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LIFTING RESULTS FOR FINITE DIMENSIONS TO THE TRANSFINITE IN SYSTEMS OF VARIETIES USING ULTRAPRODUCTS

Abstract

We redefine a system of varieties definable by a schema of equations to include finite dimensions. Then we present a technique using ultraproducts enabling one to lift results proved for every finite dimension to the transfinite. Let **Ord** denote the class of all ordinals. Let $\langle \mathbf{K}_\alpha : \alpha \in \mathbf{Ord}\rangle$ be a system of varieties definable by a schema. Given any ordinal α, we define an operator Nr_α that acts on $\mathbf{K}_\beta$ for any $\beta > \alpha$ giving an algebra in $\mathbf{K}_\alpha$, as an abstraction of taking α-neat reducts for cylindric algebras. We show that for any positive k, and any infinite ordinal α that $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k+1}$ cannot be axiomatized by a finite schema over $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$ given that the result is valid for all finite dimensions greater than some fixed finite ordinal. We apply our results to cylindric algebras and Halmos quasipolyadic algebras with equality. As an application to our algebraic result we obtain a strong incompleteness theorem (in the sense that validitities are not captured by finitary Hilbert style axiomatizations) for an algebraizable extension of $L_{\omega,\omega}$.

Keywords: algebraic logic, systems of varieties, ultraproducts, non-finite axiomaitizability.

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1. Introduction

We follow the notation of [1, 2]. Fix $2 < n < \omega$. In [5] Hirsch, Hodkinson and Maddux prove that for any positive $k \geq 1$, $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k+1} \subsetneq$

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$\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$; in fact this gap between the two varieties cannot be finitely axiomatized [4]. In [3], this result was generalized to other algebras of logic (or cylindric-like algebras) such as Pinter's substitution algebras, Halmos' polyadic algebras with and without equality; all of dimension n and for infinite dimensions for all the aforementioned algebras, together with cylindrc algebras not dealt with in [5]. In [3], the result was proved first for the finite dimensional case, then it was lifted to the transfinite using a lifting technique originating with Monk in proving non-finite axiomatizability of RCA_ω by a finite schema of equations, cf. [2]. The technique involves the use of ultraproducts, to lift results proved for all finite dimensions to the transfinite. Here we show that this technique lends itself to a much wider context.

We generalize this technique to the very general notion of *a system of varieties definable by a schema of equations* introduced by Henkin et al. cf. [2, Definition 5.6.11] to encompass all such aforementioned systems of varieties of algebras and potentially much more. A substantial new addition here is that we allow finite dimensions in our definition of a system of varieties definable by schema. What is basically characteristc of such systems $\langle \mathbf{K}_\alpha : \alpha \in \mathbf{Ord}\rangle$, is that for each ordinal $\alpha \in \mathbf{Ord}$, they define a variety of algebra of dimension α, $\mathbf{K}_\alpha$, such that if $\alpha < \beta$, and $\mathfrak{A} \in \mathbf{K}_\beta$, then the reduct of $\mathfrak{A}$ obtained by discarding the operations indexed by ordinals in β and outside α, call it $\mathfrak{Rd}_\alpha\mathfrak{A}$, is in $\mathbf{K}_\alpha$. Furthermore, one can navigate between various dimensions using more complex operators, like the *neat reduct operator* denoted by Nr_μ, where μ is any ordinal. For $\alpha < \beta$, and $\mathfrak{A} \in \mathbf{K}_\beta$ say, then $\mathsf{Nr}_\alpha\mathfrak{A} \in \mathbf{K}_\alpha$ and $\mathsf{Nr}_\alpha\mathfrak{A} \subseteq \mathfrak{Rd}_\alpha\mathfrak{A}$. Finally, for infinite dimensions the system is captured (defined) uniformy by a single schema of equations. For example the system for $\mathbf{CA} = \langle \mathsf{CA}_\alpha : \alpha \geq \omega\rangle$, the indices i, j in the operations of cylindrifcations and diagonal elements, c_i and d_{ij} $(i, j \in \omega)$ vary according to one finite schema that is finite in a two sorted sense.

2. Preliminarlies

We start with the definition counting in finite dimensions for system of varieties definable by a schema. Counting in finite dimensional algebras is new.

Definition 2.1.

