

THE CAYLEY SUM GRAPH OF IDEALS OF A LATTICE

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Abstract

Let L be a lattice, $\mathfrak{I}(L)$ be the set of ideals of L and S be a subset of $\mathfrak{I}(L)$. In this paper, we introduce an undirected Cayley graph of L , denoted by $\Gamma_{L,S}$ with elements of $\mathfrak{I}(L)$ as the vertex set and, for two distinct vertices I and J , I is adjacent to J if and only if there is an element K of S such that $I \vee K = J$ or $J \vee K = I$. We study some basic properties of the graph $\Gamma_{L,S}$ such as connectivity, girth and clique number. Moreover, we investigate the planarity, outerplanarity and ring graph of $\Gamma_{L,S}$.

Keywords: lattice, Cayley graph, ring graph, outerplanar graph.

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1. INTRODUCTION

The inquiry of graphs relevant to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [10, 13, 15, 17, 18, 20] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have also been actively

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investigated. For example, comaximal graphs of a lattice have been considered in [1], Cayley graphs of partially ordered sets have been studied in [2]. In [2] the authors introduce and investigate a new analogue of the fundamental notion of a Cayley graph for the case of lattices. The original definition of a Cayley graph was introduced by Cayley in 1878 [6] to explain the concept of abstract groups described by a set of generators. In the last 50 years, the theory of Cayley graphs has grown into a substantial branch in algebraic graph theory. We refer the reader to [7, 10–12, 14, 19] for more details.

Recently in [3], the concept of the Cayley sum graphs of a commutative ring is defined as follows.

Let R be a commutative ring, $I(R)$ be the set of all ideals of R and S be a subset of $I^*(R) = I(R) \setminus \{0\}$. The *Cayley sum graph*, denoted by $\text{Cay}^+(I(R), S)$, is an undirected graph whose vertex set is the set $I(R)$ and two distinct vertices I and J are adjacent whenever $I + K = J$ or $J + K = I$, for some ideal K in S .

In this paper we extend the concept of the Cayley sum graph of ideals of a commutative ring, for a lattice. Let L be a lattice, $\mathfrak{I}(L)$ be the set of ideals of L and S be a subset of $\mathfrak{I}(L)$. We define an undirected Cayley graph of L , denoted by $\Gamma_{L,S}$ with elements of $\mathfrak{I}(L)$ as the vertex set and, for two distinct vertices I and J , I is adjacent to J if and only if there is an element K in S such that $I \vee K = J$ or $J \vee K = I$. In Section 2, we state some preliminaries about lattices. In Section 3, we study some basic properties of the graph $\Gamma_{L,S}$ such as connectivity, girth and clique number. In Section 4, we investigate the planarity, outerplanarity and ring graph of $\Gamma_{L,S}$.

Now we recall some definitions and notations on graphs. We use the standard terminology of graphs following [5]. In a graph G , the *distance* between two distinct vertices a and b , denoted by $d(a, b)$, is the length of the shortest path connecting a and b , if such a path exists; otherwise, we set $d(a, b) := \infty$. The *diameter* of a graph G is $\text{diam}(G) = \sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of G , denoted by $\text{girth}(G)$, is the length of the shortest cycle in G , if G contains a cycle; otherwise, we set $\text{girth}(G) := \infty$. Also, for two distinct vertices a and b in G , the notation $a - b$ means that a and b are adjacent. A vertex a in a graph G is said to be a *pendant vertex* if $\text{deg}(a) = 1$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. We say that G is *totally disconnected* if no two vertices of G are adjacent. Also, G is called an *empty graph* if its vertex-set is empty. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G , denoted by $\omega(G)$, is called the *clique number* of G . The *chromatic number* of a graph G , denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A subset X of the vertices of G is called

an *independent set* if the induced subgraph on X has no edges. The maximum size of an independent set in a graph G is called the *independence number* of G and is denoted by $\alpha(G)$. For a graph G and a subset S of the vertex set $V(G)$, by $N_G[S]$ we mean the set of vertices in G which are in S or adjacent to a vertex in S . If $N_G[S] = V(G)$, then S is said to be a *dominating set* (of vertices in G). The *domination number* of a graph G , denoted by $\gamma(G)$, is the minimum size of a dominating set of the vertices in G . For a positive integer r , an *r -partite graph* is one whose vertex-set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r -partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [6, p. 153]).

