ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 8, 1996

Aleksander Waszak

ON SPACES OF DOUBLE SEQUENCES GENERATED BY MODULI OF SMOOTHNESS

To Professor Lech Włodarski on His 80th birthday

For a given φ-function φ and an element x from the space X of all real double sequences. We first introduce a sequential φ-modulus ω_φ. Next, for a given function Ψ, we define the spaces $X(\Psi)$ and X_ρ generated by ω_φ. The purpose of this paper is to investigate properties of the spaces $X(\Psi)$ and X_ρ.

1. Definitions and preliminaries

Let X be the space of all real bounded double sequences. Sequences belonging to X will be denoted by $x = (t_{\mu\nu}) = ((x)_{\mu\nu})$ or $x = (t_{\mu\nu})_{\mu\nu=0}^{\infty} = ((x)_{\mu\nu})_{\mu\nu=0}^{\infty}$, $y = (s_{\mu\nu})$, $|y| = (|s_{\mu\nu}|)$, $x_p = (t_{\mu\nu}^p)$ for $p = 1, 2, \ldots$. By a convergent sequence we shall mean a double sequence converging in the sense of Prinsgheim.

For any two nonnegative integers m and n, we may define the sets $I_1 = \{\mu, \nu) : \mu < m,\ \nu < n\}$, $I_2 = \{\mu, \nu) : \mu \geq m,\ \nu < n\}$, $I_3 = \{\mu, \nu) : \mu < m,\ \nu \geq n\}$ and $I_4 = \{\mu, \nu) : \mu \geq m,\ \nu \geq n\}$. An (m, n)-translation of a sequence $x \in X$ is defined as the sequence

$\tau_{mn}x = ((\tau_{mn}x)_{\mu\nu})_{\mu,\nu=0}^{00}$ where

$$(\tau_{\mu\nu}x)_{\mu\nu} = \begin{cases} \tau_{\mu,\nu} & \text{for } (\mu,\nu) \in I_1, \\ \tau_{\mu+m,\nu} & \text{for } (\mu,\nu) \in I_2, \\ \tau_{\mu,\nu+n} & \text{for } (\mu,\nu) \in I_3, \\ \tau_{\mu+m,\nu+n} & \text{for } (\mu\nu) \in I_4. \end{cases}$$

It is obvious that $((\tau_{00}x)_{\mu\nu})_{\mu,\nu=0}^{\infty} = ((x)_{\mu\nu})_{\mu,\nu=0}^{\infty}$ and, moreover,

$$(\tau_{m0}x)_{\mu\nu} = \begin{cases} \tau_{\mu,\nu} & \text{for } 0 \le \mu < m \text{ and all } \nu, \\ \tau_{\mu+m,\nu} & \text{for } \mu \ge m \text{ and all } \nu, \end{cases}$$
$$(\tau_{0n}x)_{\mu\nu} = \begin{cases} \tau_{\mu,\nu} & \text{for } 0 \le \nu < n \text{ and all } \mu, \\ \tau_{\mu,\nu+n} & \text{for } \nu \ge n \text{ and all } \mu. \end{cases}$$

Next, we define $M_{\mu\nu}^{mn}(x) \equiv M_{\mu\nu}(x)$ by the formulae

$$M_{\mu\nu}^{mn}(x) = |(\tau_{00}x)_{\mu\nu} - (\tau_{m0}x)_{\mu\nu} - (\tau_{0n}x)_{\mu\nu} + (\tau_{mn}x)_{\mu\nu}|$$

for all μ and ν such that $\mu \ge m \ge 1$ and $\nu \ge n \ge 1$ and, moreover,

$$M_{\mu\nu}^{00}(x) = 0 \quad \text{for any } \mu = 0,1,2,\ldots \text{ and } \nu = 0,1,2,\ldots,$$
$$M_{\mu\nu}^{m0}(x) = |(\tau_{00}x)_{\mu\nu} - (\tau_{m0}x)_{\mu\nu}| \quad \text{for any } \mu \ge 1 \text{ and } \nu \ge 0,$$
$$M_{\mu\nu}^{m0}(x) = |(\tau_{00}x)_{\mu\nu} - (\tau_{0n}x)_{\mu\nu}| \quad \text{for any } \nu \ge 1 \text{ and } \mu \ge 0.$$

Let us remark that

$$M_{\mu\nu}^{mn}(x) = \begin{cases} |\tau_{\mu,\nu} - \tau_{\mu+m,\nu} - \tau_{\mu,\nu+n} + \tau_{\mu+m,\nu+n}|, & (\mu,\nu) \in I_4, \\ 0, \quad (\mu,\nu) \in I_1 \cup I_2 \cup I_3, \end{cases}$$

and, moreover, for $m = 0$ or $n = 0$, we have $M_{\mu\nu}^{0n}(x) = |\tau_{\mu,\nu} - \tau_{\mu,\nu+n}|$ or $M_{\mu\nu}^{m0}(x) = |\tau_{\mu,\nu} - \tau_{\mu+m,\nu}|$, respectively.

By a φ-function we mean a continuous nondecreasing function $\varphi(u)$ defined for $u \ge 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. A φ-function φ is said to satisfy the condition (Δ_2) for small u if, for some constants $K > 0$, $u_0 > 0$, the inequality

$\varphi(2u) \le K\varphi(u)$ is satisfied for $0 < u \le u_0$. A φ-function φ is said to satisfy the condition (Δ_2) for all u, if there exists a positive number K such that $\varphi(2u) \le K\varphi(u)$ for all $u \ge 0$ (compare [3], [4], [5] or [9]).

