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ISOMORPHISMS IN EQ-ALGEBRAS

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Abstract

In this paper we investigate some isomorphism theorems in EQ-algebras. After establishing some basic results we give the Fundamental Homomorphism Theorem and by using it we state and prove some other isomorphism theorems. We also state and prove a correspondence theorem. Next, using some results of the theory of universal algebra we characterize subdirectly irreducible EQ-algebras.

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1. Introduction

Every many-valued logic is uniquely determined by the algebraic properties of the structure of its truth values. It is accepted that this algebraic structure should be considered as a residuated lattice fulfilling some additional properties in fuzzy logic (see [8]). In this case, both propositional and first-order logics have been developed. A natural question arises that whether also a higher-order fuzzy logic can be developed as a counterpart of the classical higher-order logic (type theory, see [1]). This question has been answered positively with the introduction of fuzzy type theory (FTT) which in [5] its the algebra of truth values is called an

EQ-algebra [6]. Many investigations has been done on EQ-algebras by authors (see [2, 4, 7]).

In this paper, we consider certain classes of EQ-algebras which are called separated EQ-algebras and state and prove some isomorphism theorems. We also state and prove a correspondence theorem. Furthermore, we investigate those EQ-algebras which are subdirectly irreducible and give some characterizations of them.

2. Preliminaries

This section is devoted to give some definitions and results from the literature. For more details, we refer to [5, 7].

Definition 1. An EQ-*algebra* is an algebra $\mathcal{E} = \langle E, \wedge, *, \sim, 1\rangle$ of type (2,2,2,0) such that for all $x, y, z, t \in E$:

(E1) $\langle E, \wedge, 1\rangle$ is a semilattice with top element 1 (the induced order is defined as $x \leq y$ if and only if $x \wedge y = x$);

(E2) $\langle E, *, 1\rangle$ is a commutative monoid and $*$ is isotone with respect to " $\leq$ ";

(E3) $x \sim x = 1$;

(E4) $((x \wedge y) \sim z) * (t \sim x) \leq z \sim (t \wedge y)$;

(E5) $(x \sim y) * (z \sim t) \leq (x \sim z) \sim (y \sim t)$;

(E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$;

(E7) $x * y \leq x \sim y$.

Definition 2. Le $\mathcal{E} = \langle E, \wedge, *, \sim, 1\rangle$ be an EQ-algebra.

- $\mathcal{E}$ is said to be *separated* if $a \sim b = 1$ implies that $a = b$, for all $a, b \in E$.
- The multiplication $*$ is said to be monotone with respect to $\to$ (or E is said to be $\to$-*monotone*) if $a \to b = 1$ implies that $a * c \to b * c = 1$, for each $c \in E$.

Example 1. The $\{\wedge, *, \leftrightarrow, 1\}$-reduct of any residuated lattice $\langle L, \wedge, \vee, *, \to, 0, 1\rangle$ is a separated EQ-algebra (see [6]).

Definition 3. A nonempty subset F of EQ-algebra $\mathcal{E}$ is called a *filter* if

(i) $1 \in F$,

(ii) $a, b \in F$ implies that $a * b \in F$,

(iii) $a, a \to b \in F$ implies that $b \in F$,

(iv) $a \to b \in F$ implies that $a * c \to b * c \in F$.

By $Fil(\mathcal{E})$ we mean the set of all filters of EQ-algebra $\mathcal{E}$.

Notice that if $\mathcal{E}$ is a separated EQ-algebra, the condition (ii) may be removed (see [7, Lemma 15]). Hence in a separated EQ-algebra a nonempty subset F which satisfies the conditions (i), (iii) and (iv) is called a filter. Any filter F of an EQ-algebra is an upset; i.e., $a \leq b$ and $a \in F$ imply $b \in F$. Moreover $a, a \sim b \in F$ imply that $b \in F$.

In an EQ-algebra $\mathcal{E}$, any filter F induces a congruence θ_F as $a\theta_F b$ if and only if $a \sim b \in F$. The set of all congruence classes, $\mathcal{E}/F$, forms an EQ-algebra with respect to the induced operations from $\mathcal{E}$. Moreover $\mathcal{E}/F$ is separated and the natural mapping $a \mapsto a/F$ is an onto homomorphism. Some additional properties are as follows.