(i) Let $2 \leq m \in \omega$. A finite m type schema is a quadruple $t = (T, \delta, \rho, c)$ such that T is a set, δ and ρ map T into ω, $c \in T$, and $\delta c = \rho c = 1$ and $\delta f \leq m$ for all $f \in T$.

(ii) A type schema as in (i) defines a signature t_n for each $n \geq m$ as follows. The domain T_n of t_n is

$$T_n = \{(f, k_0, \ldots, k_{\delta f-1}) : f \in T, k \in {}^{\delta f}n\}.$$

For each $(f, k_0, \ldots, k_{\delta f-1}) \in T_n$ we set $t_n(f, k_0, \ldots, k_{\delta f-1}) = \rho f$.

(iii) A system $(\mathbf{K}_n : n \geq m)$ of classes of algebras is of type schema t if for each $n \geq m$, $\mathbf{K}_n$ is a class of algebras having signature t_n.

Definition 2.2. Let t be a finite m type schema.

(i) With each $m \leq n \leq \beta$ we associate a language L^t_n in the signature t_n: for each $f \in T$ and $k \in {}^{\delta f}n$, we have a function symbol $f_{k0,\ldots,k(\delta f-1)}$ of rank ρf.

(ii) Let $m \leq \beta \leq n$, and let $\eta \in {}^{\beta}n$ be an injection. We associate with each term τ of L^t_β a term $\eta^+\tau$ of L^t_n. For each $\kappa \in \omega, \eta^+ v_\kappa = v_\kappa$. If $f \in T, k \in {}^{\delta f}\alpha$, and $\sigma_1, \ldots, \sigma_{\rho f-1}$ are terms of L^t_β, then

$$\eta^+ f_{k(0),\ldots,k(\delta f-1)}\sigma_0, \ldots, \sigma_{\rho f-1} = f_{\eta(k(0)),\ldots,\eta(k(\delta f-1))}\eta^+\sigma_0 \ldots \eta^+\sigma_{\rho f-1}.$$

Then we associate with each equation $e := \sigma = \tau$ of L^t_β the equation $\eta^+\sigma = \eta^+\tau$ of L^t_α, which we denote by $\eta^+(e)$. We say that $\eta^+(e)$ is an n instance of e, obtained by applying the injective map η.

(iii) A system $\mathbf{K} = (\mathbf{K}_n : n \geq m)$ of finite m type schema t is a *complete system of varieties definable by a schema*, if there is a system $(\Sigma_n : n \geq m)$ of equations such that $\mathsf{Mod}(\Sigma_n) = \mathbf{K}_n$, and for $n \leq m < \omega$ if $e \in \Sigma_n$ and $\rho : n \to m$ is an injection, then $\rho^+ e \in \Sigma_m$; $(\mathbf{K}_\alpha : \alpha \geq \omega)$ is a system of varieties definable by a schema and $\Sigma_\omega = \bigcup_{n \geq m} \Sigma_n$.

Definition 2.3.

(1) Let α, β be ordinals, $\mathfrak{A} \in \mathbf{K}_\beta$ and $\rho : \alpha \to \beta$ be an injection. We assume for simplicity of notation, and without any loss, that in addition to cylindrifiers, we have only one unary function symbol f such that $\rho(f) = \delta(f) = 1$. (The arity is one, and f has only one index.)

Then $\mathfrak{Rd}^\rho_\alpha\mathfrak{A}$ is the α-dimensional algebra obtained for $\mathfrak{A}$ by defining for $i \in \alpha$ $\mathsf{c_i}$ by $\mathsf{c}_{\rho(\mathsf{i})}$, and f_i by $f_{\rho(i)}$. $\mathfrak{Rd}_\alpha\mathfrak{A}$ is $\mathfrak{Rd}^\rho_\alpha\mathfrak{A}$ when ρ is the inclusion map.

(2) As in the first part we assume only the existence of one unary operator with one index. Let $\mathfrak{A} \in \mathbf{K}_\beta$, and $x \in A$. The dimension set of x, denoted by Δx, is the set $\Delta x = \{i \in \alpha : \mathsf{c}_i x \neq x\}$. We assume that if $\Delta x \subseteq \alpha$, then $\Delta f(x) \leq \alpha$. Then $\mathsf{Nr}_\alpha\mathfrak{B}$ is the subuniverse of $\mathfrak{Rd}_\alpha\mathfrak{B}$ consisting only of α dimensional elements.