A sequential φ-modulus of a sequence $x \in X$ is defined as

$$\omega_\varphi(x; r, s) = \sup_{m\ge r}\sup_{n\ge s} \sum_{\mu,\nu=0}^{\infty} \varphi(M_{\mu\nu}^{mn}(x)) \tag{1}$$

where φ is a given φ-function and r and s are nonnegative integers. It is easy to check that

$$\omega_\varphi(x; r, s) = \sup_{m\ge r}\sup_{n\ge s} \sum_{\mu=m,\ \nu=n}^{\infty} \varphi(M_{\mu\nu}^{mn}(x))$$

(compare e.g. [7] or [8]).

2. The space $X(\Psi)$

Let (a_{rs}) be a sequence of positive numbers with

$$s = \inf_{r,s} a_{rs} > 0. \tag{2}$$

Moreover, let Ψ be a nonnegative nondecreasing function of $u \ge 0$ such that $\Psi(u) \to 0$ as $u \to 0_+$, $\Psi(u)$ is not the identity.

We define the set

$$X(\Psi) = \{x \in X : a_{rs}\Psi(\omega_\varphi(\lambda x; r, s)) \to 0 \\ \text{as } r, s \to \infty \text{ for a } \lambda > 0\}. \tag{3}$$

Theorem 1. *Let φ be a φ-function which satisfies the condition (Δ_2) for small u, with a constant $K > 0$, and let the function Ψ satisfy the conditions $\Psi(0) = 0$ and (Δ_2) for small u, with a constant $K_1 > 0$. Then $x \in X(\Psi)$ if and only if*

$$\lim_{r,s\to\infty} a_{rs}\Psi(\omega_\varphi(\lambda x; r, s)) = 0$$

for each $\lambda > 0$.

Proof. The condition $x \in X(\Psi)$ implies that

$$\lim_{r,s\to\infty} a_{rs}\Psi(\omega_\varphi(\lambda_0 x; r, s)) = 0 \quad \text{for some } \lambda_0 > 0 \tag{4}$$

and there exists a constant $\overline{M} > 0$ such that $|t_{\mu\nu}| < \overline{M}$ for all μ and ν. For $\lambda > \lambda_0$, we choose an integer k such that $2^{k-1}\lambda_0 < \lambda < 2^k\lambda_0$ and $2^{k+2}\lambda_0\overline{M} \le u_0$. Next, we have $\lambda M_{\mu\nu}(x) \le 2^k\lambda_0 M_{\mu\nu}(x) \le 2^{k+2}\lambda_0\overline{M}$ for all μ and ν; by (Δ_2), for the function φ with with a constant $K > 0$, we have

$$\begin{aligned} \omega_\varphi(\lambda x; r, s) &= \sup_{m\ge r}\sup_{n\ge s}\sum_{\mu,\nu=0}^{\infty}\varphi(\lambda M_{\mu\nu}(x)) \\ &\le \sup_{m\ge r}\sup_{n\ge s}\sum_{\mu,\nu=0}^{\infty} K^k\varphi(\lambda_0 M_{\mu\nu}(x)) = K^k\omega_\varphi(\lambda_0 x; r, s). \end{aligned}$$

By (2) and (4), we have $\Psi(\omega_\varphi(\lambda_0 x; r, s)) \to 0$ as $r, s \to \infty$. It is seen at once that the condition $\Psi(\omega_\varphi(\lambda_0 x; r, s)) \le \delta$ for sufficiently large r and s implies that $\omega_\varphi(\lambda_0 x; r, s) \le M$ for sufficiently large r and s, where δ and M are some positive numbers. But the function Ψ satisfies (Δ_2) with the constant K_1; then

$$\Psi(2^l\omega_\varphi(\lambda_0 x; r, s)) \le K_1^l\Psi(\omega_\varphi(\lambda_0 x; r, s))$$

for sufficiently large r and s, where l is chosen so that $K^k < 2^l$. Consequently,

$$a_{rs}\Psi(\omega_\varphi(\lambda_0 x; r, s)) \le a_{rs}\Psi(2^l\omega_\varphi(\lambda_0 x; r, s)) \le K_1^l a_{rs}\Psi(\omega_\varphi(\lambda_0 x; r, s))$$

for sufficiently large r and s. Applying the above inequality and condition (4), we obtain $a_{rs}\Psi(\omega_\varphi(\lambda_0 x; r, s)) \to 0$ as $r, s \to \infty$ for each $\lambda > 0$.