Proposition 1. *In any EQ-algebra $\mathcal{E}$, the following properties hold, for all $x, y, z \in E$:*

(i) $x * y \leq x, y, \;\; x * y \leq x \wedge y$;

(ii) $x \sim y \leq x \to y$, *where* $x \to y := (x \wedge y) \sim x$;

(iii) $x \to x = 1$;

(iv) $(x \wedge y) \sim x \leq (x \wedge y \wedge z) \sim (x \wedge z)$;

- *If $\mathcal{E}$ is separated, $a \to b = 1$ implies that $a = b$.*

We recall some definitions from universal algebra. For more details we refer to [3].

Definition 4. An algebra $\mathbf{A}$ is said to be a subdirect product of an indexed family $(\mathbf{A}_i)_{i\in I}$ of algebras if $\mathbf{A}$ is a subalgebra of $\prod_{i\in I} \mathbf{A}_i$ and $\pi_i(\mathbf{A}) = \mathbf{A}_i$, for each $i \in I$.

An embedding $\phi : \mathbf{A} \to \prod_{i\in I} \mathbf{A}_i$ is called a subdirect if $\phi(\mathbf{A})$ is a subdirect product of the $\mathbf{A}_i$'s.

Definition 5. An algebra $\mathbf{A}$ is called subdirectly irreducible if for every subdirect embedding $\phi : \mathbf{A} \to \prod_{i\in I} \mathbf{A}_i$ there is an $i \in I$ such that $\pi_i\phi : \mathbf{A} \to \mathbf{A}_i$ is an isomorphism.

In the rest of the paper, $\mathcal{E}$ will denote an EQ-algebra, unless otherwise stated.

3. Isomorphism theorems

In this section we establish the Fundamental Homomorphism Theorem of universal algebra for EQ-algebras. We also state and prove some new results in this context.

Lemma 1. *Let $\mathcal{E}$ be a separated EQ-algebra and $f : \mathcal{E} \longrightarrow \mathcal{G}$ be a homomorphism of EQ-algebras. Then f is one-to-one if and only if $Ker(f) = \{x \in E : f(x) = 1\} = \{1\}$.*

Proof. It is easy. ■

Theorem 1. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be a homomorphism of EQ-algebras and F be a filter of $\mathcal{G}$. Then there exists a unique homomorphism $\tilde{f} : \mathcal{G}/F \longrightarrow \mathcal{H}$ such that $Im(\tilde{f}) = Im(f)$ and $Ker(\tilde{f}) = Ker(f)/F$. Furthermore $\tilde{f}$ is an isomorphism if and only if f is onto and $Ker(f) = F$.*

Proof. We define $\tilde{f} : \mathcal{G}/F \longrightarrow \mathcal{H}$ as $\tilde{f}(a/F) = f(a)$. Since f is a homomorphism, so $\tilde{f}$ is a homomrphism and is such that $Im(\tilde{f}) = Im(f)$. Now,

$$Ker(\tilde{f}) = \{a/F : f(a) = 1_{\mathcal{H}}\} = \{a/F : a \in Ker(f)\} = Ker(f)/F.$$

From the definition of $\tilde{f}$, it is obvious that $\tilde{f}$ is unique. Now, $\tilde{f}$ is an isomorphism if and only if it is onto and $Ker(f)/F = Ker(\tilde{f}) = F$ and this true if and only if f is onto and $Ker(f) = F$. ■

Lemma 2. *Let $\mathcal{H}$ be a separated and $\rightarrow$-monotone EQ-algebra. Then for every homomorphism $f : \mathcal{G} \longrightarrow \mathcal{H}$ of EQ-algebras, $Ker(f)$ is a filter of $\mathcal{G}$.*

Proof. Assume that $f : \mathcal{G} \longrightarrow \mathcal{H}$ is a homomorphism and $a, a \rightarrow b \in Ker(f)$, for $a, b \in E$. Then $1 \rightarrow f(b) = f(a) \rightarrow f(b) = f(a \rightarrow b) = 1$. Since $\mathcal{H}$ is separated so $f(b) = 1$ and so $b \in Ker(f)$. Also if $a \rightarrow b \in Ker(f)$, then $f(a) \rightarrow f(b) = 1$. So by $\rightarrow$-monotonicity we have $f(a * c \rightarrow b * c) = f(a) * f(c) \rightarrow f(b) * f(c) = 1$, whence $a * c \rightarrow b * c \in Ker(f)$. Moreover for $a, b \in Ker(f)$ we have $f(a * b) = f(a) * f(b) = 1$, whence $a * b \in Ker(f)$. Hence $Ker(f)$ is a filter of $\mathcal{G}$. ■