(3) For $K \subseteq \mathbf{K}_\beta$ and an injection $\rho : \alpha \to \beta$, then $\mathfrak{Rd}^\rho_\alpha K = \{\mathfrak{Rd}^\rho_\alpha\mathfrak{A} : \mathfrak{A} \in K\}$ and $\mathsf{Nr}_\alpha K = \{\mathsf{Nr}_\alpha\mathfrak{A} : \mathfrak{A} \in K\}$.

3. Lifting results to the transfinite using ultraproducts

3.1. Main result

We start with a Definition:

Definition 3.1. Let $(\mathbf{K}_\alpha : \alpha \geq 3)$ be a complete system of varieties definable by a schema. Then for $\alpha \leq \mu \leq \beta$ and $K \subseteq \mathbf{K}_\beta$, $\mathsf{Nr}_\mu K := \{\mathsf{Nr}_\mu\mathfrak{A} : \mathfrak{A} \in K\}$.

The hypothesis in the next theorem presupposes the existence of certain finite dimensional algebras that we know do exist for certain cylindric-like algebras. This will be witnessed in a while, cf. Corollary 3.3. Also, in what follows, the symbol $\mathbf{S}$ stands for the operation of forming subalgebras

Theorem 3.2. *Let $(\mathbf{K}_\alpha : \alpha \geq 3)$ be a complete system of varieties definable by a schema. Assume that for $3 \leq m < n < \omega$, there is an m dimensional algebra $\mathfrak{C}(m, n, r)$ such that*

(1) $\mathfrak{C}(m, n, r) \in \mathbf{S}\mathsf{Nr}_m\mathbf{K}_n$,

(2) $\mathfrak{C}(m, n, r) \notin \mathbf{S}\mathsf{Nr}_m\mathbf{K}_{n+1}$,

(3) $\Pi_{r\in\omega}\mathfrak{C}(m, n, r) \in \mathbf{S}\mathsf{Nr}_m\mathbf{K}_\omega$,

(4) *For $m < n$ and $k \geq 1$, there exists $x_n \in \mathfrak{C}(n, n + k, r)$ such that $\mathfrak{C}(m, m + k, r) \cong \mathfrak{Rl}_{x_n}\mathfrak{C}(n, n + k, r)$.*

Assume that for any $3 < \alpha < \beta$, $\mathsf{SNr}_\alpha\mathbf{K}_\beta$ *is a variety. Then for any ordinal* $\alpha \geq \omega$ *and finite number* $k \geq 1$, *for every ordinal* $l \geq k+1$, $\mathsf{SNr}_\alpha\mathbf{K}_{\alpha+l}$ *is not axiomatizable by a finite schema over* $\mathsf{SNr}_\alpha\mathbf{K}_{\alpha+k}$.

PROOF: The proof is divided into 3 parts.

Part 1: Let α be an infinite ordinal. Let X be any finite subset of α and let

$$I = \{\Gamma : X \subseteq \Gamma \subseteq \alpha,\ |\Gamma| < \omega\}.$$

For each $\Gamma \in I$ let $M_\Gamma = \{\Delta \in I : \Delta \supseteq \Gamma\}$ and let F be any ultrafilter over I such that for all $\Gamma \in I$ we have $M_\Gamma \in F$ (such an ultrafilter exists because $M_{\Gamma_1} \cap M_{\Gamma_2} = M_{\Gamma_1 \cup \Gamma_2}$).

For each $\Gamma \in I$ let ρ_Γ be a bijection from $|\Gamma|$ onto Γ. For each $\Gamma \in I$ let $\mathfrak{A}_\Gamma, \mathfrak{B}_\Gamma$ be $\mathbf{K}_\alpha$-type algebras.

We claim that

(*) If for each $\Gamma \in I$ we have $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma = \mathfrak{Rd}^{\rho_\Gamma}\mathfrak{B}_\Gamma$, then we have

$$\Pi_{\Gamma/F}\mathfrak{A}_\Gamma = \Pi_{\Gamma/F}\mathfrak{B}_\Gamma.$$

The proof is standard using Los' theorem.

Indeed, $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma$, $\Pi_{\Gamma/F}\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\rho$ and $\Pi_{\Gamma/F}\mathfrak{B}_\Gamma$ all have the same universe, by assumption. Also each operator o of $\mathbf{K}_\alpha$ is also the same for both ultraproducts, because $\{\Gamma \in I : \dim(o) \subseteq \mathsf{rng}(\rho_\Gamma)\} \in F$.

