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On the tangency of sets in generalized metric spaces

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Abstract. The present paper gives a certain definition of tangency of order k of arbitrary sets lying in a space having a weaker structure than the metric one. Some conditions are given for the symmetry and transitivity of tangency relation introduced. The connection between the generalization under consideration and the previously known concept of tangency in metric spaces is illustrated.

In the investigation of non-linear sets lying in linear spaces we use the method of local aproximation of those sets by some linear sets. And so, the consideration of the interval of a stright line instead of the small arc of a curve or else of a piece of a hyperplane instead of a piece of a hypersurface is a frequently used method of simplification of more complicated question. Evidently, an aproximating linear set and an aproximated set do not play the same part. More generally, one often investigates the sets of points lying in some space by way of aproximation by more regular sets. Such a manner of study seems natural. There is a separate question: what can we treat as regular sets, i. e. sets that we may use for describing some properties of other sets? It is evident that we can speak about linearization only in this case where the linear structure of the space is given. S. Gołąb and Z. Moszner in paper [2] found the definition of the tangency relation of arcs in general metric spaces. They say that an arc A is tangent to an arc B at the point p iff p is a common beginning of A and B, and the ratio of the distance from any point x of the set A to the set B and of the distance from x to p tends to zero if x tends to p. The authors of the above mentioned paper proved a very interesting theorem concerning of the tangency relation: if we assume that an arc A is Archimedean at the point p and tangent to B at that point, then the set B is tangent to A at p. The nature of the definition given by S. Gołąb and Z. Moszner has suggested a way of generalizing the concept of tangency by considering arbitrary sets instead of simple arcs. Such a generalization is due to Soós [5], and we also find it in an earlier note [6] of the present author. In that note we assume that the point p is a cluster point of the sets A and B instead of assuming that p is a comon beginning of the arcs A and B. In the note [6] a condition is formulated which is a surrogate of the Archimedean condition and next, in [7] we give some generalization of that condition. A detailed examination of the relationship between the above mentioned conditions is due to A. Chądzyńska in [1]. In the present paper we shall give a certain generalization of the previous concepts of the tangency of sets. We shall consider as a space an arbitrary set furnished with a structure more general than a metric and we shall investigate the concept of the tangency of sets which in some special cases coincides with the concept of tangency of sets in the metric space, dealt with in [2], [5] and [7].

1. Preliminaries. A system (E, l), where E is any set and l is a real non-negative function defined on the Cartesian product $E_0 \times E_0$, E_0 being the set of all non-empty subsets of E, will be called a *space*. Such a space may be treated as a certain generalization of the metric space, because the function l allows the introduction of the function l_0 by the formula

(1) $$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for } x, y \in E.$$

Making the appropriate supposition concerning the function l, we obtain the metrics l_0 on the set E.

For an arbitrary space (E, l) we may define, in the same way as in a metric space, the notion of a ball and the notion of a sphere. By the ball (sphere) of redius r about a point x we shall mean the set $B_l(x, r)$ $(S_l(x, r))$ of all points y of E such that the number defined by formula (1) is smaller than (equal to) r. Often we shall omit the subscript l and write $B(x, r)$ and $S(x, r)$ instead of $B_l(x, r)$ and $S_l(x, r)$, respectively. As for a metric space, we may introduce the notion of distance of non-empty subsets of E, setting for any A and B belonging to E_0

(2) $$l(A; B) = \inf\{l_0(x, y);\ x \in A \text{ and } y \in B\}.$$

We notice that the distance $l(A; B)$ is, in general, different from the number $l(A, B)$. So, having a space (E, l) we may define on the set E a new space, associating with any pair (A, B) of non-empty subsets of E the number $l(A; B)$. The number $l(\{p\}; A)$ will also be denoted by $l(p, A)$.