Theorem 2 (Fundamental Homomorphism Theorem)**.** *Let $\mathcal{G}$ be an EQ-algebra and $\mathcal{H}$ be a separated and $\rightarrow$-monotone EQ-algebra. If $f : \mathcal{G} \longrightarrow \mathcal{H}$ is an epimorphism, then $\mathcal{G}/Ker(f) \simeq \mathcal{H}$.*

Proof. Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an epimorphism. Since $\mathcal{H}$ is separated it follows that $Ker(f)$ is a filter of $\mathcal{G}$, by Lemma 2. So $\mathcal{G}/Ker(f)$ is a separated EQ-algebra. To prove that $\mathcal{G}/Ker(f)$ is $\rightarrow$-monotone, assume that $a/Ker(f) \rightarrow b/Ker(f) = 1 = Ker(f)$. Hence $f(a) \rightarrow f(b) = f(a \rightarrow b) = 1$ and since $\mathcal{H}$ is separated, so $f(a) = f(b)$. Now, for any $c \in G$, $f(a) * f(c) = f(b) * f(c)$ and so $f(a * c \rightarrow b * c) = 1$, whence $a * c \rightarrow b * c \in Ker(f)$. Hence

$$a/Ker(f) * c/Ker(f) \rightarrow b/Ker(f) * c/Ker(f) = Ker(f) = 1_{\mathcal{G}/Ker(f)}.$$

Thus $\mathcal{G}/Ker(f) \simeq \mathcal{H}$, by Theorem 1. ■

Corollary 1. *Let $\mathcal{H}$ be a separated EQ-algebra, $f : \mathcal{G} \longrightarrow \mathcal{H}$ be a homomorphism of EQ-algebras and A and B be filters of $\mathcal{G}$ and $\mathcal{H}$ such that $f(A) \subseteq B$. Then the mapping $\tilde{f} : \mathcal{G}/A \longrightarrow \mathcal{H}/B$ with $a/???A \mapsto f(a)/B$ is a homomorphism such that if f is onto, $f(A) = B$ and $Ker(f) \subseteq A$, then $\tilde{f}$ is an isomorphism.*

Proof. It is clear that $\mathcal{H}/B$ is a separated EQ-algebra and the natural mapping $\pi : \mathcal{H} \longrightarrow \mathcal{H}/B$ with $a \mapsto a/B$ is an epimorphism. For homomorphism $\pi f : \mathcal{G} \longrightarrow \mathcal{H}/B$, by Theorem 1, there exists a unique homomorphism $\tilde{f} : \mathcal{G}/A \longrightarrow \mathcal{H}/B$ with $\tilde{f}(a/A) = \pi f(a) = f(a)/B$ which $Im(\tilde{f}) = Im(\pi f)$ and $Ker(\tilde{f}) = Ker(\pi f)/A$. Moreover, $\tilde{f}$ is an isomorphism if and only if πf is onto and $Ker(\pi f) = A$. If f is onto, then πf is also onto. Now we show that $Ker(\pi f) = A$. If $a \in Ker(\pi f)$, then $f(a)/B = B$, whence $f(a) \in B = f(A)$. Hence $f(a) = f(b)$ for some $b \in A$. So $f(a \to b) = f(a) \to f(b) = 1$. Thus $a \to b \in Ker(f) \subseteq A$. Therefore $a \in A$. So, $Ker(\pi f) \subseteq A$. Obviously, $A \subseteq Ker(\pi f)$. ■

Proposition 2. *If F and G are filters of $\mathcal{E}$ such that $F \subseteq G$, then $G/F = \{a/F \in \mathcal{E}/F : a \in G\}$ is a filter of $\mathcal{E}/F$.*

Proof. Routine. ■

Corollary 2. *Assume that $\mathcal{E}$ is a $\to$-monotone EQ-algebra and F and G are filters of $\mathcal{E}$ such that $F \subseteq G$. Then G/F is a filter of $\mathcal{E}/F$ and $(\mathcal{E}/F)/(G/F) \simeq \mathcal{E}/G$.*

Proof. By Proposition 2, $(\mathcal{E}/F)/(G/F)$ is a separated EQ-algebra and the mapping $\psi : \mathcal{E}/F \longrightarrow \mathcal{E}/G$ defined by $a/F \mapsto a/G$ is an onto homomorphism with $Ker(\psi) = G/F$. Hence $(\mathcal{E}/F)/(G/F) \simeq \mathcal{E}/G$, by Fundamental Homomorphism Theorem. ■