Now we claim that

(**) if $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma \in \mathbf{K}_{|\Gamma|}$, for each $\Gamma \in I$, then $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \in \mathbf{K}_\alpha$. For this, it suffices to prove that each of the defining axioms for $\mathbf{K}_\alpha$ holds for $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma$.

Let $\sigma = \tau$ be one of the defining equations for $\mathbf{K}_\alpha$, the number of dimension variables is finite, say n.

Take any $i_0, i_1, \ldots, i_{n-1} \in \alpha$. We have to prove that

$$\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0 \ldots, i_{n-1}).$$

Suppose that they are all in $\mathsf{rng}(\rho_\Gamma)$, say $i_0 = \rho_\Gamma(j_0),\ i_1 = \rho_\Gamma(j_1),\ \ldots, i_{n-1} = \rho_\Gamma(j_{n-1})$, then $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma \models \sigma(j_0, \ldots, j_{n-1}) = \tau(j_0, \ldots j_{n-1})$, since $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{A}_\Gamma \in \mathbf{K}_{|\Gamma|}$, so $\mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0 \ldots, i_{n-1})$.

Hence $\{\Gamma \in I : \mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1})\} \supseteq \{\Gamma \in I : i_0, \ldots, i_{n-1} \in \mathsf{rng}(\rho_\Gamma\} \in F$. It now easily follows that

$$\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \models \sigma(i_0, \ldots, i_{n-1}) = \tau(i_0, \ldots, i_{n-1}).$$

Thus $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \in \mathbf{K}_\alpha$, and we are done.

Part 2: Let $k \geq 1$ and $r \in \omega$. Let α, I, F and ρ_Γ be as above and assume the hypothesis of the theorem. Let $\mathfrak{C}^r_\Gamma$ be an algebra similar to $\mathbf{K}_\alpha$ such that

$$\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}^r_\Gamma = \mathfrak{C}(|\Gamma|, |\Gamma| + k, r).$$

Let

$$\mathfrak{B}^r = \Pi_{\Gamma/F \in I}\mathfrak{C}^r_\Gamma.$$

Then we have

1. $\mathfrak{B}^r \in \mathbf{S}S\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$ and

2. $\mathfrak{B}^r \notin \mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k+1}$.

For each $\Gamma \in I$ let $\mathfrak{C}(|\Gamma|, k)$ be an algebra having the same signature as $\mathbf{K}_{|\Gamma|+k}$ such that $\mathsf{Nr}_{|\Gamma|}\mathfrak{C}(|\Gamma|, k) \cong \mathfrak{C}(|\Gamma|, |\Gamma| + k, r)$. Let σ_Γ be a one to one function $(|\Gamma| + k) \to (\alpha + k)$ such that $\rho_\Gamma \subseteq \sigma_\Gamma$ and $\sigma_\Gamma(|\Gamma| + i) = \alpha + i$ for every $i < k$. Let $\mathfrak{A}_\Gamma$ be an algebra similar to a $\mathbf{K}_{\alpha+k}$ such that $\mathfrak{Rd}^{\sigma_\Gamma}\mathfrak{A}_\Gamma = \mathfrak{C}(|\Gamma|, k)$. By (**) with $\alpha + k$ in place of α, $\{\alpha + i : i < k\}$ in place of X, $\{\Gamma \subseteq \alpha + k : |\Gamma| < \omega,\ X \subseteq \Gamma\}$ in place of I, and with σ_Γ in place of ρ_Γ, we know that $\Pi_{\Gamma/F}\mathfrak{A}_\Gamma \in \mathbf{K}_{\alpha+k}$.