The space (E, l) induces on the set E the topology $\mathcal{T}_l$ defined as follows. We consider $A \subset E$ as an open set in the topology $\mathcal{T}_l$ iff for any point p of A there exists a number $r > 0$ such that the ball $B(p, r)$ is contained in A. By $\bar{A}$ we shall denote the closure of the set A in the topology $\mathcal{T}_l$. We prove that

1.1. *For any point $p \in E$ and for every set $A \subset E$ if $l(p, A) = 0$, then $p \in \bar{A}$.*

The following conditions are equivalent:

(1.1.1) *For any point $p \in E$ and for every set $A \subset E$ if $p \in \bar{A}$, then $l(p, A) = 0$;*

(1.1.2) *For any point $p \in E$ and for any number $r > 0$ there exists a neighbourhood $U \subset B(p, r)$ in the topology $\mathcal{T}_l$;*

Proof. First, we assume that $l(p, A) = 0$. Let U be an arbitrary neighbourhood of the point p in the topology $\mathcal{T}_l$. Then there exists an $r > 0$ such that $B(p, r)$ is contained in U. From the definition of the number $l(p, A)$ according to (2) as the infimum of all numbers $l_0(p, x)$, where $x \in A$, it follows that there exists an $a \in A$ such that $l_0(p, a) < r$. Then $a \in B(p, r)$. Thus the set $A \cap U$ is non-empty and $p \in \bar{A}$.

Now we assume (1.1.1) and suppose that there exist a point p_0 of E and a number $r_0 > 0$ such that for any neighbourhood U of point p_0 the set $U - B(p_0, r_0)$ is non-empty. Setting $A = E - B(p_0, r_0)$ we obtain $U \cap A \neq \emptyset$. Thus $p_0 \in \bar{A}$. Hence it follows that there exists an $a \in A$ such that $l_0(p_0, a) < r_0$. Then $a \in A \cap B(p_0, r_0)$, which is impossible. Thus condition (1.1.2) is fulfilled. Now we assume (1.1.2) and take any $p \in \bar{A}$. Let $r > 0$. Then there exists a neighbourhood U of p open in $\mathcal{T}_l$ included in $B(p, r)$. This yields $A \cap U \neq \emptyset$. Therefore $a \in A \cap B(p, r)$ for some point a. Hence it follows that $l(p, A) \leqslant l_0(p, a) < r$. And, $l(p, A) = 0$.

As a corollary we obtain

1.2. *For every set $A \subset E$ the equality*

$$\bar{A} = \{p;\ p \in E \text{ and } l(p, A) = 0\} \tag{3}$$

holds if and only if the space (E, l) satisfies condition (1.1.2).

For any set $A \subset E$ and for every number $r > 0$ by A_{lr} (or shortly: A_r) we shall denote the set $\bigcup_{p \in A} B(p, r)$. The set A_r will be called the *r-neighbourhood* of A in the space (E, l). From this definition it follows that $A_r \subset A_{r'}$ when $0 \leqslant r \leqslant r'$; but the set A_r need not contain the set A.

2. Tangency of order k in the space (E, l). Let k be an arbitrary positive real, $p \in E$, and let both a and b be any non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0+]{} 0, \qquad b(r) \xrightarrow[r \to 0+]{} 0. \tag{4}$$

The pair (A, B) of subsets of E will be called *(a, b)-clustered* at the point p in the space (E, l) if 0 is a cluster point of the set Q of all real numbers $r > 0$ such that the sets

$$A \cap S(p, r)_{a(r)} \quad \text{and} \quad B \cap S(p, r)_{b(r)} \tag{5}$$

are non-empty.