Suppose that G is a graph with p vertices and q edges. Also assume that C is a cycle of G . A *chord* in G is any edge joining two nonadjacent vertices in C . A *primitive cycle* is a cycle without chords. Moreover, we say that a graph G has the *primitive cycle property (PCP)* if any two primitive cycles intersect in at most one edge. The *free rank* of G , denoted by $\text{frank}(G)$, is the number of primitive cycles of G . Also, the number $\text{rank}(G) := q - p + r$, where r is the number of connected components of G , is called the *cycle rank* of G . The cycle rank of G can be expressed as the dimension of the cycle space of G . These two numbers satisfy the inequality $\text{rank}(G) \leq \text{frank}(G)$, as is seen in [9, Proposition 2.2]. In the second section of [9], the authors provided a characterization of graphs such that the equality occurs. The precise definition of a *ring graph* can be found in Section 2 of [9]. Roughly speaking, ring graphs can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices. They showed that, for the graph G , the following conditions are equivalent:

- (i) G is a ring graph,
- (ii) $\text{rank}(G) = \text{frank}(G)$,
- (iii) G satisfies *PCP* and G does not contain a subdivision of K_4 as a subgraph.

The following lemma is useful.

Lemma 1.1 [4, Lemma 7.78]. *Let G be a graph with vertex set V . If G is 2-connected and $\deg(v) \geq 3$ for all $v \in V$, then G contains a subdivision of K_4 as a subgraph.*

2. BASIC DEFINITIONS AND PROPERTIES

In this section, firstly we recall some definitions and notations on lattices. For lattice definitions not given here, we refer to [16].

A *lattice* is a set L with two binary operation \wedge and \vee on L satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a = a$, $a \vee a = a$,
2. $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $a \vee (b \vee c) = (a \vee b) \vee c$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

Note that in every lattice $a \wedge b = a$ always implies that $a \vee b = b$. In the next theorem, we recall an equivalent definition of a lattice with respect to a partial order relation which will be used in this paper.

Theorem 2.1 [16, Theorem 2.3]. *Let L be a lattice. One can define an order \leq on L as follows:*

For any $a, b \in L$, we set $a \leq b$ if and only if $a \wedge b = a$. Then (L, \leq) is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, $\text{g.l.b.}(a, b), \text{l.u.b.}(a, b) \in P$. For each a and b in P , we define $a \wedge b := \text{g.l.b.}(a, b)$ and $a \vee b := \text{l.u.b.}(a, b)$. Then (P, \wedge, \vee) is a lattice.

Let x and y be two distinct elements of L . Whenever $x \leq y$ and there is no element z in P such that $x \leq z \leq y$, we say y *covers* x . An element x of L which covers 0 is called an *atom*, and $\text{Atom}(L)$ denotes the set of all atoms of L .

Definition 2.2 [8, Definition 39]. A non-empty subset I of a lattice L is called an *ideal* of L if and only if the following conditions are satisfied:

- (i) For $a, b \in I$, $a \vee b \in I$.
- (ii) For $a \in I$ and $c \in L$, $a \wedge c \in I$.

An ideal I of L is proper if $I \neq L$.

Theorem 2.3 [8, Theorem 59]. *For an ideal I of L , the following conditions are satisfied:*

- (i) *If $a \in I$ and $b \leq a$, then $b \in I$.*
- (ii) *If $a \vee b \in I$, then we have $a, b \in I$.*

Let I and J be ideals of a lattice L . Consider the set C of all elements c of L such that $c \leq a \vee b$, for some elements $a \in I$ and $b \in J$. Clearly, C is non-empty, because it obviously contains every element of I and of J . Also, by [8, Theorem

65], C is the least ideal (with respect to inclusion) containing I and J . We write $I \vee J$ for C . The ideal $I \vee J$ is said to be the ideal generated by the set-union $S = I \cup J$. If S consists of a single element a , then the ideal generated by the set $\{a\}$ is called the *principal ideal* generated by a ; it consists of all $x \leq a$ and will be denoted by $[a]^\ell$ (see [8, Definition 41]). It is easy to see that, for each two principal ideals $[a]^\ell$ and $[b]^\ell$, we have the following equalities:

$$[a]^\ell \wedge [b]^\ell = [a \wedge b]^\ell, \quad [a]^\ell \vee [b]^\ell = [a \vee b]^\ell.$$

A *maximal* ideal of L is a proper ideal which is maximal among all ideals of L . We denote the set of all maximal ideals of L by $\text{Max}(L)$. Also, one can easily check that the set

$$J(L) := \bigcap_{\mathfrak{m} \in \text{Max}(L)} \mathfrak{m}$$

is an ideal of L . We call it the *Jacobson radical* of L .