Part 3: Now we prove the third part of the theorem, putting the superscript r to use. Let $l \geq k + 1$, and we can assume that $l \leq \omega$. Recall that $\mathfrak{B}^r = \Pi_{\Gamma/F}\mathfrak{C}^r_\Gamma$, where $\mathfrak{C}^r_\Gamma$ has the type of $\mathbf{K}$ and $\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}^r_\Gamma = \mathfrak{C}(|\Gamma|, |\Gamma| + k, r)$. We know that

$$\Pi_{r/U}\mathfrak{Rd}^{\rho_\Gamma}\mathfrak{C}^r_\Gamma = \Pi_{r/U}\mathfrak{C}(|\Gamma|, |\Gamma| + k, r) \subseteq \mathsf{Nr}_{|\Gamma|}\mathfrak{A}_\Gamma,$$

for some $\mathfrak{A}_\Gamma \in \mathbf{K}_{|\Gamma|+\omega}$. Let $\lambda_\Gamma : |\Gamma| + l \to \alpha + l$ extend $\rho_\Gamma : |\Gamma| \to \Gamma\ (\subseteq \alpha)$ and satisfy

$$\lambda_\Gamma(|\Gamma| + i) = \alpha + i$$

for $i < l$. Now in this part, we take the l reduct of $\mathfrak{A}_\Gamma$. Accordingly, let $\mathfrak{F}_\Gamma$ be a $\mathbf{K}_{\alpha+l}$ type algebra such that $\mathfrak{Rd}^{\lambda_\Gamma}\mathfrak{F}_\Gamma = \mathfrak{Rd}_l\mathfrak{A}_\Gamma$. But now as before, $\Pi_{\Gamma/F}\mathfrak{F}_\Gamma \in \mathbf{K}_{\alpha+l}$, and

$$\begin{aligned}\Pi_{r/U}\mathfrak{B}^r &= \Pi_{r/U}\Pi_{\Gamma/F}\mathfrak{C}^r_\Gamma\\ &\cong \Pi_{\Gamma/F}\Pi_{r/U}\mathfrak{C}^r_\Gamma\\ &\subseteq \Pi_{\Gamma/F}\mathsf{Nr}_{|\Gamma|}\mathsf{c}A_\Gamma\\ &= \Pi_{\Gamma/F}\mathsf{Nr}_{|\Gamma|}\mathfrak{Rd}^{\lambda_\Gamma}\mathfrak{F}_\Gamma\\ &\subseteq \mathsf{Nr}_\alpha\Pi_{\Gamma/F}\mathfrak{F}_\Gamma.\end{aligned}$$

We are ready to prove the negative axiomatizability result. It is a Los' argument at heart, modulo some adjustments, because we are dealing with schemes, so that we will not deal with a finite set of equations, but rather α instances of a finite set of equations in the signature of $\mathbf{K}_\omega$. Let $k \geq 1$ and $l \geq k+1$. Assume for contradiction that $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ is axiomatizable by a finite schema over $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$. We can assume that there is only one equation, such that all its α instances, axiomatize $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ over $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$. So let σ be an equation in the signature of $\mathbf{K}_\omega$ and let E be its α instances such that for any $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$ we have $\mathfrak{A} \in \mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ iff $\mathfrak{A} \models E$. Then for all $r \in \omega$, there is an instance of σ, σ_r say, such that $\mathfrak{B}^r$ does not model σ_r. σ_r is obtained from σ by some injective map $\mu_r : \omega \to \alpha$. For $r \in \omega$, let $v_r \in {}^\alpha\alpha$, be an injection such that $\mu_r(i) = v_r(i)$ for $i \in \mathsf{ind}(\sigma_r)$, and let $\mathfrak{A}_r = \mathfrak{Rd}^{v_r}\mathfrak{B}^r$. Now $\Pi_{r/U}\mathfrak{A}_r \models \sigma$. But then

$$\{r \in \omega : \mathfrak{A}_r \models \sigma\} = \{r \in \omega : \mathfrak{B}^r \models \sigma_r\} \in U,$$

contradicting that $\mathfrak{B}^r$ does not model σ_r for all $r \in \omega$. □

3.2. Applications

In this section we lift results proved for all finite dimensions to the transfinite using ultraproducts. Let α be an ordinal. The next result is new:

Corollary 3.3. For any ordinal $\alpha \geq \omega$, any positive $k \geq 1$, and any ordinal $l \geq k+1$, the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+l}$ is not axiomatizable by a finite schema over the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k}$.

