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On the existence of invariant measures for piecewise convex transformations

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Abstract. A class of transformations of the interval $[0, 1]$ which do not satisfy Ulam's condition is shown to have an absolutely continuous invariant measure.

1. In 1960 S. M. Ulam [9] posed the problem of the existence of an absolutely continuous invariant measure for a transformation $\tau\colon [0, 1] \to [0, 1]$ satisfying the condition:

$$|\tau'(x)| > 1 \quad \text{if } \tau(x) = x. \tag{1}$$

In general, the answer to this problem is negative. In fact, the transformation

$$\tau(x) = \begin{cases} 1-2x, & 0 \leqslant x \leqslant \frac{5}{12}, \\ \frac{2}{7}(1-x), & \frac{5}{12} \leqslant x \leqslant 1, \end{cases} \tag{2}$$

satisfies (1) and it can be shown that τ does not have any finite absolutely continuous invariant measure (see [6]).

However, if we replace condition (1) by a more restrictive one:

$$\inf |\tau'(x)| > 1, \tag{3}$$

then there exists an absolutely continuous invariant measure for piecewise C^2-transformations satisfying condition (3), [2], [7].

In this paper we shall show that there exists an absolutely continuous invariant measure for a certain class of transformations of the closed unit interval $[0, 1]$ into itself which do not satisfy Ulam's condition (1). Our Theorem is a generalization of some results due to A. Lasota [5].

2. Let $\{[a_k, b_k]\}_{k=1}^{n;\infty}$ be an at most countable sequence of closed intervals such that

$$a_1 = 0, \quad 0 \leqslant a_k < b_k \leqslant 1, \quad \sum_k (b_k - a_k) = 1,$$
$$\text{and} \quad (a_i, b_i) \cap (a_k, b_k) = \emptyset \quad \text{for } k \neq i.$$

Let $\varphi_k\colon [a_k, b_k] \to [0, 1]$ be a sequence of convex functions, i.e. let

$$\varphi_k(ax+(1-a)y) \leqslant a\varphi_k(x)+(1-a)\varphi_k(y)$$

for $x, y \in [a_k, b_k]$ and $0 \leqslant a \leqslant 1$.

We define a transformation τ by

$$\tau(x) = \varphi_k(x) \quad \text{for } a_k < x < b_k.$$

The transformation τ is defined almost everywhere on $[0, 1]$. We have the following

THEOREM. *If the transformation τ satisfies the additional conditions:*

(a) $\varphi_k(a_k) = 0$,

(b) $\varphi_1'(0) = 1$,

(c) φ_1' *is concave,*

(d) $\dfrac{1}{\varphi_1'-1}$ *is integrable,*

(e) $\sum\limits_k \dfrac{1}{\varphi_k'(a_k)} < \infty$,

then there exists an absolutely continuous probability measure invariant under τ.

3. Let

$$b = \varphi_1(b_1), \quad a_0 = \sum_{k=2}^{n,\infty} \frac{1}{\varphi_k'(a_k)},$$

$$\psi_k(x) = \begin{cases} \varphi_k^{-1}(x) & \text{for } 0 \leqslant x \leqslant \varphi_k(b_k-0), \\ b_k & \text{for } \varphi_k(b_k-0) < x \leqslant 1. \end{cases}$$

Under the assumptions of our Theorem we have the following lemmas:

LEMMA 1. *The function*

$$\frac{1}{1-\psi_1'}$$

is integrable on $[0, 1]$.

Proof. Let $0 < d < b_1$; then

$$\infty > \big(\varphi_1'(\varphi(d))\big)^2 \cdot \int_0^{\varphi_1^{-1}(d)} \frac{1}{\varphi_1'(x)-1}\,dx \geqslant \int_0^{\varphi_1^{-1}(d)} \frac{(\varphi_1'(x))^2}{\varphi_1'(x)-1}\,dx$$

$$= \int_0^d \frac{\varphi_1'(\varphi_1^{-1}(x))}{\varphi_1'(\varphi_1^{-1}(x))-1}\,dx = \int_0^d \frac{1}{1-\psi_1'(x)}\,dx,$$

and

$$\frac{1}{1-\psi_1'(x)} \leqslant \frac{1}{1-\psi_1'(d)} < \infty \quad \text{for } d \leqslant x \leqslant 1.$$

This completes the proof of Lemma 1.