3. BASIC PROPERTIES OF THE CAYLEY GRAPH $\Gamma_{L,S}$

Let L be a lattice, $\mathfrak{I}(L)$ be the set of all ideals of L and $\mathfrak{I}^*(L) = \mathfrak{I}(L) \setminus \{L\}$. Let S be a non-empty subset of $\mathfrak{I}(L)$. We define the graph $\Gamma_{L,S}$, as an undirected graph with $\mathfrak{I}(L)$ as the vertex set, and two distinct vertices I and J are adjacent if and only if there is a vertex K in S such that $I \vee K = J$ or $J \vee K = I$. For all vertices I , $L \vee I = L$, that is, if $L \in S$, then L is adjacent to all vertices of $\mathfrak{I}(L)$ and $\Gamma_{L,S}$ is a refinement of star graph. Thus we assume that $L \notin S$.

Now suppose that $\mathfrak{I}(L)$ has at least one maximal ideal and that M_1 and M_2 are two distinct maximal ideals such that M_1 is adjacent to M_2 . Therefore there exists a vertex $K \in S$ such that $M_1 \vee K = M_2$ or $M_2 \vee K = M_1$, and hence either $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$, which is impossible. Thus the set of maximal ideals forms an independent set in $\Gamma_{L,S}$. Now let L be a lattice such that $\text{Atom}(L) \neq \emptyset$. Clearly $[a]^\ell = \{0, a\}$, where $a \in \text{Atom}(L)$. Similarly, the set of the ideals $[a]^\ell$, where $a \in \text{Atom}(L)$, forms an independent set in $\Gamma_{L,S}$.

Proposition 3.1. *Let S be a singleton subset of $\mathfrak{I}^*(L)$. Then $\Gamma_{L,S}$ is disconnected.*

Proof. Suppose that $S = \{I\}$, and that J is any vertex distinct from I . If $J \subseteq I$, then I is adjacent to J and J is not adjacent to any vertex of $\Gamma_{L,S}$, and if $I \subseteq J$, then I is not adjacent to J . Now suppose that I and J are not comparable. Then clearly I is not adjacent to J . Therefore the set $A = \{J : J \subseteq I\}$ forms a component of $\Gamma_{L,S}$ and hence the graph $\Gamma_{L,S}$ is not connected. ■

Lemma 3.2. *Let $S = \{I, J\} \subseteq \mathfrak{I}^*(L)$. Then the graph $\Gamma_{L,S}$ is connected if and only if $I \vee J = L$.*

Proof. Suppose that $I \vee J = L$. Clearly L is adjacent to both vertices I and J . We claim that $\Gamma_{L,S}$ has no isolated vertex. Now if $K \in \mathfrak{I}^*(L)$ and K is an isolated vertex, then $K \vee I = K$ and $K \vee J = K$, and hence $I, J \subseteq K$. Therefore $K = L$, which is a contradiction. Thus it is enough to show that, for any vertex K , there is a path between K and L . As K is not an isolated vertex, there is a vertex K' such that K is adjacent to K' . Hence $K \vee I = K'$ or $K' \vee I = K$ for some $I \in S$. If $K \vee I = K'$, then $I \subseteq K'$, and hence $K' \vee J = L$, which means that K' is adjacent to L . Also, if $K' \vee I = K$, then $I \subseteq K$ and $K \vee J = L$, which implies that K is adjacent to L . A similar argument for $K \vee J = K'$ or $K' \vee J = K$, shows that, for any vertex K , there is a path between K and L .

Conversely assume that $\Gamma_{L,S}$ is connected. Suppose on the contrary that $I \vee J \neq L$. Let $K = I \vee J$ and $B = \{F : F \in \mathfrak{I}(L) \text{ and } F \subseteq K\}$. Suppose that $F \in B$ and $T \notin B$. It is clear that $F \vee I$ and $F \vee J$ lies in B , and also $T \vee I$ and $T \vee J$ are not in B . Hence F is not adjacent to T . Therefore B forms a component of $\Gamma_{L,S}$ and hence the graph $\Gamma_{L,S}$ is not connected. ■

Theorem 3.3. *Let $S = \{I, J\}$ and the graph $\Gamma_{L,S}$ be connected. Then $\text{diam}(\Gamma_{L,S}) \leq 4$ and $\text{girth}(\Gamma_{L,S}) \leq 4$.*

Proof. In view of the proof of Lemma 3.2, for every vertex K that is not adjacent to L , there is a vertex K' such that K' is adjacent to K and L . Now let N and T be two distinct non adjacent vertices such that they are not adjacent to L . Then there are vertices N' and T' such that we have the path $N - N' - L - T' - T$, and hence its diameter is less than or equal to four.