Here $S(p, r)_{a(r)}$ is an $a(r)$-neighbourhood of the set $S(p, r)$ in the space (E, l) and similarly $S(p, r)_{b(r)}$. Denote by l_{pabAB} the characteristic function (relative to the set of all real numbers) of the set Q just defined, i. e. the function defined by the formulas $l_{pabAB}(r) = 1$ for $r \in Q$ and $l_{pabAB}(r) = 0$

in the opposite case. From this definition it immediately follows that

2.1. *The pair* (A, B) *of subsets of* E *is* (a, b)*-clustered at the point* p *in the space* (E, l) *if and only if*

$$\liminf_{r\to 0+}\big(1-l_{pabAB}(r)\big) = 0. \tag{6}$$

We shall say that the set A is (a, b)*-tangent of order* k *to the set* B *at the point* p in the space considered, which we shall write in the form $(A, B)\in \mathrm{T}_l(a, b, k, p)$, iff the pair (A, B) is (a, b)-clustered at the point p in (E, l) and

$$\frac{1}{r^k}\, l\big(A\cap S(p, r)_{a(r)}, B\cap S(p, r)_{b(r)}\big) \xrightarrow[r\to 0+]{} 0. \tag{7}$$

Suppose that (6) holds. For any $r > 0$ we shall denote by $\bar{l}_{pabABk}(r)$ the supremum of all numbers of the form

$$\frac{1}{t^k}\, l\big(A\cap S(p, t)_{a(t)}, B\cap S(p, t)_{b(t)}\big),$$

where $0 < t \leqslant r$. From the above it follows that the function $\bar{l}_{pabAB}$ is defined on the set of all positive real numbers and (7) is satisfied if and only if

$$\bar{l}_{pabABk}(r) \xrightarrow[r\to 0+]{} 0. \tag{8}$$

We may now define the function l_{pabABk}, setting

$$l_{pabABk}(r) = \begin{cases} \bar{l}_{pabABk}(r) & \text{for } r > 0, \text{ if (6) holds,} \\ 1 & \text{for } r > 0, \text{ in the opposite case.} \end{cases}$$

The following simple statement gives a convenient necessary and sufficient condition for the tangency of sets under consideration.

2.2. *The set* A *is* (a, b)*-tangent to the set* B *of order* k *at the point* p *if and only if*

$$l_{pabABk}(r) \xrightarrow[r\to 0+]{} 0. \tag{9}$$

Proof. Suppose that $(A, B)\in \mathrm{T}_l(a, b, k, p)$. Hence it follows (6). In this case $l_{pabABk}(r) = \bar{l}_{pabABk}(r)$ for $r > 0$ and (8) holds. So we obtain (9). If $(A, B)\notin \mathrm{T}_l(a, b, k, p)$, then we may consider two cases. In the first (6) is not satisfied. Then $l_{pabABk}(r) = 1$ for $r > 0$. In the second case (6) is fulfilled. But in that case (8) can not hold. Hence we also infer that (9) does not hold.

Now we shall assume a certain restriction concerning the space (E, l). We shall prove that

2.3. *If l_0 satisfies the triangle inequality, i. e. for $x, y, z \in E$ we have*

$$l_0(x, z) \leqslant l_0(x, y) + l_0(y, z), \tag{10}$$

then the condition $(A, B) \in \mathrm{T}_l(a, b, k, p)$, where a and b are functions fulfilling (4), *yields $p \in \bar{A} \cap \bar{B}$.*

Proof. According to the first part of 1.1 it suffices to prove that

$$l(p, A) = l(p, B) = 0. \tag{11}$$

Let $0 < \varepsilon < 1$. From (4) and the assumption that 0 is a cluster set of all $r > 0$ such that the sets (5) are non-empty it follows that there exists an $r > 0$ such that

$$r < \varepsilon/2, \quad a(r) < \varepsilon/2, \quad b(r) < \varepsilon/2, \tag{12}$$

and the points x and y belonging, respectively, to the sets $A \cap S(p, r)_{a(r)}$ and $B \cap S(p, r)_{b(r)}$. Then there exist points $x', y' \in S(p, r)$ such that

$$l_0(x', x) < a(r) \quad \text{and} \quad l_0(y', y) < b(r).$$

By the triangle inequality and (12) we obtain

$$l(p, A) \leqslant l_0(p, x) \leqslant l_0(p, x') + l_0(x', x) \leqslant r + a(r) < \varepsilon.$$

Similarly, $l(p, B) < \varepsilon$, which yields (11).

