Discussiones Mathematicae
Graph Theory 27 (2007) 565–582

14th WORKSHOP
'3in1' GRAPHS 2005
Dobczyce, November 10-12, 2005

International
Visegrad Fund

THE REPRESENTATION OF MULTI-HYPERGRAPHS BY SET INTERSECTIONS

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Abstract

This paper deals with weighted set systems $(V, \mathcal{E}, q)$, where V is a set of indices, $\mathcal{E} \subset 2^V$ and the weight q is a nonnegative integer function on $\mathcal{E}$. The basic idea of the paper is to apply weighted set systems to formulate restrictions on intersections. It is of interest to know whether a weighted set system can be represented by set intersections. An intersection representation of $(V, \mathcal{E}, q)$ is defined to be an indexed family $\mathcal{R} = (R_v)_{v \in V}$ of subsets of a set S such that

$$\Big| \bigcap_{v \in E} \mathcal{R}_v \Big| = q(E) \quad \text{for each } E \in \mathcal{E}.$$

A necessary condition for the existence of such representation is the monotonicity of q on $\mathcal{E}$ i.e., if $F \subset E$ then $q(F) \geq q(E)$. Some sufficient conditions for weighted set systems representable by set intersections are given. Appropriate existence theorems are proved by construction of the solutions.

The notion of intersection multigraphs to intersection multi-hypergraphs — hypergraphs with multiple edges, is generalized. Some conditions for intersection multi-hypergraphs are formulated.

Keywords: intersection graph, intersection hypergraph.

2000 Mathematics Subject Classification: 05C62, 05C65.

1. Introduction and Preliminaries

We use the standard notation of set theory. For a set X, the cardinality of X and the family of all subsets of X will be denoted by $|X|$ and 2^X, respectively.

A family of sets $\mathcal{E} \subset 2^X$ is called antichain if for each two distinct elements $F, E \in \mathcal{E}$ neither $F \subset E$ nor $E \subset F$. Denote by

$$Min\mathcal{E} = \{E \in \mathcal{E} : \text{ if } F \in \mathcal{E} \text{ and } F \subset E \text{ then } F = E\}$$

the subfamily of all minimal sets (lower base) of $\mathcal{E}$ and denote by

$$Max\mathcal{E} = \{E \in \mathcal{E} : \text{ if } F \in \mathcal{E} \text{ and } E \subset F \text{ then } F = E\}$$

the subfamily of all maximal sets (upper base) of $\mathcal{E}$. A family $\mathcal{E}$ is an antichain if and only if $Min\mathcal{E} = \mathcal{E} = Max\mathcal{E}$. Furthermore, we denote the lower closure and upper closure of the family $\mathcal{E}$ by

$$\mathcal{E}^- = \{F \in 2^X : \text{ there is } E \in \mathcal{E} \text{ such that } F \subset E\}$$

and

$$\mathcal{E}^+ = \{F \in 2^X : \text{ there is } E \in \mathcal{E} \text{ such that } E \subset F\},$$

respectively.

A triple $(V, \mathcal{E}, q)$ is called *weighted set system* if V is a set, $\mathcal{E} \subset 2^V$ and q is a nonnegative integer function on $\mathcal{E}$.

Consider an indexed family $\mathcal{R} = (R_v)_{v \in V}$ of subsets (not necessarily distinct or nonempty) of a set S (the "universe" of the family). The intersection of several members of $\mathcal{R}$ is determined by the set of its indices $E \subset V$. The cardinality of such intersection is called *intersection size* and is denoted by $q_{\mathcal{R}}(E)$. For example, information with respect to a triple intersection $R_{v_1} \cap R_{v_2} \cap R_{v_3}$ is given by the set E and its size $q_{\mathcal{R}}(E)$, where

$E = \{v_1, v_2, v_3\}$ and $q_{\mathcal{R}}(E) = |\, R_{v_1} \cap R_{v_2} \cap R_{v_3}\,|$. A collection of such information (called also restrictions) leads to a weighted set system $(V, \mathcal{E}, q_{\mathcal{R}})$, where $\mathcal{E} \subset 2^V$. Especially, in this way we obtain $(V, 2^V, q_{\mathcal{R}})$,

$$q_{\mathcal{R}}(E) = \Big| \bigcap_{v \in E} R_v \Big| \quad \text{for each } E \in 2^V \text{ and } E \neq \emptyset$$

$$\text{and } q_{\mathcal{R}}(\emptyset) = |S|,$$

which is called *complete intersection set system of the family* $\mathcal{R}$. We say also that the weight $q_{\mathcal{R}}$ prescribes intersection sizes of elements of $\mathcal{R}$. This system contains complete information (restrictions) on cardinality of the universe S, on cardinalities of members of $\mathcal{R}$ and sizes of their all possible intersections.

Some structures with non complete information on $\mathcal{R}$ are well known in the literature. We recall the following notions:

- *Intersection graph* (or *k-graph*) $(V, \mathcal{E})$, where $\mathcal{E}$ is a set of pairs (or exactly k) subsets of V which have nonempty intersection i.e., $E \in \mathcal{E}$ if and only if $q_{\mathcal{R}}(E) \geq 1$. Some authors assume "at least k" instead of "exactly k". We refer to Marczewski [7], Harary [5], Erdös, Goodman and Posa [4] and McKee and McMorris [9] for more details.