Theorem 2. *If Ψ satisfies (Δ_2) with a constant K_1 for small u, then $X(\Psi)$ is a vector space.*

Proof. Let $x = (t_{\mu\nu})$, $y = (s_{\mu\nu})$. From the inequality $\varphi(u+v) \leq \varphi(2u)+\varphi(2v)$ and the properties of the φ-function φ and the function Ψ we get

$$\begin{aligned}(5)\quad a_{rs}\Psi\Big(\omega_\varphi\big(\frac{1}{2}\lambda(x+y); r,s\big)\Big) &\leq a_{rs}\Psi(\omega_\varphi(\lambda x; r,s) + \omega_\varphi(\lambda y; r,s)) \\ &\leq a_{rs}\Psi(2\omega_\varphi(\lambda x; r,s)) + a_{rs}\Psi(2\omega_\varphi(\lambda y; r,s)).\end{aligned}$$

Since $x, y \in X(\Psi)$, therefore, by assumption (2),

$$\Psi(\omega_\varphi(\lambda x; r,s)) \to 0 \qquad \text{and} \qquad \Psi(\omega_\varphi(\lambda y; r,s) \to 0$$

as $r,s \to \infty$, for some $\lambda > 0$. Next, from the properties of the function Ψ we obtain that there exist indices r_0 and s_0 such that $\Psi(\omega_\varphi(\lambda x; r,s)) < \delta$ and $\Psi(\omega_\varphi(\lambda y; r,s)) < \delta$ for all $r \geq r_0$ and $s \geq s_0$, where δ is some positive number. Consequently, $\omega_\varphi(\lambda x; r,s) \leq M$ and $\omega_\varphi(\lambda y; r,s) \leq$M; moreover, $\Psi(2\omega_\varphi(\lambda x; r,s)) \leq K_1\Psi(\omega_\varphi(\lambda x; r,s))$, $\Psi(2\omega_\varphi(\lambda y; r,s)) \leq K_1\Psi(\omega_\varphi(\lambda y; r,s))$ for $r \geq r_0$ and $s \geq s_0$. Thus

$$\begin{aligned}a_{rs}\Psi\Big(\omega_\varphi\big(\frac{1}{2}\lambda(x+y); r,s\big)\Big) &\leq K_1(a_{rs}\Psi(\omega_\varphi(\lambda x; r,s) \\ &+ a_{rs}\Psi(\omega_\varphi(\lambda y; r,s))) \to 0 \qquad \text{as } r,s \to \infty,\end{aligned}$$

and $X(\Psi)$ is a vector space.

Theorem 3. *Let us suppose that a function φ satisfies the following condition:*

(a) *there exists an $\overline{\alpha} > 0$ such that for each $u > 0$ and any α satisfying the inequality $0 < \alpha \leq \overline{\alpha}$, the inequality $\varphi(\alpha u) \leq \frac{1}{2}\varphi(u)$ holds.*

Then $X(\Psi)$ is a vector space.

Proof. For $x, y \in X$ and some λ, $\alpha > 0$, we have

$$\sup_{m\geq r}\sup_{n\geq s}\sum_{\mu,\nu=0}^{\infty}\varphi(\alpha\lambda M_{\mu\nu}(x)) \leq \frac{1}{2}\sup_{m\geq r}\sup_{n\geq s}\sum_{\mu,\nu=0}^{\infty}\varphi(\lambda M_{\mu\nu}(x))$$

and, similarly,

$$\sup_{m\geq r}\sup_{n\geq s}\sum_{\mu,\nu=0}^{\infty}\varphi(\alpha\lambda M_{\mu\nu}(y))\leq\frac{1}{2}\sup_{m\geq r}\sup_{n\geq s}\sum_{\mu,\nu=0}^{\infty}\varphi(\lambda M_{\mu\nu}(y)).$$

By these two inequalities and (5),

$$\begin{aligned}a_{rs}\Psi\Big(\omega_{\varphi}\big(\frac{1}{2}\alpha\lambda(x+y);r,s\big)\Big)&\leq a_{rs}\Psi(\omega_{\varphi}(\lambda x;r,s))\\&\quad+a_{rs}\Psi(\omega_{\varphi}(\lambda y;r,s))\to 0\quad\text{as}\ \ r,s\to\infty,\end{aligned}$$

for some $\lambda>0$. Finally, $X(\Psi)$ is a vector space.

3. Pseudomodulars and pseudonorms

Let ρ be a functional defined on a real vector space Y with values $0\leq\rho(x)\leq\infty$. This functional will be called a pseudomodular if it satisfies the following conditions:

$$\begin{aligned}&\rho(0)=0,\\&\rho(-x)=\rho(x),\\&\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y),\quad\text{for all } x,y\in X \text{ and for any } \alpha,\beta\geq 0\\&\qquad\qquad\text{with } \alpha+\beta=1.\end{aligned}$$

If ρ satisfies the condition

$$\rho(x)=0\quad\text{if and only if}\quad x=0$$

instead of condition one, then ρ is called a moduler (compare e.g. [3], [4], [5] or [11]).

Now, we define in X the functional

$$\rho(x)=\sup_{r,s}a_{rs}\Psi(\omega_{\varphi}(x;r,s)). \tag{6}$$

Theorem 4. *Let a function Ψ be concave and let $\Psi(0) = 0$. Then $X(\Psi)$ is a vector space and ρ is a pseudomodular in X.*

Proof. First, let us remark that if Ψ is concave and $\Psi(0) = 0$, then Ψ satisfies (Δ_2) for all $u > 0$. Thus, by Theorem 2, the space $X(\Psi)$ is a vector space. Moreover, if $x, y \in X$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, then

$$\begin{aligned}\rho(\alpha x + \beta y) &\leq \sup_{r,s} a_{rs} \Psi\Big(\sup_{m \geq r} \sup_{n \geq s} \sum_{\mu,\nu=0}^{\infty} \varphi(\alpha M_{\mu\nu}(x) + \beta M_{\mu\nu}(y))\Big) \\ &\leq \rho(x) + \rho(y).\end{aligned}$$

