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Extended odd Frechet-exponential distribution with applications related to the environment

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Abstract

In this paper, we attempted to expand the Frechet distribution by employing the T-X family of distributions and named the newly formulated model Extended odd Frechet-exponential distribution (EOFED). Several structural properties, reliability measurements and characteristics were estimated and discussed. The study presents graphs which depict the behaviour of the probability density function, cumulative distribution function and the hazard rate function. The adaptability and flexibility of this novel distribution were achieved through the application of real-world data sets. A simulation study was performed to evaluate and compare the output efficacy of the estimators.

Key words: Frechet distribution, moments, quantile function, Renyi entropy, maximum likelihood estimation, order statistics.

1. Introduction

The Frechet distribution is a subset of the generalized extreme value distribution named after a French mathematician Maurice Rene Frechet (1878–1973), who defined it as the limiting potential distribution for sequences of extremes and published a related work in 1927. Later, Fisher and Tippett in 1928 and Gumbel in 1958 completed even more research. Rosin and Rammler employed it in 1933 to match a particle size distribution. By employing various techniques and methods, many researchers have done a lot of work by extending the Frechet distribution and thus showing its importance in various fields like Social, Medical and Engineering Sciences (such as work done by Haq et al. (2017), Pelumi et al. (2019), Penson et al. (2014), Hamed M S (2020) and so on).

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The cdf of the extended Frechet distribution is given by

$$F(t) = (1+\beta t^{-\alpha})^{\frac{-1}{\beta}} \quad ; t>0, \alpha, \beta > 0 \tag{1.1}$$

$$f(t) = \alpha t^{-\alpha-1}(1+\beta t^{-\alpha})^{\frac{-1}{\beta}-1} \quad ; t>0, \alpha, \beta > 0 \tag{1.2}$$

The exponential distribution is the continuous analogue of a geometric distribution. The exponential distribution is one parametric continuous distribution and has various attractive statistical characteristics like memoryless property. Historically, the first lifetime model for which statistical methods were extensively developed was the exponential distribution. In particular, it has its applications in various fields like biomedicine, engineering, economics, medical sciences, etc.; see (Tomitaka, Kawasaki, Ide, Akutagawa, Yamada and Furukawa (2017).

$$G(x,\theta) = 1 - e^{-\theta x}; x, \theta > 0 \tag{1.3}$$

$$\bar{G}(x,\theta) = e^{-\theta x}; x, \theta > 0 \tag{1.4}$$

The related pdf of the exponential distribution is

$$g(x,\theta) = \theta e^{-\theta x}; x, \theta > 0 \tag{1.5}$$

Transformed-Transformer (T-X) family of distributions (Alzaatreh et al. (2013)) is given by

$$F(x) = \int_0^{W[G(x)]} f(t)dt \tag{1.6}$$

where f(t) is the probability density function of a random variable T and $W[G(x)]$ is a function of cumulative density function of random variable X.

Suppose $[G, \xi]$ denotes the baseline cumulative distribution function, which depends on parameter vectorξ. Now, using T-X approach, the cumulative distribution function of the extended odd Frechet-exponential distribution (EOFED) can be derived by replacing f(t) in equation (1.6) by equation (1.2) and $W[G(x)] = \frac{G(x,\xi)}{\bar{G}(x,\xi)}$,where $\bar{G}(x,\xi) = 1 - G(x,\xi)$, which follows

$$F(x,\xi) = \int_0^{\frac{G(x,\xi)}{\bar{G}(x,\xi)}} \alpha t^{\alpha-1}(1+\beta t^{-\alpha})^{\frac{-1}{\beta}-1} dt$$

$$F(x,\xi) = \left[1+\beta\left(\frac{G(x,\xi)}{\bar{G}(x,\xi)}\right)^{-\alpha}\right]^{\frac{-1}{\beta}} \tag{1.7}$$

The corresponding pdf of (1.7) becomes

$$f(x,\xi) = \alpha g(x,\xi)\frac{(G(x,\xi))^{-\alpha-1}}{(\bar{G}(x,\xi))^{-\alpha+1}}\left[1+\beta\left(\frac{G(x,\xi)}{\bar{G}(x,\xi)}\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1} \tag{1.8}$$

2. Mixture Form

From equation (1.8) we have

$$f(x,\xi) = \alpha g(x,\xi)\frac{(G(x,\xi))^{-\alpha-1}}{(\bar{G}(x,\xi))^{-\alpha+1}}\left[1+\beta\left(\frac{G(x,\xi)}{\bar{G}(x,\xi)}\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1}$$