Now let K be a vertex distinct from L , I and J . Then we consider the following three cases:

Case 1. $K \subseteq I$ and $K \subseteq J$. In this case K is adjacent to both I and J , and hence we have the cycle, $K - I - L - J - K$ of length four.

Case 2. $K \subseteq I$ and $K \not\subseteq J$. If $K \vee J = L$, then K is adjacent to L , and hence there is a cycle $L - K - I - L$ of length three. If $T = K \vee J \neq L$, then $T \vee I = L$, and hence there is a cycle of length four as $T - K - I - L - T$.

Case 3. $K \not\subseteq I$ and $K \not\subseteq J$. Put $F = K \cap I$ and $G = F \vee J$. Therefore F is adjacent to I . If $G = L$, then there is a cycle $F - I - L - F$ of length three, and if $G \neq L$, then $G \vee I = L$ and hence we have a cycle $I - F - G - L - I$ of length four. ■

Proposition 3.4. *Let $S = \{I_1, I_2, \dots, I_n\} \subseteq \mathfrak{I}^*(L)$. Then the graph $\Gamma_{L,S}$ is connected if and only if $I_1 \vee I_2 \vee \dots \vee I_n = L$.*

Proof. First assume that $I_1 \vee I_2 \vee \dots \vee I_n = L$. Suppose that there are two ideals I_j and I_k in S such that $I_j \vee I_k = L$, for some $1 \leq j \neq k \leq n$. Therefore, by Lemma 3.2, the result holds. So we assume that for each proper subset of

S , say $\{I_{i_1}, \dots, I_{i_t}\}$, where $1 \leq i_1 \leq \dots \leq i_t \leq n$, we have $(I_{i_1} \vee \dots \vee I_{i_t}) \neq L$. Now let K be a vertex such that K is not adjacent to L . Hence $K \vee I_j \neq L$, for $j = 1, 2, \dots, n$. Put $K_j = (K \vee I_1 \vee \dots \vee I_{j-1}) \vee I_j$. Therefore there is a path of length at most n between K and $K_n = L$, and hence the graph is connected.

For the converse statement, assume that $I_1 \vee \dots \vee I_n \neq L$. Put $K = I_1 \vee \dots \vee I_n$ and let $N = \{F : F \in \mathfrak{I}(L) \text{ and } F \subseteq K\}$. Now let $F \in N$ and $T \notin N$. It is clear that $F \vee I$ and $F \vee J$ lies in N , and $T \vee I$ and $T \vee J$ are not in N , and hence F is not adjacent to T . Therefore the graph $\Gamma_{L,S}$ is not connected. ■

Corollary 3.5. *Let $S = \{I_1, I_2, \dots, I_n\} \subseteq \mathfrak{I}^*(L)$ and the graph $\Gamma_{L,S}$ be connected. Then $\text{diam}(\Gamma_{L,S}) \leq 2n$, and also $\text{girth}(\Gamma_{L,S}) \leq 4$.*

Proposition 3.6. *Let $\Gamma_{L,S}$ be connected and $K \in \mathfrak{I}^*(L)$ be a pendant vertex. Then K is adjacent to L .*

Proof. Suppose that, for some I, J in S , $K \vee I \neq K \vee J$. Then $\text{deg}(K) \geq 2$, and hence for all I, J in S , $K \vee I = K \vee J$. Put $F = K \vee I$. So, for all I in S , $I \subseteq F$ and hence $F = L$. ■

Lemma 3.7. *If $K_1 - K_2 - K_3 - K_1$ is a cycle of length three in the graph $\Gamma_{L,S}$, then $\{K_1, K_2, K_3\}$ is a chain in $\mathfrak{I}(L)$.*

Proof. If two vertices are adjacent in $\Gamma_{L,S}$, then one of them is a subset of another. Hence $\{K_1, K_2, K_3\}$ is a chain in $\mathfrak{I}(L)$. ■

Proposition 3.8. *Assume that S is a finite subset of $\mathfrak{I}(L)$ and that $\Gamma_{L,S}$ has a clique of size n . Then $|S| \geq n - 1$.*

Proof. By the definition of adjacency of vertices in $\Gamma_{L,S}$, K_1 is adjacent to K_2 only if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. Thus if the graph $\Gamma_{L,S}$ has a clique with n vertices K_1, K_2, \dots, K_n , then, by Lemma 3.7, the set $\{K_1, K_2, \dots, K_n\}$ is a chain in $\mathfrak{I}(L)$. Without loss of generality, we may assume that $K_1 \subseteq K_2 \subseteq \dots \subseteq K_n$. Hence if $|S| < n - 1$, then K_1 is not adjacent to each vertex K_i , for $i = 2, \dots, n$, and hence $\{K_1, K_2, \dots, K_n\}$ is not a clique, which is a contradiction. ■

We say that a vertex I has the property P if I is comparable with at least one of the elements in S or I is adjacent to L in $\Gamma_{L,S}$.