LEMMA 2. *The function ψ_1' is convex in $[0, b]$ and satisfies the following inequalities*:

(i) $$\psi_1'(\psi_1(x)) \leqslant \frac{1+(\psi_1')^2(x)}{2} \quad \textit{for } x \in [0, b],$$

(ii) $$\frac{-\frac{1}{2}(\psi_1')^2+2\psi_1'+\frac{1}{2}}{1+\psi_1'} \leqslant 1.$$

Proof. Let $x \in [0, b]$; then

(4) $$\psi_1'(x) = \frac{1}{\varphi_1'(\varphi_1^{-1}(x))}, \quad 0 \leqslant \psi_1' \leqslant 1.$$

Since the functions φ_1' and φ_1^{-1} are concave, we have

(5) $$\psi_1'\left(\frac{x+y}{2}\right) = \frac{1}{\varphi_1'\left(\varphi_1^{-1}\left(\frac{x+y}{2}\right)\right)} \leqslant \frac{1}{\varphi_1'\left(\frac{\varphi_1^{-1}(x)}{2}+\frac{\varphi_1^{-1}(y)}{2}\right)}$$
$$\leqslant \frac{2}{\varphi_1'(\varphi_1^{-1}(x))+\varphi_1'(\varphi_1^{-1}(y))}$$
$$\leqslant \frac{1}{2\varphi_1'(\varphi_1^{-1}(x))}+\frac{1}{2\varphi_1'(\varphi_1^{-1}(y))} = \tfrac{1}{2}\psi_1'(x)+\tfrac{1}{2}\psi_1'(y)$$

because for each $x, y > 0$
$$\frac{2}{x+y} \leqslant \frac{1}{2x}+\frac{1}{2y}.$$

From (4), (5) and the Bernstein–Doetsch Theorem it follows that ψ_1' is convex.

Proof of inequality (i). Since ψ_1'' is increasing in $[0, b]$ and $\psi_1(x) \leqslant x$, we have
$$(\psi_1'(\psi_1(x)))' = \psi_1''(\psi_1(x))\psi_1'(x) \leqslant \psi_1''(x)\psi_1'(x),$$
$$\psi_1'(\psi_1(x)) = \int_0^x \psi_1''(\psi_1(s))\psi_1'(s)ds+1 \leqslant \int_0^x \psi_1''(s)\psi_1'(s)ds+1 = \tfrac{1}{2}((\psi_1')^2x-1)+1.$$

Proof of inequality (ii). This is a consequence of the following inequality:
$$\frac{-\frac{1}{2}x^2+2x+\frac{1}{2}}{1+x} \leqslant 1 \quad \text{for } x \in [0, 1].$$

This completes the proof of Lemma 2.

Now denote by P_τ the Frobenius–Perron operator corresponding to τ; i.e., write

$$P_\tau f(x) = \sum_k f(\psi_k(x))\psi_k'(x). \tag{6}$$

It is well known that any fixed point of the operator P_τ represents the density of an invariant measure under τ.

LEMMA 3. *For any decreasing non-negative f the function $P_\tau f$ is decreasing; furthermore, the operator P_τ satisfies the following inequalities:*

(i) $$(P_\tau^m 1)(x) \leqslant 1/x, \quad m = 1, 2, \ldots,$$

(ii) $$P_\tau\big((P_\tau^m 1) 1_{[b_1,1]}\big) \leqslant a_0/b_1.$$

Proof. The first statement follows trivially from equation (6). A proof of inequality (i) is given in [1] and [5].