Theorem 5. *If a φ-function φ is convex, then $X(\Psi)$ is a vector space and ρ is a pseudomodular.*

Proof. A trivial verification shows that each convex function satisfies (a), and so, by Theorem 3, $X(\Psi)$ is a vector space. For $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, and $x, y \in X$, we have

$$\begin{aligned}\rho(\alpha x + \beta y) &\leq \sup_{r,s} a_{rs} \Psi(\omega_\varphi(\alpha x; r, s)) \\ &\quad + \sup_{r,s} a_{rs} \Psi(\omega_\varphi(\beta y; r, s)) \leq \rho(x) + \rho(y).\end{aligned}$$

Theorem 6. *If Ψ is $\overline{s}$-convex with $0 < \overline{s} \leq 1$ (i.e. $\Psi(\alpha x + \beta y) \leq \alpha^{\overline{s}} \Psi(x) \beta^{\overline{s}} \Psi(y)$ for $\alpha, \beta \geq 0$, $\alpha^{\overline{s}} + \beta^{\overline{s}} \leq 1$) and φ is convex, then ρ is an $\overline{s}$-convex pseudomodular.*

Proof. Let us notice that ρ is a pseudomodular (see Theorem 5), and that, for $x, y \in X$, we have

$$\begin{aligned}\rho(\alpha x + \beta y) &\leq \sup_{r,s} a_{rs} \Psi(\alpha \omega_\varphi(x; r, s) + \beta \omega_\varphi(y; r, s)) \\ &\leq \sup_{r,s} a_{rs} (\alpha^{\overline{s}} \Psi(\omega_\varphi(x; r, s)) + \beta^{\overline{s}} \Psi(\omega_\varphi(y; r, s))) \\ &\leq \alpha^{\overline{s}} \rho(x) + \beta^{\overline{s}} \rho(y)\end{aligned}$$

where $\alpha, \beta \geq 0$, $\alpha^{\overline{s}} + \beta^{\overline{s}} \leq 1$.

The functional ρ defines the modular space

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0_+\} \tag{7}$$

and the F-pseudonorm

$$|x|_\rho = \inf\Big\{u > 0 : \rho(\frac{x}{u}) \le u\Big\} \tag{8}$$

(compare [3], [4], [5]).

Theorem 7. *Let Ψ be an $\overline{s}$-convex function, $0 < \overline{s} \le 1$, let Ψ_{-1} be the inverse function to Ψ and, moreover, let φ be convex. Then the $\overline{s}$-homogeneous pseudonorm*

$$\|x\|_\rho^{\overline{s}} = \inf\Big\{u > 0 : \rho(\frac{x}{u^{1/\overline{s}}}) \le 1\Big\} \tag{9}$$

satisfies the inequalities

$$\|x\|_\rho^{\overline{s}} \begin{cases} \ge \sup\limits_{r,s\ge 1} \Big(\dfrac{\omega_\varphi(x;r,s)}{\Psi_{-1}(\frac{1}{a_{rs}})}\Big)^{\overline{s}} & \text{for } x \in X_\rho \text{ and } \|x\|_\rho^{\overline{s}} < 1, \\ \le \sup\limits_{r,s\ge 1} \Big(\dfrac{\omega_\varphi(x;r,s)}{\Psi_{-1}((\frac{1}{a_{rs}})}\Big)^{\overline{s}} & \text{for } x \in X_\rho \text{ and } \|x\|_\rho^{\overline{s}} > 1, \\ = 1 & \text{for } \sup\limits_{r,s\ge 1} \dfrac{\omega_\varphi(x;r,s)}{\Psi_{-1}(\frac{1}{a_{rs}}} = 1. \end{cases}$$

Proof. First, let us note that, by Theorem 6, ρ is $\overline{s}$-convex, so $\|\cdot\|_\rho^{\overline{s}}$ is an $\overline{s}$-homogeneous pseudonorm. If $\|x\|_\rho^{\overline{s}} < u < 1$, then

$$a_{r,s}\Psi\Big(\omega_\varphi\big(\frac{x}{u^{1/\overline{s}}}; r, s\big)\Big) \le 1$$

and

$$a_{r,s}\Psi\Big(\sup_{m\ge r}\sup_{n\ge s}\sum_{\mu,\nu=0}^{\infty}\varphi\big(\frac{1}{u^{1/\overline{s}}}M_{\mu\nu}(x)\big)\Big) \le a_{r,s}\Psi\Big(\frac{1}{u^{1/\overline{s}}}\omega_\varphi(x;r,s)\Big) \le 1,$$

for all r, s. Thus $\omega_\varphi(x;r,s) \le u^{1/\overline{s}}\Psi_{-1}\big(\frac{1}{a_{rs}}\big)$ and, for $u \to \|x\|_{\rho+}^{\overline{s}}$, we obtain first inequality. If $\|x\|_\rho^{\overline{s}} > u > 1$, then we have the condition

$$\sup_{r,s} a_{r,s}\Psi(u^{1/\overline{s}}\omega_\varphi(x;r,s)) > 1$$

which gives the second inequality. The last identity is evident.

