}$ and $\mathcal{S}t_l(\mathbb{I}, \mathbb{I}) = \mathbb{L}$.

(iv) $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}t_r(\mathbb{X}, \mathbb{I})$ and $\mathcal{S}_l(\mathbb{X}) \subseteq \mathcal{S}t_l(\mathbb{X}, \mathbb{I})$.

(v) If $\mathbb{L}$ is a KL-algebra, then $\mathcal{S}_r(\mathbb{X}, \{1\}) = \mathcal{S}t_r(\mathbb{X})$ and $\mathcal{S}_l(\mathbb{X}, \{1\}) = \mathcal{S}t_l(\mathbb{X})$.

(vi) If $\mathbb{X}_i \subseteq \mathbb{Y}_i$ and $\mathbb{X}_j \subseteq \mathbb{Y}_j$, then $\mathcal{S}_r(\mathbb{Y}_i, \mathbb{X}_j) \subseteq \mathcal{S}t_r(\mathbb{X}_i, \mathbb{Y}_j)$ and $\mathcal{S}_l(\mathbb{Y}_i, \mathbb{X}_j) \subseteq \mathcal{S}t_l(\mathbb{X}_i, \mathbb{Y}_j)$.

(vii) $\mathcal{S}t_r(\mathbb{X}, \bigcap_{i\in I} \mathbb{Y}_i) = \bigcap_{i\in I} \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$ and $\mathcal{S}t_l(\mathbb{X}, \bigcap_{i\in I} \mathbb{Y}_i) = \bigcap_{i\in I} \mathcal{S}t_l(\mathbb{X}, \mathbb{Y}_i)$.

Proof: (i) Assume $x \in \mathbb{X}$. Since $\mathbb{X} \subseteq \mathbb{L}$, we get $x \in \mathbb{L}$, and so $x \in \mathcal{S}t_r(\mathbb{X}, \mathbb{Y})$. Thus, for any $a \in \mathbb{X}$, $(x \to a) \to a \in \mathbb{Y}$. Consider $a = x$. Then by $(L1)$ we have

$$x = 1 \to x = (x \to x) \to x \in \mathbb{Y}.$$

Hence, $\mathbb{X} \subseteq \mathbb{Y}$. The proof of the other case is similar.

(ii) Clearly, $\mathcal{S}t_r(\mathbb{I}, \mathbb{Y}) \subseteq \mathbb{L}$. Assume $x \in \mathbb{L}$. Then for any $a \in \mathbb{I}$, by Proposition 2.5(iii), $a \lesssim (x \to a) \to a$. Thus by Proposition 2.9, $(x \to a) \to a \in \mathbb{I}$, and so $(x \to a) \to a \in \mathbb{Y}$. Hence, $x \in \mathcal{S}t_r(\mathbb{I}, \mathbb{Y})$, thus $\mathbb{L} \subseteq \mathcal{S}t_r(\mathbb{I}, \mathbb{Y})$. Therefore, $\mathbb{L} = \mathcal{S}t_r(\mathbb{I}, \mathbb{Y})$. The proof of the other case is similar.

(iii) By (ii) the proof is clear.

(iv) Let $a \in \mathcal{S}_r(\mathbb{X})$. Then for any $x \in \mathbb{X}$, $a \to x = x$ and clearly, $(a \to x) \to x = 1$. Since $\mathbb{I} \in \mathcal{I}d(\mathbb{L})$, by (I_1), $1 \in \mathbb{I}$ and so $(a \to x) \to x \in \mathbb{I}$. Thus, $a \in \mathcal{S}t_r(\mathbb{X}, \mathbb{I})$. Hence, $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}t_r(\mathbb{X}, \mathbb{I})$. The proof of the other case is similar.

(v) Since $\{1\} \in \mathcal{I}d(\mathbb{L})$, by (iv), we have $\mathcal{S}_r(\mathbb{X}) \subseteq \mathcal{S}t_r(\mathbb{X}, \{1\})$. Assume $a \in \mathcal{S}t_r(\mathbb{X}, \{1\})$. Then for any $x \in \mathbb{X}$, $(a \to x) \to x \in \{1\}$, and so $a \to x \lesssim x$. By hypothesis and Proposition 2.3, $x \lesssim a \to x$, and so $a \to x = x$. Hence, $x \in \mathcal{S}_r(\mathbb{X})$. Therefore, $\mathcal{S}_r(\mathbb{X}, \{1\}) = \mathcal{S}t_r(\mathbb{X})$. The proof of the other case is similar.