  Intersection multigraph $(V, \mathcal{E}, q_{\mathcal{R}})$, where the pair $(V, \mathcal{E})$ is the intersection graph of $\mathcal{R}$ and the multiplicity of edges determines $q_{\mathcal{R}}$ on $\mathcal{E}$. We refer to Bermond and Meyer [2], Marczyk [8] and Prisner [9].
- *Set system with prescribed intersection sizes* $(V, 2^V, q_r)$, where $q_r(E) = q_{\mathcal{R}}(E) \bmod r$ for an positive integer r, Grolmusz [6].
- *Intersection hypergraph* $(V, \mathcal{E})$, where $\mathcal{E} \subset 2^V$ and $q_{\mathcal{R}}(F) \neq 0$ if and only if $F \subset E$ for some $E \in \mathcal{E}$.

  Intersection multi-hypergraph $(V, \mathcal{E}, q_{\mathcal{R}})$, where $(V, \mathcal{E})$ is the intersection hypergraph of $\mathcal{R}$ and the multiplicity of edges determines $q_{\mathcal{R}}$ on $\mathcal{E}$.

Set systems with restricted intersection sizes play an important role in several fields of combinatorics and in computer science. For more information and for detailed literature review, we refer to Grolmusz [6] and MacKee and McMorris [9].

Problem 1. For a weighted set system $H = (V, \mathcal{E}, q)$ one may ask if we can assign subsets of a given set S to vertices of H, i.e., for each $v \in V$ assign a set $R_v \subset S$, so that

$$\Big|\bigcap_{v\in E} R_v\Big| = q(E) \quad \text{for each } E \in \mathcal{E}. \tag{1}$$

If the answer is YES then we say that H is *representable by set intersections* (*r.s.i.* for short) and that the assigned indexed family of sets $\mathcal{R} = (R_v)_{v\in V}$ is its *representation*. We say also that such representation is of the size $|S|$, the cardinality of the universe of $\mathcal{R}$. Additionally, if $\emptyset \notin \mathcal{E}$, one can look for a representation of minimal size.

The Problem 1 for $\mathcal{E} = 2^V$ is exactly the same as Problem 2 in Grolmusz [6]. Ibidem the answer is always yes, if we consider the modular version: $q(E) = q_{\mathcal{R}}(E)$ holds only modulo r for some positive integer r. For non-modular version, Grolmusz's algorithm for constructing a family with prescribed intersection sizes works only in the case described below in Section 3.2. In this paper, we investigate non modular case which seems to be more difficult.

Definition 1.1. A weighted set system $(V, \mathcal{E}, q)$ is an *intersection set system* (*i.s.s.* for short) if there exists a family $\mathcal{R} = (R_v)_{v\in V}$ such that (1) holds and the hypergraph $(V, \tilde{\mathcal{E}})$ is an intersection hypergraph of $\mathcal{R}$, where

$$\tilde{\mathcal{E}} = \{E \in \mathcal{E} : q(E) > 0\}. \tag{2}$$

It is easy to see, that $(V, \mathcal{E}, q)$ is an intersection set system of a family $\mathcal{R}$ if and only if (1) and

$$q_{\mathcal{R}}(E) = 0 \quad \text{for each } E \in (2^V \setminus \tilde{\mathcal{E}}^-) \setminus \mathcal{E}. \tag{3}$$

So we can see in (3) that restrictions applies also to some sets outside if $\mathcal{E}$. For deference between i.s.s. and r.s.i in Problem 1, see also Example 2.2 given below.

Problem 2. Is a weighted set system $H = (V, \mathcal{E}, q)$ an intersection set system? Determine if there exists a representation $\mathcal{R} = (R_v)_{v\in V}$ such that (1) and (3). Additionally, if $\emptyset \notin \mathcal{E}$, one can look for a representation as i.s.s. of minimal size.

The notion of weighted set system is closely connected with the notion of a multi-hypergraph. If there is no confusion each set from $\mathcal{E}$ we call an *edge* even if its size is zero. Several authors (for example Harary [5]) a representation means as a family of distinct sets. We omit this assumption.

2. Weighted Set Systems Representable by Set Intersections

Given a weighted set system $H = (V, \mathcal{E}, q)$, we ask about a representation $\mathcal{R}$ on a universe S such that (1) is fulfilled. If the empty set is an edge of H the size of every representation ought to be $|S| = q(\emptyset)$. Otherwise, one can ask about r.s.i. (or a representation as i.s.s.) of minimal size. The minimum size of a representation is called *weak intersection number* (or *intersection number*) of H and will be denoted by $\omega(H)$ (or by $\bar{\omega}(H)$). Of course, we have

$$\bar{\omega}(H) \geq \omega(H)$$

if an adequate representation exists.

Observation 1. *If a weighted set system $H = (V, \mathcal{E}, q)$ is r.s.i., then q is non-increasing on $\mathcal{E}$ i.e., for every two edges such that $F \subset E$ we have $q(F) \geq q(E)$.*

As the following examples show, some weighted set systems are not representable by set intersection or are not i.s.s., even if multiplicity function is non-increasing.

Example 2.1. Let $H = (V, \mathcal{E}, q)$, where $V = \{a, b, c\}$ and the set of edges $\mathcal{E} = \{\{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ with multiplicity $q(\{a\}) = q(\{a, b\}) = q(\{a, c\}) = 2$ and $q(\{b, c\}) = 1$. If $\mathcal{R} = (R_a, R_b, R_c)$ is a representation, then it satisfies: $R_a \subset R_b$ and $R_a \subset R_c$. So it is not possible to be $|R_b \cap R_c| = 1$, because $|R_a| = 2$. Therefore, H is not r.s.i..