4. Some Fréchet spaces

In the sequel, $\overline{c}$ will denote the space of all double sequences $x = (t_{\mu\nu})_{\mu,\nu=0}^{\infty}$ such that $t_{00} = t_0$, $t_{o\nu} = t_1$ for $\nu = 1, 2, \ldots$, $t_{\mu 0} = t_2$ for $\mu = 1, 2, \ldots$ and $t_{\mu\nu} = t_3$ for all $\mu \geq 1$ and $\nu \geq 1$, where t_0, t_1, t_2 and t_3 are arbitrary numbers.

It is easy to verify that:

$\overline{c}$ is a subspace of the space of all convergent double sequences;
$\overline{c} = \{x \in X : \rho(x) = 0\}$;
if φ is convex, then $x \in \overline{c}$ if and only if $|x|_\rho = 0$;
if Ψ is concave and φ is $\overline{s}$-convex with some $0 < \overline{s} \leq 1$, then $x \in \overline{c}$ if and only if $|x|_\rho = 0$

(compare e.g. [2], [7] and [10]).

Next, let one of the following two conditions hold:

φ satisfies (a),
Ψ satisfies (Δ_2) for small u.

Applying the results of [2], we shall consider quotient spaces $\widetilde{X}_\rho = X_\rho/\overline{c}$ and $\widetilde{X}(\Psi) = X(\Psi)/\overline{c}$ with elements $\widetilde{x}$, $\widetilde{y}$, etc. Moreover, we may define the modular

$$\widetilde{\rho}(\widetilde{x}) = \inf\{\rho(y) : y \in \widetilde{x}\}$$

and the pseudonorms $|\widetilde{x}|_\rho = |x|_\rho$, $\|\widetilde{x}\|_\rho^{\overline{s}} = \|x\|_\rho^{\overline{s}}$ where $x \in \widetilde{x}$.

Let $(\varphi_j)_{j=1}^{\infty}$ be a given sequence of φ-functions. By formulae (1) and (6), we may introduce sequences $(\omega_{\varphi_j}(x; r, s))$ and $(\rho_j) \equiv (\rho_{\varphi_j})$, respectively. Next, applying definitions (3) and (7), we have two sequences of spaces $(X_j(\Psi))$ and $(X_{\rho_{\varphi_j}}) \equiv (X_{\rho_j})$, respectively. Moreover, by means of the sequence (ρ_j) we shall introduce sequences $(\|x\|_j^{\overline{s}}) \equiv (\|x\|_{\rho_{\varphi_j}}^{\overline{s}})$ and $(|x|_j) \equiv (|x|_{\rho_{\varphi_j}})$ (see (8) and (9)). Arguing as in [1] and [6], we shall define the extended real-valued modulars

$$\rho_0(x) = \sup_j \rho_j(x) \quad \text{and} \quad \rho_w(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\rho_j(x)}{1 + \rho_j(x)}$$

and the countably modulared spaces X_{ρ_0} and X_{ρ_w}.

Evidently, we have $X_{\rho_0} \subset X_{\rho_w} = \bigcap_{j=1}^{n} X_{\rho_1}$, and it is easily verified that:

Theorem 8. *If Ψ is a function which satisfies the condition (Δ_2) for small u and if (φ_j) is a given sequence of φ-functions which satisfy the condition:*

(b) *there exist positive constants K, c, u_0 and in index j_0 such that*

$$\varphi_j(cu) \leq K\varphi_{j_0}(u) \quad \text{for all } \ j \geq j_0 \text{ and } 0 \leq u \leq u_0,$$

then the spaces X_{ρ_w} and X_{ρ_0} are identical.

Theorem 9. *Let φ_j for $j = 1, 2, \dots$ satisfy the conditions:*

(c) *for each $\varepsilon > 0$, there exist $A > 0$ and $\overline{\alpha} > 0$ such that, for any α and u satisfying the inequalities $0 < \alpha \leq \overline{\alpha}$, $0 < u \leq A$, the inequality $\varphi_j(\alpha u) \leq \varepsilon\varphi_j(u)$ holds for all j,*

(d) *for each $\eta > 0$, there exists an $\varepsilon > 0$ such that, for all $u > 0$ and all indices j, the inequality $\varphi_j(u) < \varepsilon$ implies $u < \eta$.*

Let Ψ be increasing, continuous, $\Psi(0) = 0$, and satisfying the condition:

(e) *for arbitrary $v_1 > 0$ and $\delta_1 > 0$, there exists an $\eta_1 > 0$ such that the inequality $\Psi(\eta u) \leq \delta_1\Psi(u)$ holds for all $0 \leq u \leq v_1$ and $0 \leq \eta \leq \eta_1$.*

Moreover, let one of the conditions hold: Ψ is concave or φ_j $(j = 1, 2, \dots)$ are convex. Then X_{ρ_0} is a Fréchet space with respect to the F-norm $|\cdot|_{\rho_0}$.