(vi) Assume $a \in \mathcal{S}_r(\mathbb{Y}_i, \mathbb{X}_j)$. Then for any $x \in \mathbb{Y}_i$, $(a \to x) \to x \in \mathbb{X}_j$. By assumption, $\mathbb{X}_i \subseteq \mathbb{Y}_i$, thus for $x \in \mathbb{X}_i$, we get $(a \to x) \to x \in \mathbb{X}_j$. In addition, $\mathbb{X}_j \subseteq \mathbb{Y}_j$, so $(a \to x) \to x \in \mathbb{Y}_j$. Hence, $a \in \mathcal{S}t_r(\mathbb{X}_i, \mathbb{Y}_j)$. Therefore, $\mathcal{S}_r(\mathbb{Y}_i, \mathbb{X}_j) \subseteq \mathcal{S}t_r(\mathbb{X}_i, \mathbb{Y}_j)$. The proof of the other case is similar.
(vii) Consider $a \in \mathcal{S}t_r(\mathbb{X}, \bigcap\limits_{i\in I} \mathbb{Y}_i)$. Then for any $x \in \mathbb{X}$, we have $(a \to x) \to x \in \bigcap\limits_{i\in I} \mathbb{Y}_i$. Thus, for all $i \in I$, we have $(a \to x) \to x \in \mathbb{Y}_i$. So, $a \in \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$. Hence, $a \in \bigcap\limits_{i\in I} \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$. Therefore, $\mathcal{S}t_r(\mathbb{X}, \bigcap\limits_{i\in I} \mathbb{Y}_i) \subseteq \bigcap\limits_{i\in I} \mathcal{S}t_r(\mathbb{X}, \mathbb{Y}_i)$. The proof of other side is similar. □

In the following example we show that the condition CKL-algebra in Proposition 3.28(ii) is necessary.

Example 3.29. According to Example 3.4(iii), $\mathbb{L}$ is not a CKL-algebra, since

$$b \to (c \to a) = b \to b = 1 \neq b = c \to a = c \to (b \to a).$$

Consider $\mathbb{I} = \{1\}$, $\mathbb{Y} = \{c, 1\}$ and $\mathbb{X} = \{a\}$. Then $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \{b, 1\} \neq \mathbb{L}$.

Proposition 3.30. Consider $\emptyset \neq \mathbb{X}, \mathbb{Y} \subseteq \mathbb{L}$. If for any $x, y \in \mathbb{L}$, $(x \to y) \to y = (y \to x) \to x$, then $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \mathcal{S}t_l(\mathbb{X}, \mathbb{Y})$.

Proof: Let $a \in \mathcal{S}t_r(\mathbb{X}, \mathbb{Y})$. Then for any $x \in \mathbb{X}$, $(a \to x) \to x \in \mathbb{Y}$. By assumption, $(x \to a) \to a \in \mathbb{Y}$, and so $a \in \mathcal{S}t_l(\mathbb{X}, \mathbb{Y})$. By the similar way, $\mathcal{S}t_l(\mathbb{X}, \mathbb{Y}) \subseteq \mathcal{S}t_r(\mathbb{X}, \mathbb{Y})$. Hence, $\mathcal{S}t_r(\mathbb{X}, \mathbb{Y}) = \mathcal{S}t_l(\mathbb{X}, \mathbb{Y})$. □

Proposition 3.31. Consider $\mathbb{L}$ be a CKL-algebra and $\mathbb{I}, \mathbb{J} \in \mathcal{I}d(\mathbb{L})$. Then $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \in \mathcal{I}d(\mathbb{L})$.