Example 2.2. Let $H = (V, \mathcal{E}, q)$, where $V = \{a, b, c\}$ and the set of edges $\mathcal{E} = \{\{a\}, \{a, b\}, \{a, c\}\}$ with multiplicity $q(\{a\}) = 3$ and $q(\{a, b\}) = q(\{a, c\}) = 2$. The family $(R_a = \{1, 2, 3\}, R_b = \{1, 2\}, R_c = \{2, 3\})$ is a representation of H by set intersection. It is easy to check that H is not i.s.s. because there is no family $\mathcal{R}$ such that H be its intersection set system. Therefore, Problem 1 has a solution but Problem 2 has no solution.

Example 2.3. Let $H = (V, \mathcal{E}, q)$, where $V = \{a, b, c, d\}$ and the set of edges $\mathcal{E} = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\}$ with multiplicity $q(\{a\}) = q(\{c\}) = 3$ and $q(\{a, b\}) = q(\{b, c\}) = q(\{c, d\}) = q(\{a, d\}) = 1$. It is easy to check that H is an i.s.s. The families $\mathcal{R}$ $(R_a = \{1, 2, 3\}, R_b = \{1\}, R_c = \{1, 2, 3\}, R_d = \{1\})$ and $\mathcal{P}$ $(P_a = \{1, 3, 5\}, P_b = \{1, 2\}, P_c = \{2, 4, 6\}, P_d = \{3, 4\})$ are its representations. We have $\omega(H) = 3$ and $\bar{\omega}(H) = 6$.

The next statement shows that Problem 2 as well as problem of characterizing intersection multi-hypergraphs is equivalent to a special case of Problem 1.

Observation 2. *Let* $H = (V, \mathcal{E}, q)$ *be a weighted set system. Let* $\bar{H} = (V, \mathcal{E} \cup (2^V \setminus \tilde{\mathcal{E}}^-), \bar{q})$*, where* $\bar{q}$ *is an extension of* q *such that*

$$\bar{q}(E) = \begin{cases} q(E) & \text{for } E \in \mathcal{E}, \\ 0 & \text{for } E \in (2^V \setminus \tilde{\mathcal{E}}^-) \setminus \mathcal{E}. \end{cases}$$

Then H *is i.s.s. if and only if* $\bar{H}$ *is r.s.i..*

Proof. If a family $\mathcal{R} = (R_v)_{v \in V}$ is a representation of $\bar{H}$ by set intersections then the weighted set system $(V, \tilde{\mathcal{E}})$ is an intersection hypergraph of $\mathcal{R}$ because (1) and (3) are satisfied. ■

The problem of characterizing intersection graphs was first posed by Marczewski [7]. He shows that any graph is an intersection graph. The same statement for multigraphs was given by McKee and McMorris [9]. Let us notice that it is not true that any multigraph with loops is an intersection multi-hypergraph (see Example 1).

3. Cliques and Clique Covers of Edges

Let $H = (V, \mathcal{E}, q)$ be a weighted set system.

A set $K \subset V$ is said to be *clique of* H if for each edge $E \in \mathcal{E}$ and $E \subset K$ we have $q(E) > 0$.

A sequence (indexed family) $\mathcal{K} = (K_1, \ldots, K_n)$ of cliques of H such that each edge E is a subset of at least $q(E)$ cliques is called *clique cover* (*of edges*) *of* H. We say that it is a *clique partition of edges of* H if, additionally,

$$|\{\, i : E \subset K_i \,\}| = q(E) \quad \text{for each } E \in \mathcal{E}.$$

Notice that a cover and a partition of a weighted set system generalize the analogous notions in graphs and hypergraphs (see Berge [1]).

Definition 3.1. By inverse of a family $\mathcal{R}$ of subsets of a set S, denoted by $Inv(\mathcal{R})$, we mean the family $\mathcal{C} = (C_s)_{s \in S}$ of subsets of the set V indexed by S such that

$$C_s = \{\, v \in V : \, s \in R_v \,\}.$$

The notion $Inv(\mathcal{R})$ is best understood as the matrix transposition on the appropriate edges-vertices incidence matrix of $\mathcal{R}$.

Observation 3. *If $\mathcal{R} = (R_v)_{v\in V}$ with the universe S is a representation of a weighted set system H by set intersection then the family $Inv(\mathcal{R})$ is a clique partition of edges of H and $\mathcal{R} = Inv(Inv(\mathcal{R}))$.*

3.1. Clique reduction of weighted set systems

Let K be a clique of $H = (V, \mathcal{E}, q)$. For each $E \in \mathcal{E}$

$$q_K(E) = \begin{cases} q(E) - 1 & \text{for } E \subset K, \\ q(E) & \text{otherwise.} \end{cases} \tag{4}$$

If q_K is nonnegative on $\mathcal{E}$ then a reduction by K is feasible. We say that *K reduces H to the weighted set system $H_K = (V, \mathcal{E}, q_K)$*. Of course, if K_1 reduces H and K_2 reduces H_{K_1}, then K_1 reduces H_{K_2} and $H_{K_1K_2} = H_{K_2K_1}$.

Let us remark that:

Observation 4. *If $\mathcal{R} = (R_v)_{v\in V}$ with the universe S is a representation of H_K, then $\tilde{\mathcal{R}} = (\tilde{R}_v)_{v\in V}$ such that for some $s \notin S$*

$$\tilde{R}_v = \begin{cases} R_v \cup \{s\} & \text{if } v \in K, \\ R_v & \text{otherwise} \end{cases} \tag{5}$$

is a representation of H (with the universe $S \cup \{s\}$).