Proof of inequality (ii):

$$P_\tau\big((P_\tau^m 1)\cdot 1_{[b_1,1]}\big)(x) \leqslant P_\tau\Big(\frac{1}{x}\, 1_{[b_1,1]}\Big)(x) \leqslant P_\tau\Big(\frac{1}{b_1}\cdot 1_{[b_1,1]}\Big)(x)$$
$$\leqslant \frac{1}{b_1} P_\tau 1_{[b_1,1]}(x) = \frac{1}{b_1}\sum_{k=2}^{n,\infty} \psi_k'(x) \leqslant \frac{1}{b_1}\sum_{k=2}^{n,\infty} \psi_k'(0) = \frac{a_0}{b_1}.$$

This completes the proof of Lemma 3.

Proof of the Theorem. Let $P_\tau^n 1 = f_n$, $a = \max(a_0, 1)$. We shall prove by induction that for every $n \in N \cup \{0\}$

$$f_n \leqslant \frac{2a}{b_1(1-\psi_1')}. \tag{7}$$

For $n = 0$ inequality (7) holds since

$$\frac{2a}{b_1(1-\psi_1')} \geqslant \frac{2a}{b_1} \geqslant 1.$$

Now, for n such that

$$f_n \leqslant \frac{2a}{b_1(1-\psi_1')}$$

we have from Lemmas 2, 3

$$f_{n+1} = P_\tau f_n = P_\tau f_n 1_{[0,b_1]} + P_\tau f_n 1_{[b_1,1]} \leqslant P_\tau f_n 1_{[0,b_1]} + \frac{a_0}{b_1}$$
$$\leqslant f_n(\psi_1)\psi_1' 1_{[0,b]} + f_n(\psi_1)\psi_1' 1_{[b,1]} + \frac{a}{b_1} = f_n(\psi_1)\psi_1' 1_{[0,b]} + \frac{a}{b_1}$$

$$\leqslant \frac{2a\psi_1'}{b_1(1-\psi_1'(\psi_1))} 1_{[0,b]} + \frac{a}{b_1} \leqslant \frac{a}{b_1}\left(\frac{2\psi_1'}{1-\frac{1+(\psi_1')^2}{2}}+1\right)$$

$$= \frac{a}{b_1}\left(\frac{4\psi_1'}{1-(\psi_1')^2}+1\right) = \frac{a}{b_1}\left(\frac{4\psi_1'+1-\psi_1'^2}{1-(\psi_1')^2}\right)$$

$$= \frac{2a}{b_1(1-\psi_1')}\cdot\frac{-\frac{1}{2}(\psi_1')^2+2\psi_1'+\frac{1}{2}}{1+\psi_1'} \leqslant \frac{2a}{b_1(1-\psi_1')}.$$

From (7) it follows that the sequence

$$\left\{\frac{1}{n}\sum_{k=1}^{n-1} P_\tau^k f\right\}_{n=1}^{\infty}$$

is weakly relatively compact in L^1 for $0 \leqslant f \leqslant 1$. Thus, by the Kakutani–Yosida ergodic theorem there exists the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} P_\tau^k f = f_0$$

and $P_\tau f_0 = f_0$. Consequently, by the definition of P_τ, $d\mu = f_0\,dx$ is an absolutely continuous invariant measure, and this completes the proof.

4. From the proof of our Theorem it follows easily that the density f_0 of the invariant measure is dominated by the integrable function $2a/b_1(1-\psi_1')$. In general, f_0 is not bounded by a constant (cf. [5], [8]).

EXAMPLE. Consider the transformation

$$\tau(x) = \begin{cases} x+\frac{2}{3}x^{3/2} & \text{for } 0 \leqslant x \leqslant \frac{1}{3}, \\ \frac{3}{2}x-\frac{1}{2} & \text{for } \frac{1}{3} < x \leqslant 1. \end{cases}$$

It is easy to see that τ satisfies the conditions of our Theorem. In this case we have

$$\frac{1}{\varphi_1'-1} = \frac{1}{\sqrt{x}}.$$

Therefore, from our results it follows that τ admits an invariant probability measure μ such that

$$0 \leqslant \frac{d\mu}{dx} \leqslant 6\left(\sqrt{1+\sqrt{\tau^{-1}(x)}}+1\right) \quad \text{(from (7))},$$

where τ^{-1} denotes the inverse function to τ restricted to the interval $(0, \frac{1}{3})$.

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