Proof. Let $x_p \in \widetilde{x}_p$, $x_p = (t^p_{\mu\nu})^{\infty}_{\mu,\nu=0}$ be such that $t^p_{1,\nu} = t^p_{\mu,1} = 0$ for all μ, ν and p, let $(\widetilde{x}_p)$ be a Cauchy sequence in $\widetilde{X}_{\rho_j}$ and, moreover, let j be an arbitrary index. For each $\varepsilon > 0$, one can find an N such that $|x_p - x_q|_\rho < a\Psi(\varepsilon)$ for $p, q > N$, where a is defined by (2). Thus there exists u_ε such that $0 < u_\varepsilon < a\Psi(\varepsilon)$ and

$$a_{rs}\Psi\Big(\omega_{\varphi_j}\big(\frac{x_p - x_q}{u_\varepsilon}; r, s\big)\Big) \leq u_\varepsilon$$

for $p, q > N$ and all r, s. Hence

$$\omega_{\varphi_j}\Big(\frac{x_p - x_q}{u_\varepsilon}; r, s\Big) \leq \Psi_{-1}\Big(\frac{u_\varepsilon}{a_{rs}}\Big) \leq \Psi_{-1}\Big(\frac{u_\varepsilon}{a}\Big) < \varepsilon$$

for $p, q > N$ and all r, s, where Ψ_{-1} denotes the inverse function to Ψ. Applying (1), we have

$$\sum_{\mu=m,\ \nu=n}^{\mu=\overline{m},\ \nu=\overline{n}} \varphi_j\Big(\frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x_q)\Big) < \Psi_{-1}\Big(\frac{u_\varepsilon}{a}\Big) < \varepsilon \tag{10}$$

for $p, q > N$, $\overline{m} \geq \mu \geq m \geq r$ and $\overline{n} \geq \nu \geq n \geq s$. By (d), for each $\eta > 0$, one can find an $\varepsilon > 0$ such that

$$\frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x_q) < \eta \tag{11}$$

for $p, q > N$, $\mu \geq m \geq 1$, $\nu \geq n \geq 1$. Next, we have

$$|t^p_{\mu+m,\,\nu+n} - t^q_{\mu+m,\,\nu+n}| < A_1 + A_2 + A_3 + M_{\mu\nu}(x_p - x_q)$$

where $A_1 = |t^p_{\mu,\nu} - t^q_{\mu,\nu}|$, $A_2 = |t^p_{\mu+m,\nu} - t^q_{\mu+m,\nu}|$, $A_3 = |t^p_{\mu,\nu+n} - t^q_{\mu,\nu+n}|$. First, let us remark that, by the definitions of $t^p_{1,\mu}$ and $t^p_{\mu,1}$, we have $A_1 = A_2 = A_3 = 0$ for $r = s = 1$ and $\mu = \nu = 1$ and we see that $(t^p_{2,2})_{p=1}^\infty$ is a Cauchy sequence. Next, by induction we obtain that $(t^p_{\mu\nu})_{p=1}^\infty$ are Cauchy sequences for all μ, ν. Hence these sequences are convergent. We write $x = (t_{\mu\nu})_{\mu\nu=0}^\infty$ where $t_{\mu\nu} = 0$ for $\mu = 0$ or $\nu = 0$ and $t_{\mu\nu} = \lim_{p\to\infty} t^p_{\mu\nu}$ for $\mu, \nu = 1, 2, \ldots$. Taking $q \to \infty$ in (10), we have

$$\sum_{\mu=m,\ \nu=n}^{\mu=\overline{m},\ \nu=\overline{n}} \varphi_j\Big(\frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x)\Big) \leq \Psi_{-1}\Big(\frac{u_\varepsilon}{a_{rs}}\Big)$$

for $p > N$, $\overline{m} \geq m \geq r$, $\overline{n} \geq n \geq s$; and, for $\overline{m}, \overline{n} \to \infty$, we obtain

$$\sum_{\mu=m,\ \nu=n}^{\infty} \varphi_j\Big(\frac{1}{u_\varepsilon} M_{\mu\nu}(x_p - x)\Big) \leq \Psi_{-1}\Big(\frac{u}{a_{rs}}\Big)$$

for $p > N$, $m \geq r \geq 1$ and $n \geq s \geq 1$. Consequently,

$$\omega_{\varphi_j}\Big(\frac{x_p - x}{u_\varepsilon}; r, s\Big) \leq \Psi_{-1}\Big(\frac{u_\varepsilon}{a_{rs}}\Big)$$

for $p > N$ and $r, s \geq 1$, so

$$a_{rs}\Psi\Big(\omega_{\varphi_j}\big(\frac{1}{u_\varepsilon}(x_p - x); r, s\big)\Big) \leq u_\varepsilon \quad \text{for } p > N \text{ and all } r,\ s. \tag{12}$$

We are going to prove that $\rho(\lambda(x_p - x)) \to 0$ as $\lambda \to 0_+$ for large p. Let N be chosen as above. For $\varepsilon, \lambda > 0$ and $p > N$, we have

$$\begin{aligned}\omega_{\varphi_j}(\lambda(x_p - x); r, s) &= \omega_{\varphi_j}\Big(\lambda u_\varepsilon \frac{x_p - x}{u_\varepsilon}; r, s\Big) \\ &= \sup_{m \geq r}\sup_{n \geq s}\sum_{\mu,\nu}^{\infty}\varphi_j\Big(\lambda u_\varepsilon M_{\mu\nu}\big(\frac{x_p - x}{u_\varepsilon}\big)\Big).\end{aligned}$$