Observation 5. *If K is a clique of a weighted set system $H = (V, \mathcal{E}, q)$ and H_K is r.s.i., then H is r.s.i.. On the other hand, if H is r.s.i. weighted set system and $q \not\equiv 0$, then there is a clique K such that H_K is r.s.i. and $q_K \not\equiv q$.*

The last implication is true because for a given representation $\mathcal{R} = (R_v)_{v\in V}$ with the universe S, as a possible clique we can take

$$K = \bigcup \Big\{ E \in \mathcal{E} : s \in \bigcap_{v\in E} R_v \Big\} \quad \text{for some } s \in S.$$

Namely, the family $\tilde{\mathcal{R}} = (\tilde{R}_v)_{v\in V}$, where $\tilde{R}_v = R_v \setminus \{s\}$ is such a representation of H_K.

Definition 3.2. A sequence $\mathcal{K} = (K_1, \ldots, K_n)$ of cliques of H forms a *completely reducing sequence of* H if the obtained system $H_{K_1 \ldots K_n}$ has the weight $q_{K_1 \ldots K_n} \equiv 0$.

Example 3.1. Let $H = (V, \mathcal{E}, q)$, where $V = \{a, b, c, d, e, f\}$, $\mathcal{E} = \{\{a, b, c\}, \{a, d, f\}, \{a, e, f\}, \{b, c\}, \{e, f\}, \{c, d\}, \{a, d\}, \{d\}, \{e\}\}$ and the weight of edges q is described in the Table 1.

Each successive reduction by the following cliques: $K_1 = \{a, d, e, f\}$, $K_2 = \{a, d, f\}$, $K_3 = \{b, c\}$ and $K_4 = \{b, c, e, f\}$ is feasible — see Table 1.

Table 1

E	$\{abc\}$	$\{adf\}$	$\{aef\}$	$\{bc\}$	$\{ef\}$	$\{cd\}$	$\{ad\}$	$\{d\}$	$\{e\}$
$q(E)$	0	2	1	2	2	0	2	2	2
$q_{K_1}(E)$	0	1	0	2	1	0	1	1	1
$q_{K_1K_2}(E)$	0	0	0	2	1	0	0	0	1
$q_{K_1 \ldots K_3}(E)$	0	0	0	1	1	0	0	0	1
$q_{K_1 \ldots K_4}(E)$	0	0	0	0	0	0	0	0	0

Therefore, $\mathcal{K} = (K_1, K_2, K_3, K_4)$ is a partition of edges of H that forms a completely reducing sequence. It leads to the representation given in Table 2.

Table 2

V	a	b	c	d	e	f
R_v	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$	$\{1,2\}$	$\{1,4\}$	$\{1,2,4\}$

Theorem 3.2. *A weighted set system H is r.s.i. if and only if there exists a completely reducing sequence of H. If $\mathcal{K} = (K_1, \ldots, K_n)$ completely reduces H, then $Inv(\mathcal{K})$ with the universe $S = \{1, \ldots, n\}$ is a representation of H.*

Proof. The family $\mathcal{R}^0$ with $R_v^0 = \emptyset$ for each $v \in V$ represents the weighted set system $H_{K_1 \ldots K_n}$ by set intersection because this weighted set system has the weight of edges equal to zero. By (4) and Observation 5, $\mathcal{R}^i = (R_v^i)_{v \in V}$ with the universe $S^i = \{n - i + 1, \ldots n\}$ such that

$$R_v^i = \begin{cases} R_v^{i-1} \cup \{n - i + 1\} & \text{if } v \in K_{n-i+1}, \\ R_v^{i-1} & \text{otherwise} \end{cases} \tag{6}$$

represents the weighted set system $H_{K_1 \ldots K_{n-i}}$ by set intersection for $i = 1$ and so on successively for $i = 2, \ldots, n-1$. Additionally, $\mathcal{R}^n$ (equal to $Inv(\mathcal{K})$) with the universe $S^n = \{1, \ldots, n\}$ is a representation of H. ■

3.2. Intersection numbers of weighted set systems

According to Observation 5, we have the following corollary of Theorem 3.2.

Corollary 3.3. *An indexed family of cliques $\mathcal{K}$ is a clique partition of edges of a weighted set system H if and only if H is completely reduced by $\mathcal{K}$. Additionally, $\omega(H) \leq |\mathcal{K}|$.*

Theorem 3.4. *If a weighted set system $H = (V, \mathcal{E}, q)$ is r.s.i., then*

$$\max_{E \in \mathcal{E}} q(E) \;\leq\; \omega(H) \;\leq \sum_{E \in Min\mathcal{E}} q(E) \tag{7}$$

and the bounds are sharp.

Proof. If $\mathcal{K}$ is a complete reduction sequence of H then the sum of weights of all edges from $Min\mathcal{E}$ decreases at least 1 in every step of the reduction. Then

$$\max_{E \in \mathcal{E}} q(E) \;\leq\; |\mathcal{K}| \;\leq \sum_{E \in Min\mathcal{E}} q(E)$$

and (7) follows from Theorem 3.2. For weighted set systems satisfying $|Min\mathcal{E}| = 1$ we have $\max_{E \in \mathcal{E}} q(E) = q(\bar{E})$ for $\bar{E} \in Min\mathcal{E}$. Therefore, the equalities are obtained simultaneously. ■

The lower bound in (7) is obtained for the following case:

Property 3.5. *Let $H = (V, \mathcal{E}, q)$. Assume that for each edge E and every $\mathcal{F} \subset \mathcal{E}$ such that $E \subset \bigcup \mathcal{F}$ we have*

$$q(E) \geq \min_{F \in \mathcal{F}} q(F). \tag{8}$$

Then H is r.s.i. and $\omega(H) = \max_{E \in \mathcal{E}} q(E)$.