Using binomial expansion

$$(1+x)^{-a}=\sum_{k=0}^{\infty}\binom{-a}{k}x^k|x|<1$$

$$f(x,\xi)=\alpha g(x,\xi)\frac{(G(x,\xi))^{-\alpha-1}}{(\bar{G}(x,\xi))^{-\alpha+1}}\sum_{p=0}^{\infty}\binom{-\frac{1}{\beta}-1}{p}\beta^p\left(\frac{G(x,\xi)}{\bar{G}(x,\xi)}\right)^{-p\alpha}$$

Again, using binomial expansion we get

$$f(x,\xi)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q\binom{-\frac{1}{\beta}-1}{p}\binom{(p+1)\alpha-1}{q}\alpha\beta^p g(x,\xi)(G(x,\xi))^{q-(p+1)\alpha-1}$$

Using (1.3) and (1.5) in the above expression we get

$$f(x,\xi)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^q\binom{-\left(\frac{1}{\beta}+1\right)}{p}\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{s}(-1)^s\alpha\beta^p\theta e^{-\theta x}\left(e^{-\theta xs}\right)$$

$$f(x,\xi)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\varsigma_{pqs}\alpha\beta^p\theta e^{-\theta x(s+1)} \tag{2.1}$$

where $\varsigma_{pqs}=(-1)^q(-1)^s\binom{-\left(\frac{1}{\beta}+1\right)}{p}\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{s}$

3. Formation of Extended Odd Frechet-Exponential Distribution

The extended odd Frechet-exponential distribution (EOFED) is developed by using the T-X family of distributions as described by Alzaatreh et al. (2013). We formulated the distribution as follows.

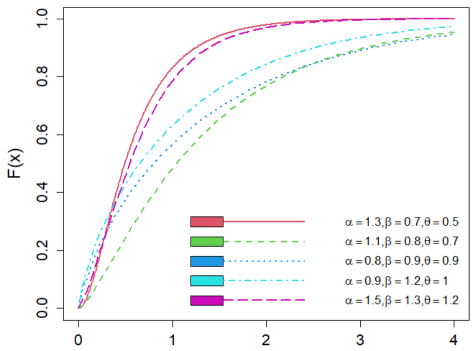
Substituting equation (1.3) and (1.4) in equation (1.7), we obtain the cdf of formulated distribution, which follows

$$F(x,\alpha,\beta,\theta)=\left[1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}};x>0,\alpha,\beta,\theta>0 \tag{3.1}$$

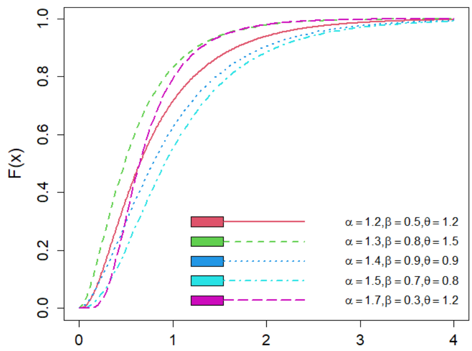
The related pdf of (3.1) is given by

$$f(x,\alpha,\beta,\theta)=\alpha\theta e^{-\theta x}\left(e^{-\theta x}-1\right)^{-\alpha-1}\left[1+\beta\left(e^{-\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1};x>0,\alpha,\beta,\theta>0 \tag{3.2}$$

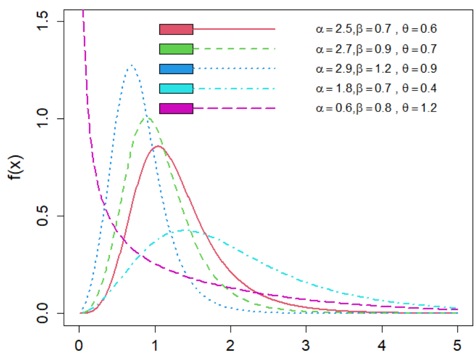
Figures (3.1), (3.2), (3.3) and (3.4) expound some of possible contours of cdf and pdf for a distinct choice of parameters respectively.