If we take $p \to \infty$ in (11), then $M_{\mu\nu}\big(\frac{x_p - x}{u_\varepsilon}\big) \leq \eta$. By (c) with $\overline{\varepsilon} = \varepsilon$, $\eta = A$, $\alpha = \lambda u_\varepsilon \leq \overline{\alpha}$ for $u = \frac{1}{u_\varepsilon}M_{\mu\nu}(x_p - x)$, we have

$$\varphi\Big(\lambda u_\varepsilon M_{\mu\nu}\big(\frac{x_p - x}{u_\varepsilon}\big)\Big) \leq \overline{\varepsilon}\varphi\Big(\big(\frac{1}{u_\varepsilon}M_{\mu\nu}(x_p - x)\big)$$

for $p > N$ and $\mu \geq m \geq 1$, $\nu \geq n \geq 1$. Hence

$$\omega_{\varphi_j}(\lambda(x_p - x); r, s) \leq \overline{\varepsilon}\omega_{\varphi_j}\Big(\frac{x_p - x}{u_\varepsilon}; r, s\Big) \leq \overline{\varepsilon}\Psi_{-1}\Big(\frac{u_\varepsilon}{a_{rs}}\Big) \leq \overline{\varepsilon}\varepsilon.$$

Finally, for $0 < \lambda < \frac{\overline{\alpha}}{u_\varepsilon}$, we have

$$\rho_j(\lambda(x_p - x)) \leq \sup_{r,s} a_{rs}\Psi\Big(\overline{\varepsilon}\Psi_{-1}\big(\frac{u_\varepsilon}{a_{rs}}\big)\Big).$$

Next, we apply condition (e) with $v_1 = \Psi_{-1}\big(\frac{u_\varepsilon}{a}\big)$ and $u = \Psi_{-1}\big(\frac{u_\varepsilon}{a_{rs}}\big)$. For $\delta_1 > 0$ and $\overline{\varepsilon} = \eta_1$, we have

$$\Psi\Big(\overline{\varepsilon}\Psi_{-1}\big(\frac{u_\varepsilon}{a_{rs}}\big)\Big) \leq \delta_1\Psi\Big(\Psi_{-1}\big(\frac{u_\varepsilon}{a_{rs}}\big)\Big) = \delta_1\frac{u_\varepsilon}{a_{rs}}.$$

Thus

$$\rho_j(\lambda(x_p - x)) \leq \sup_{r,s} a_{rs}\delta_1 \frac{u_\varepsilon}{a_{rs}} = \delta_1 u_\varepsilon$$

for $0 < \lambda u_\varepsilon \leq \overline{\alpha}$. Since u_ε is fixed, this implies $\rho_0(\lambda(x_p - x)) \to 0$ as $\lambda \to 0_+$, for $p > N$, i.e. $x_p - x \in X_{\rho_0}$ for sufficiently large p. Since X_{ρ_j} is a vector space, $x \in X_{\rho_j}$. By (12), $\rho_0\big(\frac{1}{u_\varepsilon}(x_p - x)\big) \leq u_\varepsilon$ for $p > N$. Thus $|x_p - x|_{\rho_0} < u_\varepsilon a\Psi(\varepsilon)$ for $p > N$. Finally, $|x_p - x|_{\rho_0} \to 0$ as $p \to \infty$, which proves the completeness of the space X_{ρ_0}.

Theorem 10. *Let a function Ψ satisfy the same assumptions as in Theorems 1 and 9 and let φ-functions (φ_j), where $\varphi = (\varphi_j)$, satisfy conditions (c), (d) and the condition (Δ_2) (i.e. $\varphi = \varphi_j(u)$ satisfies the condition (Δ_2) for small u with a constant $K > 0$ independent of j). Then $\widetilde{X}_j(\Psi) \cap \widetilde{X}_{\rho_j}$ is a Fréchet space with respect to the F-norm $|\cdot|_{\rho_j}$ for $j = 1, 2, \ldots$.*

Proof. Let j be an arbitrary positive integer. It is sufficient to remark that $\widetilde{X}_j(\Psi) \cap \widetilde{X}_{\rho_j}$ is a closed subspace of $\widetilde{X}_{\rho_j}$ with respect to the F-norm $|\cdot|_{\rho_j}$. Let $\widetilde{x}_p \to \widetilde{x}$ in $\widetilde{X}_{\rho_j}$, $\widetilde{x}_p \in \widetilde{X}_j(\Psi) \cap \widetilde{X}_{\rho_j}$, $x_p \in \widetilde{x}_p$, $x \in \widetilde{x}$. Then, for each $\lambda > 0$,

$$a_{rs}\Psi(\omega_{\varphi_j}(\lambda(x_p - x); r, s)) \to 0 \quad \text{as} \quad p \to \infty$$

uniformly with respect to r and s. Applying the property of ω_{φ_j} and the condition (Δ_2) for φ with a constant $K > 0$, we obtain

$$\begin{aligned}\omega_{\varphi_j}(\lambda x; r, s) &\leq \omega_{\varphi_j}(2\lambda(x_p - x); r, s) + \omega_{\varphi_j}(2\lambda x; r, s)\\ &\leq K(\omega_{\varphi_j}(\lambda(x_p - x); r, s) + \omega_{\varphi_j}(\lambda x; r, s)).\end{aligned}$$