Proof: By $(L1)$, since for any $a \in \mathbb{I}$, $(1 \to a) \to a = 1 \in \mathbb{J}$, we get $1 \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Assume $a, a \to b \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Then for any $x \in \mathbb{I}$, $(a \to x) \to x \in \mathbb{J}$ and $((a \to b) \to x) \to x \in \mathbb{J}$. Since $x \in \mathbb{I}$, by assumption and Proposition 2.5(ii), $x \lesssim a \to x$, and by Proposition 2.9 we get $a \to x \in \mathbb{I}$. So, $((a \to b) \to (a \to x)) \to (a \to x) \in \mathbb{J}$. In addition, by Proposition 2.5(viii) we have $b \to x \lesssim (a \to b) \to (a \to x)$, and by Proposition 2.5(vii), we have

$$((a \to b) \to (a \to x)) \to (a \to x) \lesssim (b \to x) \to (a \to x),$$

Since $((a \to b) \to (a \to x)) \to (a \to x) \in \mathbb{J}$ and $\mathbb{J} \in \mathcal{I}d(\mathbb{L})$, by Proposition 2.9, $(b \to x) \to (a \to x) \in \mathbb{J}$. Moreover, by Proposition 2.5(xii), $a \to x = ((a \to x) \to x) \to x$. Thus

$$\begin{aligned}((a \to x) \to x) \to ((b \to x) \to x) &= (b \to x) \to (((a \to x) \to x) \to x)\\ &= (b \to x) \to (a \to x) \in \mathbb{J}.\end{aligned}$$

From $\mathbb{J} \in \mathcal{I}d(\mathbb{L})$ and $(a \to x) \to x \in \mathbb{J}$, by (I_2), we have $(b \to x) \to x \in \mathbb{J}$. Hence, $b \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Therefore, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \in \mathcal{I}d(\mathbb{L})$. □

THEOREM 3.32.

(*i*) *For any* $\mathbb{I}, \mathbb{J} \in \mathcal{I}d(\mathbb{L})$, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$.

(*ii*) *If* $\mathbb{L}$ *is a* CKL*-algebra, then* $\mathcal{S}t_r(\mathbb{I}, \mathbb{J})$ *is the greatest ideal of* $\mathbb{L}$ *such that* $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$.

PROOF: (*i*) Let $a \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I}$. Then $a \in \mathbb{I}$ and $a \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Thus for any $x \in \mathbb{I}$, $(a \to x) \to x \in \mathbb{J}$. Consider $x = a$, so by $(L1)$, $(a \to a) \to a = a \in \mathbb{J}$. Thus, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$.

(*ii*) By (*i*), obviously, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$. Suppose there exists $\mathbb{K} \in \mathcal{I}d(\mathbb{L})$, where $\mathbb{K} \cap \mathbb{I} \subseteq \mathbb{J}$. We show that $\mathbb{K} \subseteq \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. For this, assume $a \in \mathbb{K}$ and $x \in \mathbb{I}$. Thus by Proposition 2.5(*ii*) and (*iii*), $a, x \lesssim (a \to x) \to x$. Since $\mathbb{I}, \mathbb{K} \in \mathcal{I}d(\mathbb{L})$, by Proposition 2.9, we get $(a \to x) \to x \in \mathbb{K} \cap \mathbb{I}$, and so $(a \to x) \to x \in \mathbb{J}$. Thus, $a \in \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$, and so $\mathbb{K} \subseteq \mathcal{S}t_r(\mathbb{I}, \mathbb{J})$. Therefore, $\mathcal{S}t_r(\mathbb{I}, \mathbb{J})$ is the greatest ideal of $\mathbb{L}$ such that $\mathcal{S}t_r(\mathbb{I}, \mathbb{J}) \cap \mathbb{I} \subseteq \mathbb{J}$. □

COROLLARY 3.33. Assume $\mathbb{L}$ is a CKL-algebra. Then $\langle \mathcal{I}d(\mathbb{L}), \sqcap, \sqcup, \{1\}, \mathbb{L} \rangle$ is a relative pseudo-complement lattice where $\mathcal{S}t_r(\mathbb{I}, \mathbb{J})$ is the relative pseudo-complement of $\mathbb{I}$ with respect to $\mathbb{J}$ in $\mathcal{I}d(\mathbb{L})$ such that $\mathbb{I} \sqcap \mathbb{J} = \mathbb{I} \cap \mathbb{J}$ and $\mathbb{I} \sqcup \mathbb{J}$ is a generated ideal of $\mathbb{L}$ contains $\mathbb{I} \cup \mathbb{J}$.