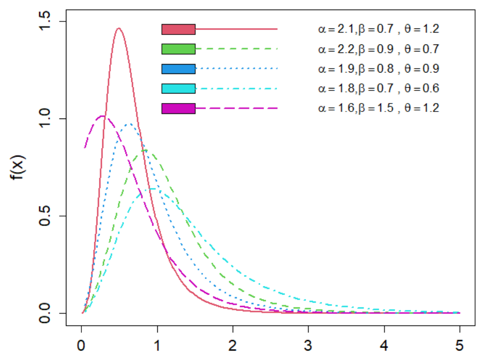
3.1. CDF plots of EOFED(α,β,θ)



3.2 CDF plots of EOFED(α,β,θ)



3.3 PDF plots of EOFED(α,β,θ)



3.4 PDF plots of EOFED(α,β,θ)

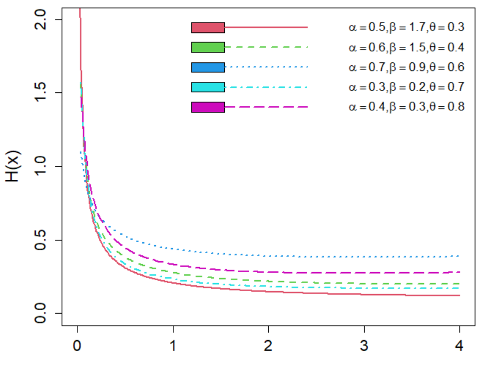
4. Reliability Measures

Survival function, hazard function and reverse hazard function of extended odd Frechet-exponential distribution (EOFED) are given by

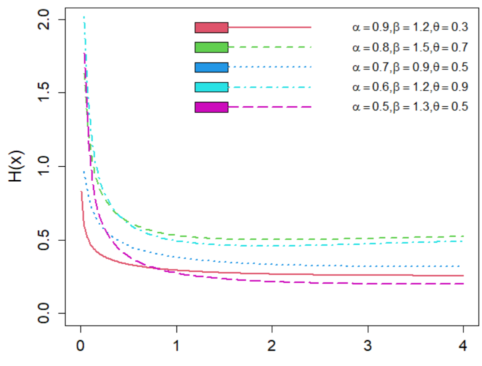
$$S_x(x) = 1 - \left[1 + \beta\left(e^{\theta x} - 1\right)^{-\alpha}\right]^{\frac{-1}{\beta}}$$

$$H(x) = \frac{\alpha\theta e^{\theta x}\left(e^{\theta x}-1\right)^{-\alpha-1}\left(1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right)^{\frac{-1}{\beta}-1}}{1-\left\{1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right\}^{\frac{-1}{\beta}}}$$

Figure (4.1) and (4.2) shows some of possible shapes of extended odd Frechet-exponential distribution's hazard rate function for distinct choice of parameters respectively.



4.1. HRF plots of EOFED(α,β,θ)



4.2. HRF plots of EOFED(α,β,θ)

5. Structural properties of extended odd Frechet-exponential distribution (EOFED)

Theorem 1: $If \rightleftarrows x \sim E\,OFED(\alpha,\beta,\theta)$ then its r[th] moment is given by

$$E(x)^r = \mu_r{}' = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty} \varsigma_{pqs} \frac{\Gamma(r+1)}{((s+1)\theta)^{r+1}}$$

Proof: We know that r[th] moment about origin is given by

$$\mu_r{}' = E(x^r) = \int_0^{\infty} x^r f(x,\alpha,\beta,\theta)dx$$

$$= \int_0^{\infty} x^r \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^q \binom{-\left(\frac{1}{\beta}+1\right)}{p}\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{s}(-1)^s\alpha\beta^p\theta e^{-\theta x}\left(e^{-\theta x s}\right)dx$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}(-1)^q(-1)^s \binom{-\left(\frac{1}{\beta}+1\right)}{p}\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{s}\alpha\beta^p\theta\int_0^{\infty} x^r e^{-\theta x}\left(e^{-\theta x s}\right)dx$$

After solving the above integral, we get the following expression

$$\mu_r{}' = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\varsigma_{pqs}\frac{\Gamma(r+1)}{((s+1)\theta)^{r+1}} \tag{5.1}$$

where $\varsigma_{pqs} = (-1)^q(-1)^s\binom{-\left(\frac{1}{\beta}+1\right)}{p}\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{s}\alpha\beta^p\theta$

Putting r=1,2,3,4 in the above equation we get first four moments about origin.