Taking $\lambda > 0$ fixed, by the properties of Ψ, we may find some $\overline{p}$ such that $\Psi(\omega_{\varphi_j}(\lambda(x_p - x); r, s)) < \delta$ for $p \geq \overline{p}$ and for all r and s, where δ is some positive constant. Hence there exists $M > 0$ such that $\omega_{\varphi_j}(\lambda(x_p - x); r, s) \leq M$ for $p \geq \overline{p}$ and all r and s. If k is chosen so that $K \leq 2^k$, then, from the inequality $\Psi(u + v) \leq \Psi(2u) + \Psi(2v)$ and the condition (Δ_2) for Ψ, for small u with a constant $K_1 > 0$, we obtain

$$\begin{aligned} a_{rs}\Psi(\omega_{\varphi_j}(\lambda x; r, s)) &\leq a_{rs}\Psi(2K\omega_{\varphi_j}(\lambda(x_p - x); r, s)) \\ &+ a_{rs}\Psi(2K\omega_{\varphi_j}(\lambda x_p; r, s)) \\ &\leq K_1^{k+1} a_{rs}(\Psi(\omega_{\varphi_j}(\lambda(x_p - x)x; r, s)) \\ &+ \Psi(\omega_{\varphi_j}(\lambda x_p; r, s))) \end{aligned}$$

for $p \geq \overline{p}$ and all r and s. Let us fix $\varepsilon > 0$. There is an index $p_0 > \overline{p}$ such that

$$a_{rs}\Psi(\omega_{\varphi_j}(\lambda(x_{p_0} - x); r, s)) < \frac{1}{2}\varepsilon K_1^{-(k+1)}.$$

But $x_{p_0} \in X_j(\Psi)$, and so, by Theorem 1, we obtain

$$a_{rs}\Psi(\omega_{\varphi_j}(\lambda x_{p_0}; r, s)) \to 0 \quad \text{as} \quad r, s \to \infty.$$

Thus, there exist r_0 and s_0 such that

$$a_{rs}\Psi(\omega_{\varphi_j}(\lambda(x_{p_0}; r, s)) < \frac{1}{2}\varepsilon K_1^{-(k+1)}$$

for all $r \geq r_0$ and $s \geq s_0$. Finally,

$$a_{rs}\Psi(\omega_{\varphi_j}(\lambda x; r, s)) \leq K_1^{k+1}\Big(\frac{1}{2}\varepsilon K_1^{-(k+1)} + \frac{1}{2}\varepsilon K_1^{-(k+1)}\Big) = \varepsilon$$

for all $r \geq r_0$ and $s \geq s_0$, which shows that $x \in X_j(\Psi)$. Since, by Theorem 9, $x \in X_{\rho_j}$, therefore $x \in X_j(\Psi) \cap X_{\rho_j}$, and so, $x \in \widetilde{X}_j(\Psi) \cap \widetilde{X}_{\rho_j}$.

We may also consider Theorems 9 and 10 with modular convergence (with respect to the modular $\widetilde{\rho}(\widetilde{x})$) in place of F-norm convergence. In the subsequent paper an application to problems of two-modular convergence of sequences will be shown.

REFERENCES

[1] J. Albrycht and J. Musielak, *Countably modulared spaces*, Studia Math. **31** (1968), 331–337.

[2] T. Jędryka and J. Musielak, *Some remarks on F-modular spaces*, Functiones et Approx. **2** (1976), 83–100.

[3] A. Kufner, O. John and S. Fučik, *Function spaces*, ACADEMIA, Prague, 1977.

[4] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. 1034, Springer Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[5] ——— and W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 49–65.

[6] ——— and A. Waszak, *On some countably modulared spaces*, Studia Math. **38** (1970), 51–57.

[7] ———, *Generalized variation and translation operator in some sequence spaces*, Hokkaido Math. Journal **17** (1988), 345–353.

[8] ———, *Sequence spaces generated by moduli of smoothness*, Revista Math. Comp. (Madrid) **8** (1995), 1–15.

[9] H. Nakano, *Generalized modular spaces*, Studia Math. **31** (1968), 439–449.

[10] A. Waszak, *On some modular spaces of double sequences I*, Commentationes Math. **30** (1991), 255–264.

Aleksander Waszak

PRZESTRZENIE CIĄGÓW PODWÓJNYCH GENEROWANE MODUŁAMI GŁADKOŚCI

Dla danej φ-funkcji φ oraz elementu $x = ((x)_{\mu,\nu})$ z przestrzeni X ciągów rzeczywistych podwójnych, najpierw wprowadzony został ciągowy φ-moduł ω_φ wzorem

$$\omega_\varphi(x; r, s) = \sup_{m\geq r}\sup_{n\geq s}\sum_{\mu,\nu=0}^{\infty}\varphi(|(\tau_{00}x)_{\mu\nu} - (\tau_{m0}x)_{\mu\nu} - (\tau_{0n}x)_{\mu\nu} + (\tau_{mn}x)_{\mu\nu}|)$$

gdzie τ_{mn} oznacza (m,n)-translację ciągu $x \in X$. W dalszym ciągu dla danej funkcji Ψ zdefiniowane zostały przestrzenie

$$X(\Psi) = \{x \in X : a_{r,s}\Psi(\omega_\varphi(x;r,s)) \to 0 \text{ dla } \lambda > 0 \text{ oraz } r,s \to \infty\},$$
$$X_\rho = \{x \in X : \rho(\lambda x) = \sup_{r,s} a_{r,s}\Psi(\omega_\varphi(x;r,s)) \to 0 \text{ gdy } \lambda \to 0^+\},$$

gdzie (a_{rs}) oznacza ciąg liczb dodatnich. Celem prezentowanej pracy jest podanie własności przestrzeni $X(\Psi)$ oraz X_ρ.

Faculty of Mathematics and Computer Science
Adam Mickiewicz University
ul. Matejki 48/49, 60-769 Poznań, Poland