Lemma 3. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be a homomorphism of EQ-algebras.*

(i) *$Ker(f) \subseteq K$ if and only if $f^{-1}(f(K)) = K$, for any filter K of $\mathcal{G}$.*

(ii) *The inverse image under f of any filter of $\mathcal{H}$ is again a filter of $\mathcal{G}$ containing $Ker(f)$.*

(iii) *If f is onto, then the image of any filter of $\mathcal{G}$ is a filter of $\mathcal{H}$.*

Proof. Routine. ■

Theorem 3 (Correspondence Theorem)**.** *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an onto homomorphism of separated EQ-algebras. Then the assignment $K \mapsto f(K)$ defines a bijection correspondence between $S_f(\mathcal{G})$ of all filters of $\mathcal{G}$ containing $Ker(f)$ and the set $S(\mathcal{H})$ of all filters of $\mathcal{H}$.*

Proof. We define the mappings $\Phi : S_f(\mathcal{G}) \longrightarrow S(\mathcal{H})$ and $\Psi : S(\mathcal{H}) \longrightarrow S_f(\mathcal{G})$ by $K \mapsto f(K)$ and $M \mapsto f^{-1}(M)$. Φ and Ψ are well-defined and $\Phi\Psi = id_{S(\mathcal{H})}$ and $\Psi\Phi = id_{S_f(\mathcal{G})}$, by Lemma 3(ii) and (iii). Hence Φ is a bijection. ■

Theorem 4. *Let $\mathcal{E}$ be a separated EQ-algebra. Then $\mathcal{E}$ is a subdirect product of a family $\{E_i : i \in I\}$ of separated and $\rightarrow$-monotone EQ-algebras if and only if for each $i \in I$ there exists a filter $F_i \subseteq E$ such that $\bigcap_{i\in I} F_i = \{1\}$ and $E/F_i \simeq E_i$.*

Proof. Assume that $\{\mathcal{E}_i : i \in I\}$ is a family of separated and $\rightarrow$-monotone EQ-algebras. If $\mathcal{E}$ is a subdirect product of $\mathcal{E}_i$'s, there exists a monomorphism $\phi : \mathcal{E} \rightarrow \prod \mathcal{E}_i$ such that $g_i =_{def} \pi_i\phi(E) = E_i$. So, by Fundamental Homomorphism Theorem, $E/Kerg_i \simeq E_i$. Now, we show that $\bigcap_{i\in I} Kerg_i = \{1\}$. Let $a \in E$. Then $a \in \bigcap_{i\in I} Kerg_i$ if and only if $\phi(a)(i) = \pi\phi(a) = 1_{E_i}$ if and only if $a = 1$.

Conversely, assume that for each $i \in I$ there exists a filter F_i of $\mathcal{E}$ such that $\bigcap_{i\in I} F_i = \{1\}$ and $\mathcal{E}/F_i \simeq \mathcal{E}_i$. Now, we consider the mapping $\phi : \mathcal{E} \rightarrow \prod \mathcal{E}_i$ by $\phi(a)(i) = \phi_i(a/F_i)$. It is easy to check that ϕ is a homomorphism. Moreover, for $a \in E$, $\phi(a) = 1$ if and only if $\phi_i(a/F_i) = \phi(a)(i) = 1$, for each $i \in I$, if and only if $a/F_i = 1/F_i$, for each $i \in I$, if and only if $a = a \sim 1 \in F_i$, for each $i \in I$. This implies that $a = 1$ and so $\bigcap_{i\in I} F_i = \{1\}$. ■

Lemma 4. *Let $\phi_i : \mathcal{E} \rightarrow \mathcal{E}_i$ $(i \in I)$ be a family of homomorphisms of EQ-algebras. Then the natural homomorphism $\phi : \mathcal{E} \rightarrow \prod_{i\in I} \mathcal{E}_i$ is an embedding if and only if $\bigcap_{i\in I} Ker\phi_i = \{1\}$.*

Proof. Assume that ϕ is an embedding and $a \in \bigcap_{i\in I} Ker\phi_i$, for $a \in E$. Then $\phi(a)(i) = \phi_i(a) = 1$, for all $i \in I$. This implies that $a \in Ker\phi$ and so $a = 1$. Conversely, assume that $\phi(a) = \phi(b)$, for $a, b \in E$. Then for all $i \in I$ we have $\phi_i(a) = \phi(a)(i) = \phi(b)(i) = \phi_i(b)$. This implies that $\phi_i(a \rightarrow b) = 1$, for all $i \in I$ and so $a \rightarrow b = 1$. Since $\mathcal{E}$ is separated so $a = b$, whence ϕ is an embedding. ■