PROOF: By Theorem 3.32 and [5, Definition 3.5 and Proposition 3.6], the proof is straightforward. □

4. Conclusion

The aim of this paper is to introduce the notion of stabilizers in L-algebras and develop stabilizer theory in L-algebras. In this paper, the notions of left and right stabilizers are introduced and some properties related to them has been investigated. Then, the relations among stabilizers, ideals and co-annihilators are discussed. Also, it was shown that the set of all ideals in a CKL-algebra forms a relative pseudo-complemented lattice. Also, it was proved that all right stabilizers in CKL-algebra are ideals. Then by using the right stabilizers, a basis for a topology on L-algebra was produced. Finally, it was proved that the generated topology by this basis is Baire, connected, locally connected and separable and the other properties of this topology are investigated.

In future, we can introduce the notions of fuzzy left and right stabilizers and investigate their related properties and discuss the relations among fuzzy stabilizers, fuzzy ideals and fuzzy co-annihilators.

Acknowledgements. The authors are very indebted to the editor and anonymous referees for their careful reading and valuable suggestions which helped to improve the readability of the paper.

References

[1] M. Aaly Kologani, *Relations between L-algebras and other logical algebras*, **Journal of Algebraic Hyperstructures and Logical Algebras**, vol. 4(1) (2023), pp. 27–46, DOI: https://doi.org/10.52547/HATEF.JAHLA.4.1.3.

[2] M. Aaly Kologani, *Some results on L-algebars*, **Soft Computing**, vol. 27 (2023), pp. 13765–13777, DOI: https://doi.org/10.1007/s00500-023-08965-5.

[3] R. A. Borzooei, M. Aaly Kologani, *Stabilizers topology of hoops*, **Algebraic Structures and Their Applications**, vol. 1(1) (2014), pp. 35–48, DOI: https://doi.org/10.1090/S0002-9904-1935-06166-X.

[4] X. Y. Cheng, M. Wang, W. Wang, J. T. Wang, *Stabilizers in EQ-algebras*, **Open Mathematics**, vol. 17 (2019), pp. 998–1013, DOI: https://doi.org/10.1515/math-2019-0078.

[5] L. C. Ciungu, *Results in L-algebras*, **Algebra Univers**, vol. 87 (2021), p. 7, DOI: https://doi.org/10.1007/s00012-020-00695-1.

[6] M. Haveshki, M. Mohamadhasani, *Stabilizer in BL-algebras and its properties*, **International Mathematical Forum**, vol. 5(57) (2010), pp. 2809–2816, DOI: https://doi.org/10.1090/S0002-9904-1935-06166-X.

[7] B. Meng, X. L. Xin, *Generalized co-annihilator of BL-algebras*, **Open Mathematics**, vol. 13(1) (2015), pp. 639–654, DOI: https://doi.org/10.1515/math-2015-0060.

[8] W. Rump, *L-algebras, self-similarity, and ℓ-groups*, **Journal of Algebra**, vol. 320 (2008), pp. 2328–2348, DOI: https://doi.org/10.1016/j.jalgebra.2008.05.033.

[9] W. Rump, *A general Glivenko theorem*, **Algebra Universalis**, vol. 61 (2009), pp. 455–473, DOI: https://doi.org/10.1007/s00012-009-0018-y.

[10] W. Rump, Y. Yang, *Interval in ℓ-groups as L-algebras*, **Algebra Universalis**, vol. 67(2) (2012), pp. 121–130, DOI: https://doi.org/10.1007/s00012-012-0172-5.

[11] W. J. Tao, A. B. Saeid, P. F. He, *Stabilizers in MTL-algebras*, **Journal of Intelligent and Fuzzy Systems**, vol. 35 (2018), pp. 717–727, DOI: https://doi.org/10.3233/JIFS-171105.

[12] Y. L. Wu, J. Wang, Y. C. Yang, *Lattice-ordered effect algebras and L-algebras*, **Fuzzy Sets and Systems**, vol. 369 (2019), pp. 103–113, DOI: https://doi.org/10.1016/j.fss.2018.08.013.

[13] Y. L. Wu, Y. C. Yang, *Orthomodular lattices as L-algebras*, **Soft Computing**, vol. 24 (2020), pp. 14391–14400, DOI: https://doi.org/10.1007/s00500-020-05242-7.

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