As a corollary we obtain

2.4. *If the real functions a and b satisfy condition* (4) *and the function l satisfies the triangle inequality, i. e.*

$$l(A, C) \leqslant l(A, B) + l(B, C) \quad \text{for } A, B, C \in E_0, \tag{13}$$

then from $(A, B) \in \mathrm{T}_l(a, b, k, p)$ it follows that $p \in \bar{A} \cap \bar{B}$.

3. Some questions related to the symmetry and transitivity of the relation $\mathrm{T}_l(a, b, k, p)$. The function l is, in general, not symmetric, i. e. the equality $l(A, B) = l(B, A)$ need not hold for all $A, B \in E_0$. Moreover, the function a need not be equal to b. From these two causes it follows that relation $\mathrm{T}_l(a, b, k, p)$ must be symmetric. But we may consider a class $\mathscr{C}$ contained in E_0 and ask about the conditions under which this relation is symmetric in $\mathscr{C}$. Related to this question is the problem of formulating of necessary and sufficient condition concerning the set B under which for an arbitrary A of $\mathscr{C}$ if A is (a, b)-tangent of order k to B at the point p, then B is (a, b)-tangent of order k to A at p. Before formulating that condition we introduce some notations. By sgn we shall denote the real function defined by the formula

$$\operatorname{sgn} t = \begin{cases} t/|t| & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Denote by $l_{pabk,\mathscr{C}}(B)$ the infimum of the set of all reals of the form

$$(14)\qquad \operatorname{sgn}\limsup_{r\to 0+} l_{pabABk}(r)-\operatorname{sgn}\limsup_{r\to 0+} l_{pabBAk}(r),$$

where the set A is varying in the set $\mathscr{C}$.

3.1. *The following two conditions are equivalent:*

(3.1.1) *For any $A\in\mathscr{C}$ if A is (a,b)-tangent of order k to B at p, then B is (a,b)-tangent of order k to A at p;*

$$(3.1.2)\qquad l_{pabk,\mathscr{C}}(B)\geqslant 0.$$

Proof. Let us suppose that (3.1.1) is fulfilled. From 2.2 it follows that for every set $A\in\mathscr{C}$ if

$$(15)\qquad \limsup_{r\to 0+} l_{pabABk}(r) = 0,$$

then

$$(16)\qquad \limsup_{r\to 0+} l_{pabBAk}(r) = 0.$$

From (15) it follows (16) if and only if number (14) is non-negative. Hence we obtain (3.1.2). Conversely, from (3.1.2) it follows that number (14) is non-negative for any $A\in\mathscr{C}$. Then for every $A\in\mathscr{C}$ identity (15) yields (16). Therefore (3.1.1) is satisfied.

We now give some conditions which are sufficient for the transitivity of the relation $\mathrm{T}_l(a,b,k,p)$. First, we prove the following lemma:

3.2. *If in the space (E,l) the triangle inequality is satisfied and non-negative real functions b and b_1 fulfil the following condition:*

(3.2.1) *There exists $\delta>0$ such that*

$$(17)\qquad \operatorname{sgn}\big(l(B_1,A)-l(B,A)\big)\geqslant \operatorname{sgn}\big(b_1(r)-b(r)\big),$$

$$(18)\qquad \operatorname{sgn}\big(l(A,B)-l(A,B_1)\big)\leqslant \operatorname{sgn}\big(b_1(r)-b(r)\big),$$

where $A\in E_0,\ \emptyset\neq B\subset B_1\subset S(p,r)_{\max\{b(r),b_1(r)\}}$, *and* $0<r<\delta$

then:

(3.2.2) *For any A, B, C, if $(A,B)\in\mathrm{T}_l(a,b,k,p)$, $(B,C)\in\mathrm{T}_l(b_1,c,k,p)$, and 0 is a cluster point of the set of all numbers $r>0$ such that the sets $A\cap S(p,r)_{a(r)}$ and $C\cap S(p,r)_{b(r)}$ are non-empty, then $(A,C)\in\mathrm{T}_l(a,c,k,p)$.*