Proof. If (8) holds then $K = \bigcup \{E \in \mathcal{E} : q(E) > 0\}$ reduces H and H_K also satisfies (8). So we can construct a sequence of the length $\max_{E \in \mathcal{E}} q(E)$ which completely reduces H. It is enough to use Corollary 3.3. ■

Let us denote

$$\mathcal{E}_E^+ = \{F \in \mathcal{E} \ : \ E \subset F \ \text{ and } F \neq E\}. \tag{9}$$

In the same manner as above, we can show:

Property 3.6. *Let $H = (V, \mathcal{E}, q)$. Assume that for each edge E and the antichain $\mathcal{F} = Min\ \mathcal{E}_E^+$ we have*

$$q(E) \geq \sum_{F \in \mathcal{F}} q(F). \tag{10}$$

Then H is r.s.i. and a representation can be obtained as the inverse family of a set of edges of H.

Proof. Every $K \in Max\{E \in \mathcal{E} : q(E) > 0\}$ reduces H to H_K which also satisfies the assumption of the Property. So we can construct a sequence $\mathcal{K}$ of elements of $\mathcal{E}$ which completely reduces H. It is enough to use Theorem 3.2. ∎

It is easy to check that weighted set systems in Examples 2.3 and 3.1 satisfy the assumption of Property 3.6. The solutions – the representations $\mathcal{P}$ of Example 2.3 and $\mathcal{P}$ of Example 3.1 (presented in Table 4 below) are obtained by the algorithm given in the proofs of Theorem 3.2 and Property 3.6.

Let us remark that the construction of representation by set intersections in Grolmusz [6] works for non-modular case only under the assumption $q(E) \geq \sum_{F \in \mathcal{E}_E^+} q(F)$ for each $E \subset V$ — stronger than the above assumption in Property 3.6.

We say that the set of edges $\mathcal{E}$ is *primitively covered* if the following implication holds: If $\mathcal{F} \subset \mathcal{E}$ and $E \subset \bigcup \mathcal{F}$ then $E \subset F$ for some $F \in \mathcal{F}$.

Property 3.7. *The set of edges $\mathcal{E} \subset 2^V$ is primitively covered if and only if for every non-increasing weight function q on $\mathcal{E}$ the weighted set system $H = (V, \mathcal{E}, q)$ is r.s.i.. Additionally, $\omega(H) = \max\{q(E) \mid E \in \mathcal{E}\}$.*

Proof. The implication ($\Rightarrow$) follows from Property 3.5.

To the proof the other one, suppose that for every non-increasing weight function q on $\mathcal{E}$ the weighted set system $H = (V, \mathcal{E}, q)$ is r.s.i. and $\mathcal{E}$ be not primitive covered. There exist $F_1, \ldots, F_n \in \mathcal{E}$ and $F \in \mathcal{E}$ such that

$$F \subset \bigcup_{i=1}^{n} F_i \quad \text{and} \quad F \setminus F_i \neq \emptyset \quad \text{for each } i = 1, \ldots, n.$$

Define q on $\mathcal{E}$ as

$$q(E) = \begin{cases} 1 & \text{if there exists } i \le n \text{ such that } \ E \subset F_i, \\ 0 & \text{otherwise.} \end{cases}$$

There is no sequence of cliques which completely reduces $(V, \mathcal{E}, q)$. By Theorem 3.2, this weighted set system is not r.s.i., a contradiction. ■

4. Set Systems with Tightly Non-increasing Weight

In opposite to $\mathcal{E}_E^+$ given in (9), let us denote

$$\mathcal{E}_E^- = \{F \in \mathcal{E} : F \subset E \ \text{ and } F \neq E\}. \tag{11}$$

For a given $H = (V, \mathcal{E}, q)$ we define inductively the function q^* on $\mathcal{E}$, called *sub-weight in* H in the following way:

$$q^*(E) = 0 \quad \text{for every} \quad E \in Max\mathcal{E}$$

and if q^* has been defined on edges in $\mathcal{E}_E^+$ then

$$q^*(E) = \sum_{F \in \mathcal{E}_E^+} [q(F) - q^*(F)] \quad \text{for} \quad E \in \mathcal{E} \setminus Max\mathcal{E}. \tag{12}$$

Definition 4.1. We call a weight function q *tightly non-increasing* if

$$q(E) \ge q^*(E) \quad \text{for every} \quad E \in \mathcal{E}. \tag{13}$$

It is easy to check, that if q is tightly non-increasing then it is non-increasing. For every two edges $E \subset F$ we have

$$q(E) \ge q^*(E) \ge [q(F) - q^*(F)] + q^*(F) = q(F).$$

Theorem 4.1. *Let $H = (V, \mathcal{E}, q)$. If q is tightly non-increasing, then H is r.s.i. weighted set system. Further, a representation can be obtained as the inverse family of a subset of $\mathcal{E}$.*

Proof. Suppose that $K \in \mathcal{E}$ satisfies

$$q(K) > 0 \quad \text{and} \quad q(E) = 0 \quad \text{for each} \quad E \in \mathcal{E}_K^+. \tag{14}$$

The function q_K, given by (4), is nonnegative. So a reduction by K is feasible and K reduces H to $H_K = (V, \mathcal{E}, q_K)$ with the sub-weight of edges q_K^*.