Lemma 5. *If $F_i \in Fil(\mathcal{E})$ and $\bigcap_{i\in I} F_i = \{1\}$, then the natural homomorphism $\nu : \mathcal{E} \rightarrow \prod_{i\in I} \mathcal{E}/F_i$ defined by $\nu(a)(i) = a/F_i$ is a subdirect embedding.*

Proof. Consider the natural homomorphism $\nu_i : \mathcal{E} \rightarrow \mathcal{E}/F_i$. Since $Ker\nu_i = F_i$, from Lemma 4 it follows that ν is an embedding. On the other hand, since every ν_i is surjective, so ν is subdirect embedding. ■

Theorem 5. *A nontrivial EQ-algebra $\mathcal{E}$ is subdirectly irreducible if and only if the intersection of all members of $Fil(\mathcal{E}) - \{1\}$ differs from $\{1\}$.*

Proof. Assume that $\cap\,(Fil(\mathcal{E}) - \{1\}) = \{1\}$ and let $I = Fil(\mathcal{E}) - \{1\}$. Then the mapping $\phi : \mathcal{E} \rightarrow \prod_{F\in I} \mathcal{E}/F$ is a subdirect embedding, by Lemma 5 and since the mapping $\mathcal{E} \rightarrow \mathcal{E}/F$ is not injective, so $\mathcal{E}$ is not subdirectly irreducible.

Conversely, assume that $F = \cap\,(Fil(\mathcal{E}) - \{1\} \neq \{1\}$. Let $a \in E$ be such that $a \in F$ but $a \neq 1$. If $\phi : \mathcal{E} \rightarrow \prod_{i\in I} \mathcal{E}_i$ is a subdirect embedding, on one hand the mapping $\pi_i\phi : \mathcal{E} \rightarrow \mathcal{E}_i$ is an empimorphism, on the other hand $\phi(a)(i) \neq 1$, for some $i \in I$. This implies that $\pi_i\phi(a) \neq 1$ and so $F \nsubseteq Ker(\pi_i\phi)$, whence $Ker(\pi_i\phi) = \{1\}$, means that $\pi_i\phi$ is injective. Hence $\pi_i\phi$ is an isomorphism. ■

Corollary 3. *A nontrivial EQ-algebra $\mathcal{E}$ is subdirectly irreducible if and only if when $\mathcal{E}$ is isomorphic to a subdirect product of the family $\{\mathcal{E}_i : i \in I\}$ of EQ-algebras, then $\mathcal{E} \simeq \mathcal{E}_i$, for some $i \in I$.*

Proof. Assume that $\mathcal{E}$ is subdirectly irreducible and is isomorphic by a subdirect product of a family $\{\mathcal{E}_i : i \in I\}$. By Theorem 4, for each $i \in I$ there is $F_i \in Fil(\mathcal{E})$ such that $\mathcal{E}/F_i \simeq \mathcal{E}_i$ and $\bigcap_{i\in I} F_i = \{1\}$. Considering Theorem 5, we conclude that $F_i = \{1\}$, for some $i \in I$, and hence $\mathcal{E} \simeq \mathcal{E}_i$, for some $i \in I$.

Conversely, assume that the intersection of all nontrivial filters of $\mathcal{E}$ is trivial and $I = FiL(\mathcal{E}) - \{1\}$. Then the homomorphism $\phi_i : \mathcal{E} \to \mathcal{E}/F_i$ induces the homomorphism $\phi : \mathcal{E} \to \prod_{F_i \in I} \mathcal{E}/F_i$ which is an embedding because $\bigcap_{i\in I} F_i = \{1\}$. Obviuosly, $\pi_i\phi(E) = E/F_i$. Hence $\mathcal{E}$ is isomorphic by a subdirect product of the family of $\{\mathcal{E}/F_i : i \in I\}$. So, by hypothesis, $\mathcal{E} \simeq \mathcal{E}/F_i$, for some $i \in I$. This implies that $F_i = \{1\}$, which is a contradiction. ■

4. Conclusion

Homomorphism theorems are a useful and applicable tool to characterize and classify algebras of the same type. In this paper we investigated some isomorphism theorems to characterize EQ-algebras such as Fundamental Homomorphism Theorem. We also investigate some other characterizations for EQ-algebras by using isomorphism theorems.

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