Proof. Let us suppose that in (E,l) the triangle inequality is satisfied and (3.2.1) holds. Let $(A,B)\in\mathrm{T}_l(a,b,k,p)$, $(B,C)\in\mathrm{T}_l(b_1,c,k,p)$. From these suppositions it follows that (7) holds and

$$(19)\qquad \frac{1}{r^k}\, l\big(B\cap S(p,r)_{b_1(r)},\ C\cap S(p,r)_{c(r)}\big)\xrightarrow[r\to 0+]{} 0.$$

We set

$$(20)\qquad \varphi(r) = \frac{1}{r^k}\Big(l\big(A\cap S(p,r)_{a(r)},\, B\cap S(p,r)_{b(r)}\big)+l\big(B\cap S(p,r)_{b_1(r)},\, C\cap S(p,r)_{c(r)}\big)\Big).$$

We have $\varphi(r)\to 0$ when $r\to 0^+$. From the triangle inequality we obtain

$$(21)\qquad \frac{1}{r^k}\Big(l\big(A\cap S(p,r)_{a(r)},\, C\cap S(p,r)_{c(r)}\big)\Big) \leqslant \frac{1}{r^k}\Big(l\big(A\cap S(p,r)_{a(r)},\, B\cap S(p,r)_{b(r)}\big)+l\big(B\cap S(p,r)_{b(r)},\, C\cap S(p,r)_{c(r)}\big)\Big)$$

and

$$(22)\qquad \frac{1}{r^k}\Big(l\big(A\cap S(p,r)_{a(r)},\, C\cap S(p,r)_{c(r)}\big)\Big) \leqslant \frac{1}{r^k}\Big(l\big(A\cap S(p,r)_{a(r)},\, B\cap S(p,r)_{b_1(r)}\big)+l\big(B\cap S(p,r)_{b_1(r)},\, C\cap S(p,r)_{c(r)}\big)\Big).$$

Let $\delta>0$ be a number such that inequalities (17) and (18) are satisfied when $0<r<\delta$ and B is a non-empty subset of B_1, B_1 being contained in $S(p,r)_{\max\{b(r),\,b_1(r)\}}$. If $b_1(r)\geqslant b(r)$, we have $B\cap S(p,r)_{b_1(r)}\subset B\cap S(p,r)_{b(r)}$. Setting in (17) the set $B\cap S(p,r)_{b(r)}$ instead of B and the set $B\cap S(p,r)_{b_1(r)}$ instead of B_1, we obtain from (21)

$$(23)\qquad \frac{1}{r^k}\,l\big(A\cap S(p,r)_{a(r)},\, C\cap S(p,r)_{c(r)}\big)\leqslant\varphi(r).$$

Similarly in the case of $b_1(r)<b(r)$, using (18) and (22), we obtain the same inequality (23). Hence it follows that the set A is (a,c)-tangent of order k to C at the point p. This ends the proof.

Now we shall define a function l_p which will be useful in formulating a sufficient condition of transitivity of the relation $\mathrm{T}_l(a,b,k,p)$. Let a be any non-negative real function defined on some right-hand side neighbourhood of the number 0, and let A be an arbitrary subset of E and $r>0$. Put

$$(24)\qquad l_p(A,a,r) = l_{paaAA}(r),$$

where $l_{paaAA}(r)$ has the same meaning as in Section 2. In other words, the number $l_p(A,a,r)$ is the supremum of all values of the characteristic function of the set $A\cap S(p,r)_{a(r)}$. We have $l_p(A,a,r)=1$ if and only if this set is non-empty.