Point 1. The function q_K^* satisfies:

$$q_K^*(E) = \begin{cases} q^*(E) - 1 & \text{for} \quad E \in \mathcal{E}_K^-, \\ q^*(E) & \text{otherwise.} \end{cases} \tag{15}$$

For $E \in Max\mathcal{E}$ we have $q_K^*(E) = 0 = q^*(E)$ and (15) holds. Suppose that $E \in \mathcal{E}$ and for each $F \in \mathcal{E}_E^+$ (15) holds. If $E \in \mathcal{E}_K^-$ then

$$\begin{aligned} q^*(E) &= \sum_{F \in \mathcal{E}_E^+ \cap \mathcal{E}_K^-} [q_K(F) - q_K^*(F)] + [q_K(K) - q_K^*(K)] \\ &+ \sum_{F \in \mathcal{E}_E^+ \setminus (\mathcal{E}_K^- \cup \{K\})} [q_K(F) - q_K^*(F)] \\ &= \sum_{F \in \mathcal{E}_E^+ \cap \mathcal{E}_K^-} [q(F) - 1 - q^*(F) + 1] + [q(K) - 1 - q^*(K)] \\ &+ \sum_{F \in \mathcal{E}_E^+ \setminus (\mathcal{E}_K^- \cup \{K\})} [q(F) - q^*(F)] = q^*(E) - 1. \end{aligned}$$

If $E \notin \mathcal{E}_K^-$ then $\mathcal{E}_E^+ \cap (\mathcal{E}_K^- \cup \{K\}) = \emptyset$ and $q_K^*(E) = q^*(E)$. Therefore E satisfies (15).

Point 2. The function q_K^* is tightly non-increasing. For $E \in \mathcal{E}_K^-$, by (15) we have

$$q_K(E) = q(E) - 1 \geq q^*(E) - 1 = q_K^*(E).$$

The same holds for $E \in \mathcal{E}_K^+$. For $E = K$ we have

$$q_K(K) = q(K) - 1 \geq 0 = q_K^*(K)$$

because of the assumption on K.

In order to prove the Theorem we construct step by step cliques for successive reduction. We take $K_1 \in \mathcal{E}$ which satisfies (14) for q. So on, for $i = 1, \ldots, n-1$ if only $q_{K_1 \ldots K_i} \not\equiv 0$, we take $K_{i+1} \in \mathcal{E}$ which satisfies (14) for $q_{K_1 \ldots K_i}$. There exists n such that $q_{K_1 \ldots K_n} \equiv 0$. All functions $q_{K_1 \ldots K_i}$ are nonnegative and, by Point 2, tightly non-increasing.

The constructed sequence $\mathcal{K} = (K_1, \ldots, K_n)$ of cliques of H completely reduces H. From Theorem 3.2 follows that $Inv(\mathcal{K})$ with the universe $S = \{1, \ldots, n\}$ is a desired representation of H. ∎

It is easy to check that weighted set systems in Examples 2.3 and 3.1 have tightly non-increasing weight functions. The weighted set system in Example 2.2 does not satisfy this property.

As a corollary we have a generalization of the Marczewski existence theorem.

Corollary 4.2. *If* $(V, \mathcal{E})$ *is a hypergraph such that* $\mathcal{E}$ *is an antichain (in particular a uniform hypergraph), then for every weight* q *the weighted set system* $H = (V, \mathcal{E}, q)$ *is r.s.i..*

Example 4.3. Consider weighted set system given in Example 3.1, i.e., $H = (V, \mathcal{E}, q)$, where $V = \{a, b, c, d, e, f\}$, $\mathcal{E}$ and the weight of edges q are given in the Table 1. We have (see (2)):

$$\tilde{\mathcal{E}} = \{\{a, d, f\}, \{a, e, f\}, \{b, c\}, \{e, f\}, \{a, d\}, \{d\}, \{e\}\}$$

and

$$\tilde{\mathcal{E}}^- = \tilde{\mathcal{E}} \cup \{\{a, f\}, \{d, f\}, \{a, e\}, \{a\}, \{b\}, \{c\}, \{f\}, \emptyset\}.$$

Define $\bar{H} = (V, \mathcal{E} \cup (2^V \setminus \tilde{\mathcal{E}}^-), \bar{q})$, where

$$\bar{q}(E) = \begin{cases} q(E) & \text{for } E \in \mathcal{E}, \\ 0 & \text{for } E \in (2^V \setminus \tilde{\mathcal{E}}^-) \setminus \mathcal{E}. \end{cases}$$

The sub-weight q^* is given in Table 3:

Table 3

E	$\{abc\}$	$\{adf\}$	$\{aef\}$	$\{bc\}$	$\{ef\}$	$\{cd\}$	$\{ad\}$	$\{d\}$	$\{e\}$
$q(E)$	0	2	1	2	2	0	2	2	2
$q^*(E)$	0	0	0	0	1	0	2	2	2

It is easy to see that $\bar{q}^*(E) = q^*(E)$ for each $E \in \mathcal{E}$ and $\bar{q}^*(E) = 0$ for each other edge of $\bar{H}$. Therefore, $\bar{q}$ is tightly non-increasing weight function of $\bar{H}$. The sequence $\mathcal{K} = (\{adf\}, \{adf\}, \{aef\}, \{bc\}, \{bc\}, \{ef\})$ is a partition of edges of $\bar{H}$ which forms a completely reducing sequence. It leads to the representation $\mathcal{P} = (P_v)_{v\in V}$ with the universe $S = \{1, \ldots, 6\}$ of H, given in Table 4.