Proof: Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m,n,r)$ be the algebra $\mathfrak{Ca}(\mathbf{H})$ where $\mathbf{H} = H^{n+1}_m(\mathfrak{A}(n,r),\omega))$, is the CA_m atom structure consisting of all $n+1$–wide m–dimensional wide ω hypernetworks [4, Definition 12.21] on $\mathfrak{A}(n,r)$ as defined in [4, Definition 15.2]. Then $\mathfrak{C}(m,n,r) \in \mathsf{CA}_m$. Then for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{CA}_n$, $\mathfrak{C}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{n+1}$

and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RCA}_m$, cf. [4, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13]. Take

$$x_n = \{f \in H_n^{n+k+1}(\mathfrak{A}(n,r),\omega); m \le j < n \to \exists i < m, f(i,j) = \mathsf{Id}\}.$$

Then $x_n \in \mathfrak{C}(n, n+k, r)$ and $\mathsf{c}_i x_n \cdot \mathsf{c}_j x_n = x_n$ for distinct $i, j < m$. Furthermore (*), $I_n : \mathfrak{C}(m, m+k, r) \cong \mathfrak{Rl}_{x_n}\mathfrak{Rd}_m\mathfrak{C}(n, n+k, r)$ via the map, defined for $S \subseteq H_m^{m+k+1}(\mathfrak{A}(m+k,r),\omega))$, by

$$I_n(S) = \{f \in H_n^{n+k+1}(\mathfrak{A}(n,r),\omega) : f \restriction {}^{\le m+k+1}m \in S,$$
$$\forall j(m \le j < n \to \exists i < m, f(i,j) = \mathsf{Id})\}.$$

Applying Theorem 3.2 we get the required. □

We let QEA_α stand for the class of quasipolyadic equality algebras of dimension α as defined in [2]. We use the formalism given in the appendix of [3] following Sain and Thompson [6] where this variety is denoted by FPEA_α short for *finitary polyadic equality algebras of dimension* α. For $\alpha < \omega$, QEA_α is definitionally equivalent to Halmos' polyadic algebras of dimension α denoted in [2] by PEA_α. A *quasi-polyadic equality set algebra* is an algebra of the form $\langle \mathfrak{B}({}^\alpha U), \mathsf{C}_i, \mathsf{S}_{[i|j]}, \mathsf{S}_{[i,j]}, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$ where for $i, j \in \alpha$, $\mathsf{S}_{[i,j]}$ is the unary operation of substitution corresponding to the transposition $[i,j]$ defined for $X \subseteq^\alpha U$ as follows: $S_{[i,j]}X = \{s \in {}^\alpha U : s \circ [i,j] \in X\}$. The abstract variety $\mathsf{QEA}_\alpha(\mathsf{FPEA}_\alpha)$ is defined by a finite schema of equations (in [6]) which holds in the class of quasipolyadic set algebras of the same dimension. This schema is recalled in the appendix of [3].

Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m,n,r)$ be the algebra $\mathfrak{Ca}(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n,r),\omega))$, is the CA_m atom structure consisting of all $n+1$–wide m–dimensional wide ω hypernetworks [4, Definition 12.21] on $\mathfrak{A}(n,r)$ as defined in [4, Definition 15.2]. Then $\mathfrak{C}(m,n,r) \in \mathsf{CA}_m$, and it can be easily expanded to a QEA_m, since $\mathfrak{C}(m,n,r)$ is 'symmetric', in the sense that it allows a polyadic equality expansion by defining substitution operations corresponding to transpositions. This follows by observing that $\mathbf{H}$ is obviously symmetric in the following exact sense: For $\theta : m \to m$ and $N \in \mathbf{H}$, $N\theta \in \mathbf{H}$, where $N\theta$ is defined by $(N\theta)(x,y) = N(\theta(x),\theta(y))$. Hence, the binary polyadic operations defined on the atom structure $\mathbf{H}$ the obvious way (by swapping co–ordinates) lifts to polyadic operations of its complex algebra $\mathfrak{C}(m,n,r)$. In more detail, for a transposition $\tau : m \to m$, and $X \subseteq \mathbf{H}$, define $\mathsf{s}_\tau(X) = \{N \in \mathbf{H} : N\tau \in X\}$. Furthermore, for any $r \in \omega$

and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m,n,r) \in \mathsf{Nr}_m\mathsf{QEA}_n$, $\mathfrak{Rd}_{ca}\mathfrak{C}(m,n,r) \notin \mathbf{S}\mathsf{Nr}_m\mathsf{CA}_{n+1}$ and $\Pi_{r/U}\mathfrak{C}(m,n,r) \in \mathsf{RQEA}_m$ by easily adapting [4, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13] to the QEA context.