Theorem 2: $If \rightleftarrows x \sim E\,OFED(\alpha,\beta,\theta)$, then it shows that the moment generating function of the extended odd Frechet-exponential distribution (EFOED) is given by

$$M_x(t) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}\varsigma_{pqs}\frac{\Gamma(r+1)}{((s+1)\theta)^{r+1}}$$

Proof: We know that the moment generating function is given by

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x;\alpha,\beta,\theta)dx$$

Applying Taylor's theorem

$$M_x(t) = E(e^{tx}) = \int_0^\infty \left\{1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots\right\} f(x;\alpha,\beta,\theta)dx$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty \varsigma_{pqs} \frac{\Gamma(r+1)}{((s+1)\theta)^{r+1}}$$

Using (5.1)

$$M_x(t) = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty \sum_{r=0}^\infty \frac{t^r}{r!} \varsigma_{pqs} \frac{\Gamma(r+1)}{((s+1)\theta)^{r+1}} \tag{5.2}$$

The characteristic function of the extended odd Frechet-exponential distribution (EOFED) is obtained by replacing $t$by itin equation (5.2) as follows:

$$\Phi_x(it) = E(e^{itx}) = \int_0^\infty e^{itx} f(x;\alpha,\beta,\theta)dx$$

$$\Phi_x(it) = \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty \sum_{r=0}^\infty \frac{(it)^r}{r!} \varsigma_{pqs} \frac{\Gamma(r+1)}{((s+1)\theta)^{r+1}} \tag{5.3}$$

6. Quantile Function

The quantile function of the above distribution can be put in the following form:

$$Q_n(u) = X_q = F^{-1}(u)$$

where $Q_n(u)$ represents the quantile function of $F(x)$ for $u \in (0,1)$.

Suppose we put $F(x) = \left[1 + \beta\left(e^{\theta x} - 1\right)^{-\alpha}\right]^{\frac{-1}{\beta}} = u$ (6.1)

After solving equation (6.1), we obtain the quantile function of the extended odd Frechet-exponential distribution as given by

$$Q_n(u) = X_q = \frac{1}{\theta} log\left[1 + \left(\frac{\mu^{-\beta} - 1}{\beta}\right)^{\frac{-1}{\alpha}}\right]$$

7. Renyi Entropy

If x is a continuous random variable following EOFED with pdf $f(x;\alpha,\beta,\theta)$, then

$$R_e = \frac{1}{1-\rho} log\left\{\int_0^\infty f^\rho(x,\alpha,\beta,\theta)dx\right\}$$

$$R_e = \frac{1}{1-\rho} log\left\{\sum_{p=0}^\infty \sum_{q=0}^\infty \binom{-\rho\left(\frac{\beta+1}{\beta}\right)}{p} \binom{(p+\rho)\alpha - 1}{q} \alpha^\rho \beta^p \int_0^\infty \left(\theta e^{-\theta x}\right)^\rho \left(1 - e^{-\theta x}\right)^{q-(\rho+p)\alpha-1} dx\right\}$$

$$R_e = \frac{1}{1-\rho} log \sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty (-1)^q (-1)^s \binom{-\rho\left(\frac{\beta+1}{\beta}\right)}{p} \binom{(p+\rho)\alpha - 1}{q} \binom{q-(p+\rho)\alpha - 1}{s} \alpha^\rho \beta^p \int_0^\infty \left(\theta e^{-\theta x}\right)^\rho e^{-\theta xs} dx$$

$$R_e = \frac{1}{1-\rho} log\left\{\sum_{p=0}^\infty \sum_{q=0}^\infty \sum_{s=0}^\infty \varsigma_{pqs} \frac{\theta^\rho}{(s+\rho)\theta}\right\} \tag{7.1}$$

8. Tsallis Entropy

If x is a continuous random variable following EOFED with pdf $f(x\,;\alpha\,,\beta,\theta)$, then

$$s_e = \frac{1}{\rho-1}\left\{1-\int_0^\infty f^\rho(x,\alpha,\beta,\theta)dx\right\}$$

$$s_e = \frac{1}{\rho-1}\left\{1-\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\varsigma_{pqs}\frac{\theta^\rho}{(s+\rho)\theta}\right\} \quad (8.1)$$

9. Incomplete Moments

We know that

$$I_q(x) = \int_0^s x^s f(x\,;\alpha\,,\beta,\theta)dx$$

$$I_q(x) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^q(-1)^r\left(-\binom{\frac{\beta+1}{\beta}}{p}\right)\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{r}\alpha\beta^p\theta\int_0^s x^s e^{-\theta x(s+1)}\,dx$$

$$I_q(x) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\varsigma_{pqs}\int_0^s x^s e^{-\theta x(s+1)}\,dx$$