Proposition 3.9. *Let $S = \{I, J\}$ and $\Gamma_{L,S}$ be connected. If all vertices of $\Gamma_{L,S}$ has the property P , then $S \cup \{L\}$ is a dominating set in $\Gamma_{L,S}$.*

Proof. Let F be an arbitrary vertex in $\Gamma_{L,S}$. Then we show that F is adjacent to L, I or J . Since F has the property P , there is a vertex in S , say I , such that $I \subseteq F$ or $F \subseteq I$. If $F \subseteq I$, then clearly F is adjacent to I . Also if $I \subseteq F$, then, since $I \vee J = L$, we have that $F \vee J = L$, which means that F is adjacent to L . ■

Lemma 3.10. *Let $S = \{I\} \subseteq \mathfrak{J}(L)$. Then there is no path of length greater than 2 in $\Gamma_{L,S}$.*

Proof. First we claim that if there is a path $K_1 - K_2 - K_3$ of length 2 in $\Gamma_{L,S}$, then $K_1, K_3 \subseteq K_2$. Since K_1 is adjacent to K_2 , we have $K_1 \vee I = K_2$ or $K_2 \vee I = K_1$. Also K_3 is adjacent to K_2 . So $K_3 \vee I = K_2$ or $K_2 \vee I = K_3$. Assume that $K_2 \vee I = K_1$. Thus we have $K_3 \vee I = K_2$ and this is impossible. Hence $K_1 \vee I = K_2$ and $K_3 \vee I = K_2$. Therefore $K_1, K_3 \subseteq K_2$. Now suppose that there is a path $K_1 - K_2 - K_3 - K_4$ of length three in $\Gamma_{L,S}$. By the above discussion, we have $K_1, K_3 \subseteq K_2$ and $K_2, K_4 \subseteq K_3$ and this is impossible. ■

Proposition 3.11. *Let $S \subseteq \mathfrak{J}(L)$. Then $\Gamma_{L,S}$ has no cycle if and only if $S = \{I\}$ for some $I \in \mathfrak{J}(L)$.*

Proof. Assume that $|S| \geq 2$ and $I, J \in S$. Put $F = I \cap J$ and $G = I \cup J$. Then it is clear that $F - I - G - J - F$ is a cycle in $\Gamma_{L,S}$. Now let $S = \{I\}$. Then, by Lemma 3.10, it is clear that there is no cycle in $\Gamma_{L,S}$. ■

4. PLANARITY OF $\Gamma_{L,S}$

In this section we assume that $S \subseteq \text{Max}(L)$, $|S| \geq 2$ and $0 \in L$.

Notation 4.1. To simplify of notations, let $S = \{M_1, M_2, \dots, M_n\}$ be a subset of maximal ideals of $\mathfrak{J}^*(L)$. We set $S_i := \{F | F \subseteq M_i \text{ and } F \not\subseteq \bigcup_{j \neq i} M_j\}$ and $S_{ij} := \{F | F \subseteq M_i \cap M_j \text{ and } F \not\subseteq \bigcup_{k \neq i,j} M_k\}$ and similarly $S_{12 \dots n} := \{F | F \subseteq M_1 \cap M_2 \cap \dots \cap M_n\}$.

Remark 4.2. Let $S = \{M_1, M_2, \dots, M_n\}$ be a subset of maximal ideals of $\mathfrak{J}^*(L)$. If, for all $2 \leq k \leq n$, $S_{i_1 i_2 \dots i_k} = \{0\}$, then the graph $\Gamma_{L,S}$ is a planar bipartite graph as it is shown in Figure 1. In the case that $n = 2$, if $S_1 \cup S_2 = S$ and $S_{12} \neq \{0\}$, then $\Gamma_{L,S}$ is also a planar bipartite graph as it is shown in Figure 2, where, for $1 \leq k \leq \ell$, $A_k \in S_{12}$.

In the rest of this section, we assume that, for some $1 \leq k \leq n$, $S_{i_1 i_2 \dots i_k} \neq \{0\}$.