THEOREM 3.4. *Let $2 < m < n < \omega$. For $\mathbf{K} \in \{\mathsf{CA}, \mathsf{QEA}\}$, any positive $k \geq 1$,for any ordinal $l \geq k+1$, the variety $\mathbf{S}\mathsf{Nr}_m\mathbf{K}_{m+l}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_m\mathbf{K}_{m+k}$.*

Now from Theorem 3.2 we get (the known [3, Corollary 14]):

THEOREM 3.5. *For any ordinal $\alpha \geq \omega$, for $\mathbf{K} \in \{\mathsf{CA}, \mathsf{QEA}\}$, for any positive $k \geq 1$, and for any ordinal $l \geq k+1$, the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+l}$ is not finitely axiomatizable over the variety $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$.*

We denote by L_ω the basic algebraizable typeless extension of $L_{\omega,\omega}$ with usual Tarskian square semantics dealt with in [2, § 4.3]. For provability we use the basic proof system in [2, p. 157, § 4.3] which is a natural algebrazable (in the standard Blok-Pigizzi sense) extension of a complete calculas for $L_{\omega,\omega}$ expressed in terms of so-called restricted formulas. A restricted formula is one in which the variables in its atomic subformulas appear only in their natural order. We write $\vdash_{\omega+k}$ for provability using $\omega+k$ variables where k is any positive number. As an immediate corollary to the result proved in Corollary 3.3, we get:

THEOREM 3.6. *For any positive number $k \geq 1$, there is no finite schemata of L_ω whose set Σ of instances satisfies $\Sigma \vdash_{\omega+k} \phi \Longleftrightarrow \vdash_{\omega+k+1} \phi$.*

The last Theorem says that using only one extra variable to proofs adds an 'infinite' strength to the proof system which is certainly an oddity and a telling 'finite-infinite' discrepancy if read only this way. This result (formulated in an entirely abstract form) seems to us centered at the very core of the so many non-finite axiomatizability results of varieties of representable algebras recurring in algebraic logic. This stems from the observation that for CAs (and many cylindric-like algebras such as quasi polyadic algebras with and without equality also dealt with above), we have that for any ordinal α, $\mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+\omega} = \mathsf{RCA}_\alpha$, and that for any ordinal $\alpha > 2$, the sequence $\langle \mathbf{S}\mathsf{Nr}_\alpha\mathsf{CA}_{\alpha+k+1} : k \geq 1\rangle_{k\in\omega}$ is a strictly decreasing sequence with respect to class inclusion with the minimum gaps (of length only one, namely, from $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k+1}$ to $\mathbf{S}\mathsf{Nr}_\alpha\mathbf{K}_{\alpha+k}$ for any positive k and any ordinal $\alpha > 2$) allowing no finite schema axiomatization.

References

[1] L. Henkin, J. Monk, A. Tarski, **Cylindric Algebras, Part I**, vol. 64 of Studies in Logic and the Foundations of Mathematics, North Holland, Amsterdam (1971).

[2] L. Henkin, J. Monk, A. Tarski, **Cylindric Algebras, Part II**, vol. 115 of Studies in Logic and the Foundations of Mathematics, North Holland, Amsterdam (1985), URL: https://www.sciencedirect.com/bookseries/studies-in-logic-and-the-foundations-of-mathematics/vol/115/suppl/C.

[3] R. Hirsch, T. S. Ahmed, *The neat embedding problem for algebras other than cylindric algebras and for infinite dimensions*, **Journal of Symbolic Logic**, vol. 79(1) (2014), pp. 208–222, DOI: https://doi.org/10.1017/jsl.2013.20.

[4] R. Hirsch, I. Hodkinson, **Relation algebras by games**, vol. 147 of Studies in Logic and the Foundations of Mathematics, Elsevier, Amsterdam (2002), URL: https://www.sciencedirect.com/bookseries/studies-in-logic-and-the-foundations-of-mathematics/vol/147/suppl/C.

[5] R. Hirsch, I. Hodkinson, R. Maddux, *Relation algebra reducts of cylindric algebras and an application to proof theory*, **Journal of Symbolic Logic**, vol. 67(1) (2002), pp. 197–213, DOI: https://doi.org/10.2178/jsl/1190150037.

[6] I. Sain, R. Thompson, *Strictly finite schema axiomatization of quasi-polyadic algebras*, [in:] H. Andréka, D. Monk, I. Németi (eds.), **Algebraic Logic**, North Holland, Amsterdam (1991), pp. 539–572.

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