After solving the above integral, we will get the following equation:

$$I_q(x) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\varsigma_{pqs}\frac{\gamma((s+1),\theta(s+1)s)}{(\theta(s+1))^{s+1}} \quad (9.1)$$

10. Mean Residual Function

We know that

$$m_r(x) = \frac{1}{S_x(x)}\int_x^\infty t\,f(t\,;\alpha\,,\beta,\theta)dt - x$$

$$m_r(x) = \frac{1}{S_x(x)}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^q\left(-\binom{\frac{\beta+1}{\beta}}{p}\right)\binom{(p+1)\alpha-1}{q}\alpha\beta^p\int_x^\infty t\left(\theta e^{-\theta t}\right)\left(1-e^{-\theta t}\right)^{q-(p+1)\alpha-1}dt - x$$

After solving the above equation, we get

$$m_r(x) = \frac{1}{1-\left[1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\varsigma_{pqs}\frac{\Gamma(2,\theta(s+1)x)}{\{\theta(s+1)\}^2} - x \quad (10.1)$$

11. Mean deviation from Mean

We know that

$$D(\mu) = E(|x-\mu|)$$

$$D(\mu) = 2\mu F(\mu) - 2\int_0^\mu xf(x)dx \quad (11.1)$$

We know that

$$\int_0^{\mu} xf(x)dx$$
$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^q(-1)^r\binom{-\left(\frac{\beta+1}{\beta}\right)}{p}\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{r}\alpha\beta^p\theta\int_0^{\mu} xe^{-(s+1)\theta x}dx$$

$$\int_0^{\mu} xf(x)dx=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\varsigma_{pqs}\frac{\gamma(2,\theta(s+1)\mu)}{(\theta(s+1))^2} \quad (11.2)$$

Now, putting the above value in equation (11.1), we get the required equation as given below:

$$D(\mu)=2\mu[1+\beta(e^{\mu}-1)^{-\alpha}]^{\frac{-1}{\beta}}-2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\varsigma_{pqs}\frac{\gamma(2,\theta(s+1)\mu)}{(\theta(s+1))^2} \quad (11.3)$$

12. Mean deviation from Median

We know that

$$D(M)=E(|x-M|)$$
$$D(M)=\mu-2\int_0^{M} xf(x)dx \quad (12.1)$$

We have

$$\int_0^{M} xf(x)dx$$
$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^q(-1)^r\binom{-\left(\frac{\beta+1}{\beta}\right)}{p}\binom{(p+1)\alpha-1}{q}\binom{q-(p+1)\alpha-1}{r}\alpha\beta^p\theta\int_0^{M} e^{(s+1)\theta x}dx$$

$$\int_0^{M} xf(x)dx=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\varsigma_{pqs}\frac{(\gamma(2,\theta(s+1)M))}{(\theta(s+1))^2} \quad (12.2)$$

Now, putting the above value in equation (12.1), we get the required equation as given below:

$$D(M)=\mu-2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\varsigma_{pqs}\frac{(\gamma(2,\theta(s+1)M))}{(\theta(s+1))^2} \quad (12.3)$$

13. Order Statistics

Let $x_{(1)},x_{(2)},x_{(3)},\ldots,x_{(n)}$ denote the order statistics of n random samples drawn from the extended odd Frechet-exponential distribution. Then, the pdf of $x_{(k)}$is given by

$$f_{x(k)}(x;\theta)=\frac{n!}{(k-1)!\,(n-k)!}f_X(x)[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}$$

$$f_{x(k)}(x;\alpha,\beta,\theta)=\left[\begin{array}{c}\frac{n!}{(k-1)!\,(n-k)!}\alpha\theta e^{-\theta x}\left(e^{-\theta x}-1\right)^{-\alpha-1}\left[1+\beta\left(e^{-\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1}\\ *\left\{\left[1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}}\right\}^{k-1}\left(1-\left[1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1}\right)\end{array}\right]^{n-k}$$

Then, the pdf of first order $X_{(1)}$ of the extended odd Frechet-exponential distribution is given by

$$f_{x(1)}(x;\alpha,\beta,\theta) = n\alpha\theta e^{-\theta x}\left(e^{-\theta x}-1\right)^{-\alpha-1}\left[1+\beta\left(e^{-\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1}\left(1 - \left[1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1}\right)^{n-1}$$

and the pdf of nth order $X_{(n)}$ of the extended odd Frechet-exponential distribution is given by

$$f_{x(n)}(x;\alpha,\beta,\theta) = n\alpha\theta e^{-\theta x}\left(e^{-\theta x}-1\right)^{-\alpha-1}\left[1 + \beta\left(e^{-\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}-1}\left\{\left[1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right]^{\frac{-1}{\beta}}\right\}^{n-1}$$