Theorem 4.3. *Let S be a subset of maximal ideals of $\mathfrak{J}^*(L)$. Then $\Gamma_{L,S}$ is a 3-partite graph and $\text{diam}(\Gamma_{L,S}) \leq 3$.*

Proof. Put $X_1 = \{L\}$, $X_2 = S$ and $X_3 = \mathfrak{J}(L) \setminus (S \cup \{L\})$. As elements of S are maximal ideals, X_2 is an independent set and we claim that X_3 is also an independent set. For, if there are two vertices I and J in X_3 such that I is adjacent to J , then there is a vertex $M_i \in S$ such that $I \vee M_i = J$ or $J \vee M_i = I$, and therefore either $M_i \subseteq I$ or $M_i \subseteq J$. But M_i is a maximal ideal, and this is a contradiction.

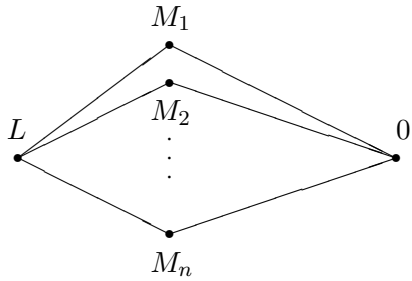


Figure 1

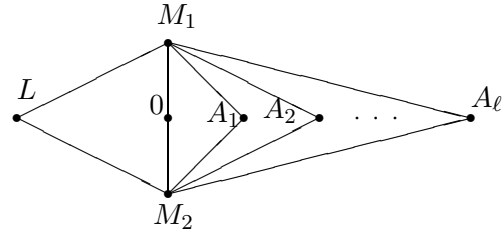


Figure 2

Now, let I and J be two non adjacent vertices. We know that any vertex in S is adjacent to L . So it is enough to consider the following cases:

Case 1. $I \in X_1$. In this case J is in X_3 , and since $\Gamma_{L,S}$ is connected, J is adjacent to a vertex of X_2 , and hence $d(I, J) = 2$.

Case 2. $I \in X_2$ and $J \in X_3$. In this situation we can easily see that $d(I, J) = 2$.

Case 3. $I, J \in X_3$. In this case if $I, J \in S_{12\dots n}$, then for all vertices M_k in X_2 , I, J are adjacent to M_k , and if $I \in S_{12\dots n}$ and $J \notin S_{12\dots n}$, then for each maximal ideal M_k in X_2 , I is adjacent to M_k . If J and M_k are not comparable, then we have the path $J - L - M_k - I$, and if $J \subseteq M_k$, then $d(I, J) = 2$. If $I, J \notin S_{12\dots n}$, then there are elements M_i and M_j in X_2 such that $I \not\subseteq M_i$ and $J \not\subseteq M_j$. Thus we have the path $I - L - J$.

By considering the above cases we have $\text{diam}(\Gamma_{L,S}) \leq 3$. ■

The following corollary follows from Theorem 4.3.

Corollary 4.4. *Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$. Then $S \cup \{L\}$ is a dominating set for $\Gamma_{L,S}$.*

Now we study the planarity of $\Gamma_{L,S}$ in the case that S is a subset of maximal ideals of $\mathfrak{I}^*(L)$. Let $|S| \geq 3$ and $J(L) \neq 0$. As $0, L$ and $J(L)$ are adjacent to all elements of S , we have $\Gamma_{L,S}$ has a subgraph isomorphic to $K_{3,3}$. Thus $\Gamma_{L,S}$ is not planar. Therefore we assume that $J(L) = 0$.

Lemma 4.5. *Let $S = \{M_1, M_2, M_3\}$. If $|S_{123}| > 1$, then $\Gamma_{L,S}$ is not planar, and if $|S_{123}| = 1$, then we have the following statements:*

1. *If, for some i, j , $|S_{ij}| \geq 2$, then $\Gamma_{L,S}$ is not planar.*
2. *If, for all i, j , $|S_{ij}| \leq 1$, then $\Gamma_{L,S}$ is a planar graph.*

Proof. As $|S_{123}| > 1$, there is a nonzero ideal $F \in S_{123}$, and hence $0, L, F$ are adjacent to every element of S . Therefore $\Gamma_{L,S}$ has a subgraph isomorphic to $K_{3,3}$ and it is not planar. Now suppose that $|S_{123}| = 1$. For the first statement, without loss of generality, we may assume that $|S_{12}| = 2$ and $F_1, F_2 \in S_{12}$. Therefore we have a subdivision of $K_{3,3}$ in $\Gamma_{L,S}$ as it is pictured in Figure 3, and hence the graph $\Gamma_{L,S}$ is not planar.