Table 4

V	a	b	c	d	e	f
P_v	$\{1,2,3\}$	$\{4,5\}$	$\{4,5\}$	$\{1,2\}$	$\{3,6\}$	$\{1,2,3,6\}$

It is worth to remark that both families of sets $\mathcal{R}$ (given in Table 2) and $\mathcal{P}$ (in Table 4) are representations of H by set intersection. Namely, H is an intersection set system only of the family $\mathcal{P}$ because $(V, \tilde{\mathcal{E}})$ is not intersection hypergraph of $\mathcal{R}$. Therefore, see Theorem 2.4, H is i.s.s..

Theorem 4.4. *If the weight function of a weighted set system $H = (V, \mathcal{E}, q)$ is tightly non-increasing, then it is an intersection set system. Further,*

$$\bar{\omega}(H) = \sum_{E\in\mathcal{E}} [q(E) - q^*(E)].$$

Proof. According to Theorem 2.4, we construct the weighted set system $\bar{H} = (V, \mathcal{E} \cup (2^V \setminus \tilde{\mathcal{E}}^-), \bar{q})$, where $\bar{q}$ is an extension of q such that $\bar{q}(E) = 0$ for each $E \in (2^V \setminus \tilde{\mathcal{E}}^-) \setminus \mathcal{E}$. It is easy to check that $\bar{q}$ is tightly non-increasing in $\bar{H}$ and $\bar{q}^*(E) = q^*(E)$ for each $E \in \mathcal{E}$. From Theorems 2.4 and 4.1 follows that H is i.s.s..

Let $\mathcal{P}$ with the universe S be a representation of $\bar{H}$. Its weight function $q_{\mathcal{P}}$ satisfies:

$$q_{\mathcal{P}}(E) = \bar{q}(E) \quad \text{for each} \quad E \in \mathcal{E} \cup (2^V \setminus \tilde{\mathcal{E}}^-)$$

and

$$q_{\mathcal{P}}(E) \neq 0 \quad \Leftrightarrow \quad E \in \tilde{\mathcal{E}}^-.$$

Therefore, $q^*_{\mathcal{P}}(E) = \bar{q}^*(E)$ for each $E \in \mathcal{E} \cup (2^V \setminus \tilde{\mathcal{E}}^-)$ and $|S| = q_{\mathcal{P}}(\emptyset)$ with

$$(16) \qquad q_{\mathcal{P}}(\emptyset) \geq q^*_{\mathcal{P}}(\emptyset) = \sum_{E\in\mathcal{E}} [q(E) - q^*(E)] + \sum_{E\in(\tilde{\mathcal{E}}\setminus\mathcal{E})} [q_{\mathcal{P}}(E) - q^*_{\mathcal{P}}(E)].$$

Let $\tilde{q}$ be the unique extension of $\bar{q}$ on 2^V such that $\tilde{q} = \tilde{q}^*$. The weight function $\tilde{q}$ is tightly non-increasing. If $\mathcal{R}$ is a representation with the universe W of $(V, 2^V, \tilde{q})$ then it is also a representation of $\bar{H}$. Additionally, H is an intersection set system of $\mathcal{R}$. We have

$$|W| = q_{\mathcal{R}}(\emptyset) = \sum_{E \in 2^V} [\tilde{q}(E) - \tilde{q}^*(E)] = \sum_{E \in \mathcal{E}} [q(E) - q^*(E)] = \bar{\omega}(H)$$

because the construction and (16). ■

For the case considered in Example 4.1, the unique extension of $\bar{q}$ on 2^V such that $\tilde{q} = \tilde{q}^*$ is presented in Table 5.

Table 5

E	abc	adf	aef	bc	ef	cd	ad	d	e	af	df	ae	f	a	b	c	$\emptyset$
$\tilde{q}(E)$	0	2	1	2	2	0	2	2	2	3	2	1	4	3	2	2	6
$\tilde{q}^*(E)$	0	0	0	0	1	0	2	2	2	3	2	1	4	3	2	2	6

4.1. Representations of complete weighted set systems

In the remainder of this section we assume $H = (V, 2^V, q)$. We ask for which q the weighted set system H is r.s.i.. We begin with some observations.

If $\mathcal{R} = (R_v)_{v \in V}$ is an indexed family of subsets of V and $E \subset V$, then the set

$$\Lambda_{\mathcal{R}}(E) = \Big(\bigcap_{v \in E} R_v\Big) \setminus \Big(\bigcup_{v \in V \setminus E} R_v\Big)$$

is called an *atom* of $\mathcal{R}$ generated by E.

Lemma 1. *Let $\mathcal{R} = (R_v)_{v \in V}$ be an indexed family. We have:*

(i) *If $E, F \in 2^V$ and $E \neq F$, then $\Lambda_{\mathcal{R}}(E) \cap \Lambda_{\mathcal{R}}(F) = \emptyset$,*

(ii) *for each $v \in V$*

$$R_v = \bigcup \{\Lambda_{\mathcal{R}}(E) : E \subset V \quad \textit{and} \quad v \in E\}.$$

Proof. Suppose (i) of the lemma is false and $E \setminus F \neq \emptyset$. Then we could find $v \in E \setminus F$ and $s \in \Lambda_{\mathcal{R}}(E) \cap \Lambda_{\mathcal{R}}(E)$. It follows that $s \in R_v$ because $v \in E$ and $s \in \Lambda_{\mathcal{R}}(E)$. On the contrary, $s \notin R_v$ because $v \notin F$ and $s \in \Lambda_{\mathcal{R}}(F)$.