14. Maximum Likelihood Estimation

Let $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ be n random samples from the extended odd Frechet-exponential distribution, and then its likelihood function is given by

$$l = \prod_{i=1}^{n} f(x,\alpha,\beta,\theta)$$

$$l = (\alpha\theta)^n e^{\sum_{i=1}^{n}\theta x}\prod_{i=1}^{n}\left(e^{\theta x}-1\right)^{-\alpha-1}\left(1+\beta\left(e^{\theta x}-1\right)^{-\alpha}\right)^{-\frac{1}{\beta}-1}$$

Taking log on both sides we get

$$log\, l = n\, log\, \alpha + n\, log\, \theta + \sum_{i=1}^{n}\theta x_i - (\alpha+1)\sum_{i=1}^{n} log\left(e^{\theta x_i}-1\right) - \left(\frac{1}{\beta}+1\right)\sum_{i=1}^{n} log\left(1+\beta\left(e^{\theta x_i}-1\right)^{-\alpha}\right)$$

Differentiating w.r.t α,β and θ the above equation we get

$$\frac{\partial\, log\, l}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} log\left(e^{\theta x_i}-1\right) + (\beta+1)\sum_{i=1}^{n}\frac{\left(e^{\theta x_i}-1\right)^{-\alpha}}{1+\beta(e^{\theta x_i}-1)^{-\alpha}} log\left(e^{\theta x_i}-1\right)$$

$$\frac{\partial\, log\, l}{\partial\beta} = -\left(\frac{1}{\beta}+1\right)\sum_{i=1}^{n}\frac{\left(e^{\theta x_i}-1\right)^{-\alpha}}{1+\beta(e^{\theta x_i}-1)^{-\alpha}} + \left(\frac{1}{\beta^2}\right)\sum_{i-1}^{n} log\left(1+\beta\left(e^{\theta x_i}-1\right)^{-\alpha}\right)$$

$$\frac{\partial\, log\, l}{\partial\theta} = \frac{n}{\theta} + \sum_{i-1}^{n} x_i - (\alpha+1)\theta\sum_{i=1}^{n}\frac{e^{\theta x_i}}{e^{\theta x_i}-1} + \alpha(\beta+1)\theta\sum_{i=1}^{n}\frac{e^{\theta x_i}\left(e^{\theta x_i}-1\right)^{-\alpha-1}}{1+\beta(e^{\theta x_i}-1)^{-\alpha}}$$

The above mentioned equations are non-linear equations that cannot be expressed in compact form, and solving them directly for $\alpha, \beta$and θ is difficult. The MLE of the parameters labelled as $\hat{\varsigma}(\hat{\alpha},\hat{\beta},\hat{\theta})$of $\varsigma(\alpha,\beta,\theta)$ may be derived by employing iterative methods such as Newton–Raphson method, secant method, regula–falsi method, and so on.

15. Application

In this segment, the efficacy of the newly developed distribution has been assessed using two realistic sets of data. As the new distribution is compared to new modified Weibull distribution (NMWD), extended Frechet distribution (EFD), Frechet distribution (FD), exponential distribution (ED), inverse Weibull Burr distribution (IWBD) and Kumarswamy power Frechet distribution (KPFD). It is revealed that the newly developed distribution offers an appropriate fit.

Various criteria including the AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion), HQIC (Hannan-Quinn information criteria) and KS (Kolmogorov-Smirnov) are used to compare the fitted models. The p-value of each model is also recorded.

Data Set 1: The following observations contain 30 successive march precipitations (in inches). These observations were studied by Hankley 1977. The observations are as follows: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.55, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05

Table 15.1: Descriptive statistics of data set 1

Min.	Q_1	Median	Mean	Q_3	Max.	S.D	Skew.	Kurt.
0.320	0.915	1.470	1.676	2.087	4.750	0.999	1.092	4.216

Table 15.2: The ML Estimates of the unknown parameters for data set 1

Model	Estimates				Standard Error			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
EOFED	1.743	0.695	0.501	-----	0.729	0.806	0.117	----
NMWD	0.126	0.193	1.631	2.551	0.089	0.144	0.473	0.789
EFD	1.899	0.241	-----	------	0.384	0.256	-----	-----
FD	0.802	1.162	-----	-----	0.164	0.177	------	-----
ED	0.597	-----	-----	-----	0.108	-----	-----	-----
IWBD	0.314	1.813	1.000	-----	0.164	0.372	0.290	-----
KPFD	0.569	0.106	66.183	48.659	0.442	0.418	264.9	158.9