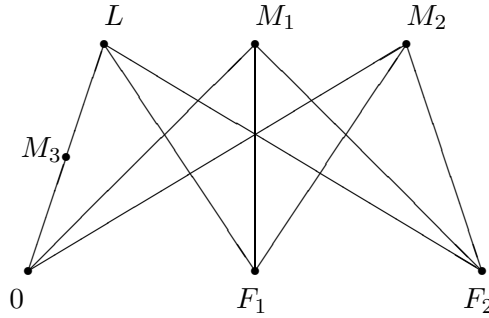


Figure 3

For the second statement, let $|S_{ij}| \leq 1$ for all i, j . Then $\Gamma_{L,S}$ is a planar graph, as it is shown in Figure 4, where $F_{ij} \in S_{ij}$.

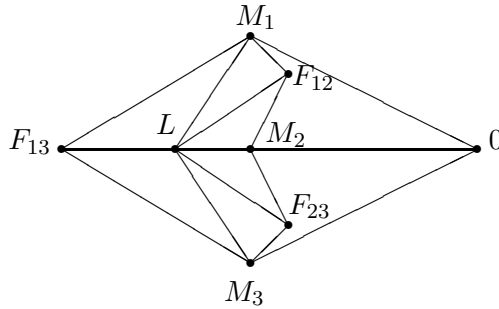


Figure 4

■

Proposition 4.6. Let S be a subset of maximal ideals of $\mathfrak{T}^*(L)$ with $|S| \geq 4$. Then we have the following statements:

- (1) If, for some i_1, i_2, \dots, i_k , with $3 \leq k \leq n - 1$, $S_{i_1 i_2 \dots i_k} \neq \{0\}$ or $S_{12 \dots n} \neq \{0\}$, then $\Gamma_{L,S}$ is not planar.
- (2) If, for all j_1, j_2, \dots, j_k , with $3 \leq k \leq n - 1$, $S_{j_1 j_2 \dots j_k} = \{0\}$, $S_{12 \dots n} = \{0\}$ and for some i, j , $|S_{ij}| \geq 2$, then $\Gamma_{L,S}$ is not planar.

- (3) If, for all i, j , $|S_{ij}| \leq 1$ and there are integers i_1, i_2, \dots, i_k , $k \geq 3$ such that $S_{i_1 i_2}, S_{i_2 i_3}, \dots, S_{i_{k-1} i_k}, S_{i_1 i_k}$ are non empty or there are integers i, j_1, j_2, \dots, j_k such that $S_{i j_l} \neq \{0\}$, where $l = 1, 2, \dots, k$ and $k \geq 3$, then $\Gamma_{L,S}$ is not a planar graph.

Proof. (1) Let $S = \{M_1, M_2, \dots, M_n\}$. If, for some j_1, j_2, \dots, j_k , $3 \leq k \leq n - 1$, $S_{j_1 j_2 \dots j_k} \neq \{0\}$ or $S_{1,2,\dots,n} \neq \{0\}$, then $\Gamma_{L,S}$ has a subgraph isomorphic to $K_{3,3}$, and hence $\Gamma_{L,S}$ is not planar.

(2) If, for some i, j , $|S_{ij}| \geq 2$ and $F_1, F_2 \in S_{ij}$, then we have a subdivision of $K_{3,3}$ in $\Gamma_{L,S}$ as it is shown in Figure 5. Thus $\Gamma_{L,S}$ is not planar.

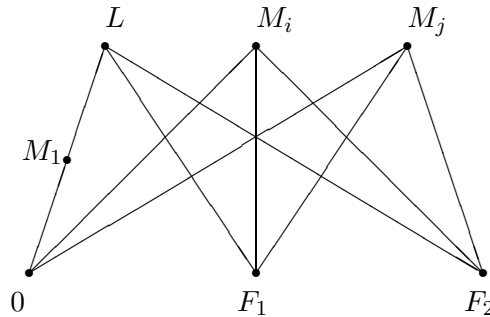


Figure 5

- (3) Suppose that there are integers i_1, i_2, \dots, i_k , with $k \geq 3$ such that $S_{i_1 i_2}, S_{i_2 i_3}, \dots, S_{i_{k-1} i_k}, S_{i_1 i_k}$ are non empty. Then $\Gamma_{L,S}$ has a subdivision of $K_{3,3}$ as it is shown in Figure 6. Now, assume that there are integers i, j_1, j_2, \dots, j_k such that $S_{i j_l} \neq \{0\}$, $l = 1, 2, \dots, k$ and $k \geq 3$. Then $\Gamma_{L,S}$ has a subdivision of $K_{3,3}$ as it is pictured in Figure 7. ■