Let $v \in V$ and denote $A_v = \bigcup\{\Lambda_{\mathcal{R}}(E) : E \subset V \text{ and } v \in E\}$. If $s \in R_v$ then $s \in \Lambda_{\mathcal{R}}(\{u \in V : s \in R_u\}) \subset A_v$. If $s \in A_v$ then there exists $F \subset V$ such that $s \in \Lambda_{\mathcal{R}}(F)$ and $v \in F$. It implies $s \in R_v$. Then we have (ii). ■

Lemma 2. *If $\mathcal{R}$ is a representation of $H = (V, 2^V, q)$ then for every edge $E \subset V$ we have*

$$|\Lambda_{\mathcal{R}}(E)| = q(E) - q^*(E). \tag{17}$$

Proof. We prove it step by step starting from $E = V$.

$$|\Lambda_{\mathcal{R}}(V)| = \Big|\bigcap_{v \in V} R_v\Big| = q(V) = q(V) - q^*(V).$$

Suppose (17) is true for each F such that $E \subset F$ and $F \neq E$. From Lemma 1 we have

$$\begin{aligned} q(E) &= \Big|\bigcap_{v\in E} R_v\Big| = \Big|\bigcap_{v\in E}\bigcup\{\Lambda_{\mathcal{R}}(F) : F \subset E \text{ and } v \in F\}\Big| \\ &= \Big|\bigcup\{\Lambda_{\mathcal{R}}(F) : F \subset V \text{ and } E \subset F\}\Big| = |\Lambda_{\mathcal{R}}(E)| + \sum_{F \in (2^V)_E^+} |\Lambda_{\mathcal{R}}(F)| \\ &= |\Lambda_{\mathcal{R}}(E)| + \sum_{F \in (2^V)_E^+} [q(F) - q^*(F)] = |\Lambda_{\mathcal{R}}(E)| + q^*(E). \end{aligned}$$

■

Theorem 4.5. *A weighted set system $H = (V, 2^V, q)$ is r.s.i. if and only if its weight function is tightly non-increasing. Further, there exists at most one representation.*

Proof. The first part of the theorem follows from Theorem 4.1 and Lemma 2. Let $\mathcal{R} = (R_v)_{v\in V}$ and $\mathcal{P} = (P_v)_{v\in V}$ be representations of H by set intersections. By Lemma 1(i), each of $S = \bigcup R_v$ and $W = \bigcup P_v$ can be partitioned into disjoint subsets;

$$S = \bigcup_{E\subset V} \Lambda_{\mathcal{R}}(E) \quad \text{and} \quad W = \bigcup_{E \subset V} \Lambda_{\mathcal{P}}(E).$$

For every $E \subset V$ we have $|\Lambda_{\mathcal{R}}(E)| = |\Lambda_{\mathcal{P}}(E)|$, because Lemma 1(i). Let φ_E be a bijection from $\Lambda_{\mathcal{R}}(E)$ on $\Lambda_{\mathcal{P}}(E)$ and $\varphi : S \to W$ such that

$$\varphi(s) = \varphi_E(s) \quad \text{if and only if} \quad s \in \Lambda_{\mathcal{R}}(E).$$

Of course, φ is a bijection and for every $v \in V$ we have $P_v = \varphi(R_v)$. ■

4.2. Intersection hypergraphs

For a given hypergraph $(V, \mathcal{E})$ we define a class $Q_{\mathcal{E}}$ of adequate weight functions on 2^V such that $q \in Q_{\mathcal{E}}$ if and only if $\{E \subset V : q(E) > 0 = \mathcal{E}^-\}$.

The problem if a hypergraph $(V, \mathcal{E})$ is an intersection hypergraph is equivalent to the following problem:

For which hypergraphs $(V, \mathcal{E})$ there exists a weight $q \in Q_{\mathcal{E}}$ such that the weighted set system $(V, 2^V, q)$ is r.s.i.. We define the intersection number of $(V, \mathcal{E})$ as

$$w((V, \mathcal{E})) = \min\{\omega((V, 2^V, q)) : q \in Q_{\mathcal{E}}\}.$$

In this context, there is a natural question about uniquely intersectable hypergraphs (see Bylka and Komar [3] for intersection graphs).

Theorem 4.6. *Every hypergraph $(V, \mathcal{E})$ is an intersection hypergraph. Further $w((V, \mathcal{E})) = |Max\mathcal{E}|$ with unique representation by set intersections.*

Proof. There exists exactly one function $q \in Q_{\mathcal{E}}$ such that

$$q_{\mathcal{E}}(E) = \begin{cases} 0 & \text{for } \ E \in 2^V \setminus \mathcal{E}^-, \\ 1 & \text{for } \ E \in Max\mathcal{E}, \\ q^*(E) & \text{for } \ E \in \mathcal{E}^- \setminus Max\mathcal{E}. \end{cases}$$

Of course, to know q on $Max\mathcal{E} \cup (2^V \setminus \mathcal{E}^-)$ we can construct q^* on 2^V. The weight q is tightly non-increasing and every complete reduction sequence has $|Max\mathcal{E}|$ elements. The theorem follows from Theorem 4.5 and the construction. ■

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Received 13 February 2006
Revised 24 October 2007
Accepted 24 October 2007