Table 15.3: Measures of goodness-of-fit statistics for the data set 1

Model	EOFED	NMWD	EFD	FD	ED	IWBD	KPFD
-2logl	76.501	76.789	82.398	108.26	90.985	84.303	76.044
AIC	80.501	84.789	86.398	112.26	92.985	83.003	84.033
CAIC	82.945	86.389	86.842	112.71	93.429	98.117	85.633
HQIC	81.397	86.583	87.294	113.16	95.881	96.336	85.827
BIC	86.704	90.394	89.200	115.07	94.386	89.001	89.638

Table 15.4: The K-S and p-value for data set 1

Model	EOFED	NMWD	EFD	FD	ED	IWBD	KPFD
K-S value	0.0680	0.0612	0.2102	0.2160	0.5159	0.1881	0.0608
p-value	0.9999	0.9991	0.1409	0.1216	2.319 e-07	0.2393	0.9998

Data Set 2: The data is obtained from Hinkley (1977) and consists of 30 successive values of March precipitation (in inches) in Minneapolis/St. Paul. The data sets are: 2.2, 3.37, 1.43, 0.74, 1.2, 1.95, 1.2, 0.81, 1.74, 0.77, 0.81, 0.59, 0.32, 1.31, 1.62, 0.52, 2.1, 1.51, 3.09, 3.00, 2.05, 0.9, 1.89, 0.96, 2.48, 4.75, 1.35, 1.18, 1.87, 2.81.

Table 15.5: Descriptive statistics of data set 2

Min.	Q_1	Median	Mean	Q_3	Max.	S.D	Skew.	Kurt.
0.320	0.915	1.470	1.684	2.087	4.750	0.9905	1.1260	4.2882

Table 15.6: The ML Estimates of the unknown parameters for data set 2

Model	Estimates				Standard Error			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
EOFED	1.710	0.589	0.508	-----	0.640	0.682	0.107	----
NMWD	0.117	0.192	1.682	2.683	0.083	0.133	0.474	0.816
EFD	1.941	0.216	-----	------	0.385	0.235	-----	-----
FD	0.855	1.185	-----	-----	0.173	0.178	------	-----
ED	0.594	-----	------	------	0.108	-----	-----	-----
IWBD	0.305	1.843	1.000	-----	0.159	0.372	0.287	-----
KPFD	0.621	0.151	43.21	35.01	0.623	1.005	291.9	123.8

Table 15.7: Measures of goodness-of-fit statistics for the data set 2

Model	EOFED	NMWD	EFD	FD	ED	IWBD	KPFD
-2logl	75.552	76.039	81.584	105.42	91.270	76.039	76.223
AIC	81.552	84.039	85.584	109.43	93.270	84.040	83.207
CAIC	81.996	85.639	86.028	109.87	93.715	85.639	84.807
HQIC	80.448	85.833	86.480	110.32	96.167	85.833	84.999
BIC	85.755	89.645	88.386	112.23	94.672	89.645	88.812

Table 15.8: The K-S and p-value for data set 2

Model	EOFED	NMWD	EFD	FD	ED	IWBD	KPFD
K-S value	0.0696	0.0612	0.2097	0.1953	0.5307	0.2817	0.0640
p-value	0.9999	0.9986	0.1423	0.2025	9.2e-08	0.01713	0.9997