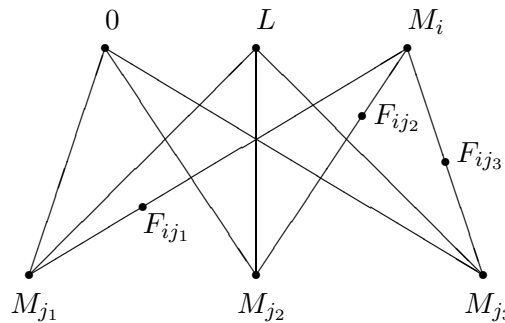


Figure 6

In the sequel of this section, we deal with the outerplanarity of $\Gamma_{L,S}$. By [9], we know that every outerplanar graph is a ring graph and every ring graph is a

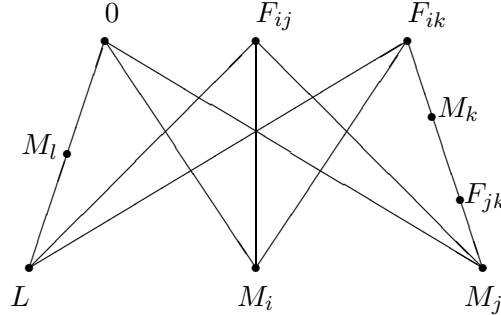


Figure 7

planar graph. Let S be a subset of maximal ideals of $\mathfrak{J}^*(L)$ with $|S| \geq 3$ and $\Gamma_{L,S}$ is a planar graph. By Proposition 4.5, for all i, j , we have $|S_{ij}| \leq 1$, and if at least one S_{ij} is non-empty, then $\Gamma_{L,S}$ has an induced subgraph H that is satisfied in the conditions of Lemma 1.1. Therefore $\Gamma_{L,S}$ has a subdivision isomorphic to K_4 , as it is shown in Figure 8. Hence it is not a ring graph.

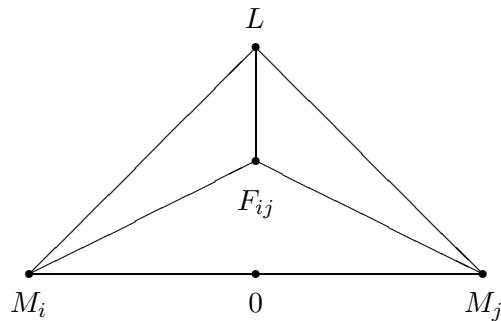


Figure 8

By [9, Lemma 2.9], for $n \geq 3$, $K_{2,n}$ is not a ring graph. Now if, for all i, j , $S_{ij} = \{0\}$ and at least one of the S_i 's is non-empty, then $\Gamma_{L,S}$ has an induced subgraph isomorphic to $K_{2,n}$, $n \geq 3$, and by [9, Corollary 2.15], $\Gamma_{L,S}$ is not a ring graph. Assume that $|S| = 2$. If $S_{12} \neq \{0\}$, then $\Gamma_{L,S}$ has an induced subgraph isomorphic to $K_{2,3}$, which is not a ring graph, and if $S_{12} = \{0\}$, then $\text{rank}(\Gamma_{L,S}) = \text{frank}(\Gamma_{L,S}) = |S_1| + |S_2| + 1$. Therefore $\Gamma_{L,S}$ is a ring graph.

By the above discussion we have the following theorem.

Theorem 4.7. *Let S be a subset of maximal ideals of $\mathfrak{J}^*(L)$. Then $\Gamma_{L,S}$ is a ring graph if and only if $|S| = 2$ and $S_{12} = \{0\}$.*

Proposition 4.8. *Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$ with $|S| = 2$. Then $\Gamma_{L,S}$ is an outerplanar graph if and only if $S_{12} = \{0\}$ and, for $i = 1, 2$, $|S_i| \leq 1$.*

Proof. Assume that $S_{12} \neq \{0\}$ or, for some i , $|S_i| \geq 2$. Therefore $\Gamma_{L,S}$ has a subdivision isomorphic to $K_{2,3}$, and hence $\Gamma_{L,S}$ is not an outerplanar graph. It is clear that if $S_{12} = \{0\}$ and, for $i = 1, 2$, $|S_i| \leq 1$, then $\Gamma_{L,S}$ is outerplanar. ■

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