Fig 15.1 Estimated pdf's of the fitted models for data set1

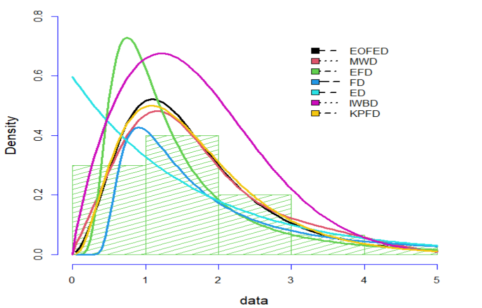


Fig 15.2 Empirical cdf versus fitted cdf's for data set1

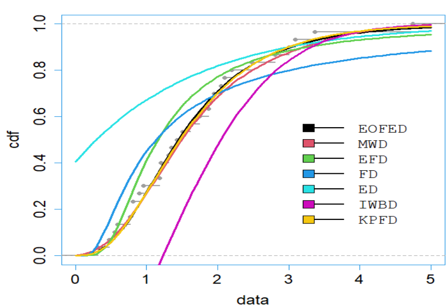


Fig 15.3 Estimated pdf's of the fitted models for data set2

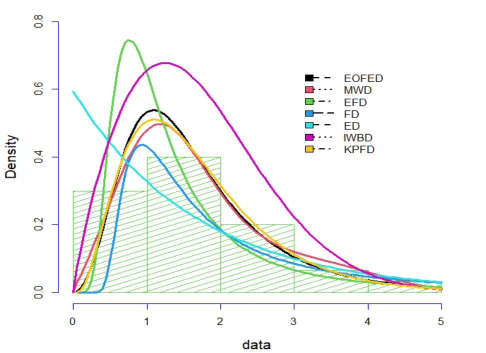
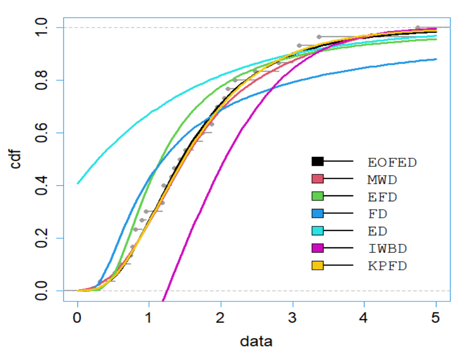


Fig 15.4 Empirical cdf versus fitted cdf's for data set2



Figures (15.1), (15.2), (15.3) and (15.4) represent the estimated densities and cdfs of the fitted distributions to data set 1 and 2.

16. Simulation Study

In this section, we study the performance of ML estimators for different sample sizes (n=150, 250,500, 800). We have employed the inverse CDF technique for data simulation for EOFED distribution using R software. The process was repeated 1000 times for calculation of bias, variance and MSE. It is evident from the tables that a decreasing trend is being observed in bias, variance and MSE as we increase the sample size. Hence, the performance of ML estimators is quite well, consistent in the case of the extended odd Frechet-exponential distribution.

Table 16.1: sThe Bias, Variance and MSEs of 1,000 simulations of EOFED for parameter values $\alpha = 1.0$ $\beta = 0.9$ and $\theta = 2.3$

Sample Size	Parameters	Bias	Variance	MSE
150	α	-0.064	7.41e-05	0.0212
	β	-0.125	8.11e-05	0.0157
	θ	-0.216	7.07e-03	0.0531
250	α	-0.062	7.23e-05	0.0210
	β	-0.123	6.81e-05	0.0156
	θ	-0.213	5.24e-05	0.0497
500	α	-0.061	5.43e-05	0.0200
	β	-0.122	3.11e-05	0.0152
	θ	-0.209	3.15e-05	0.0470
800	α	-0.059	4.22e-05	0.0199
	β	-0.121	2.98e-05	0.0149
	θ	-0.209	1.33e-03	0.0453

Table 16.2: The Bias, Variance and MSEs of 1,000 simulations of EOFED for parameter values $\alpha = 1.1$ $\beta = 0.8$ and $\theta = 2.0$

Sample Size	Parameters	Bias	Variance	MSE
150	α	-0.156	6.88e-o5	0.0266
	β	-0.029	6.04e-05	0.0009
	θ	-0.199	6.59e-03	0.0459
250	α	-0.155	5.29e-05	0.0253
	β	-0.027	5.05e-05	0.0008
	θ	-0.191	5.33e-03	0.0399
500	α	-0.149	2.38e-05	0.0222
	β	-0.026	2.70e-05	0.0007
	θ	-0.187	2.74e-03	0.0376
800	α	-0.148	2.09e-05	0.0219
	β	-0.026	1.89e-05	0.0007
	θ	-0.185	1.23e-03	0.0352

17. Result and Discussion

In this paper, a new distribution named “extended odd Frechet-exponential distribution”, which is obtained by T-X method, is introduced. Several mathematical quantities for the newly developed distribution are derived. The method of maximum likelihood estimation is used to estimate the unknown parameters of the established model. It is obvious from Tables 15.3, 15.4, 15.7 and 15.8 that the extended odd Frechet-exponential distribution has smaller values for AIC, AICC, BIC, HQIC and K-S statistics as well as higher p-value when compared with other competitive models. Accordingly, we arrive at the conclusion that the extended odd Frechet-exponential distribution provides a more adequate fit than the compared ones.

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