An important case in Lemma 3.2 is that of $b_1 = a$. We prove that

3.3. *If in the space (E, l) the triangle inequality is satisfied and the real functions a and b fulfil the following conditions:*

(3.3.1) *There exists $\delta > 0$ such that*

$$\operatorname{sgn}\big(l(A, B) - l(A, B_1)\big) \leqslant \operatorname{sgn}\big(a(r) - b(r)\big)$$
$$\leqslant \operatorname{sgn}\big(l(B_1, A) - l(B, A)\big),$$

when $A, B \in E_0$, $B \subset B_1 \subset S(p, r)_{\max\{a(r), b(r)\}}$, $0 < r < \delta$;

(3.3.2) *For any sets A, B, if the set A is (a, b)-tangent of order k to the set B at the point p, then $l_p(A, b, r) \leqslant l_p(B, b, r)$;*

then the relation $\mathrm{T}_l(a, b, k, p)$ is transitive.

Proof. Let $(A, B), (B, C) \in \mathrm{T}_l(a, b, k, p)$. From the definition of the relation $\mathrm{T}_l(a, b, k, p)$ it follows that for any $\eta > 0$ there exists a positive $r < \eta$ such that sets (5) are non-empty. Then we have $l_p(B, b, r) = 1$. Hence, by (3.3.2), we obtain $l_p(C, b, r) = 1$. In other words, the set $C \cap S(p, r)_{b(r)}$ is non-empty. From Lemma 3.2 it follows that (A, C) belongs to $\mathrm{T}_l(a, b, k, p)$.

4. Some special cases. The most important cases are obtained when we consider the concept of tangency in a metric space or else, more generally, in a pseudometric one. Let (E, ϱ) be a bounded *metric* (or else *pseudometric*) space. In the investigation touching the tangency of sets the assumption that the space is bounded makes no essential restriction. The metric ϱ induces some functions ϱ_i such that (E, ϱ_i) is a space in the meaning of Section 1. Namely, we may define the function ϱ_0 by the formula

$$(25) \qquad \varrho_0(A, B) = \sup\{\varrho(x, B);\ x \in A\} \quad \text{ for } A, B \in E_0,$$

where $\varrho(x, B)$ is the *distance* from the point x to the set B, i. e. $\varrho(x, B)$ is the infimum of $\varrho(x, y)$ for $y \in B$. It is evident that the function ϱ_0 satisfies the triangle inequality and that $\varrho_0(A, B) = 0$ if and only if the non-empty set A is contained in the closure of B. The function ϱ_0 defined above leads to the function ϱ_1 defined as follows:

$$(26) \qquad \varrho_1(A, B) = \max\{\varrho_0(A, B), \varrho_0(B, A)\} \quad \text{ for } A, B \in E_0.$$

This function is a pseudometric of the set E_0 of all non-empty subsets of E. If we restrict the function to the set of all closed non-empty sets of the metric space (E, ϱ), we obtain a metric space, the so-called Hausdorff metric space of closed sets.

By $\operatorname{diam}_\varrho A$ we shall denote the diameter of the set A in the metric space (E, ϱ). We define the function ϱ_i for $i = 2, \ldots, 6$ by the formulas:

$$\varrho_2(A, B) = \inf\{\operatorname{diam}_\varrho(\{x\} \cup B);\ x \in A\}, \tag{27}$$

$$\varrho_3(A, B) = \max\{\varrho_2(A, B), \varrho_2(B, A)\}, \tag{28}$$

$$\varrho_4(A, B) = \min\{\varrho_2(A, B), \varrho_2(B, A)\}, \tag{29}$$

$$\varrho_5(A, B) = \operatorname{diam}_\varrho(A \cup B), \tag{30}$$

$$\varrho_6(A, B) = \inf\{\varrho(x; B);\ x \in A\}. \tag{31}$$

Let us remark that if φ is any non-negative real function of seven variables, then we may define the function ϱ_7 as follows:

$$\varrho_7(A, B) = \varphi\big(\varrho_0(A, B), \varrho_1(A, B), \ldots, \varrho_6(A, B)\big) \tag{32}$$

for any $A, B \in E_0$. If the real function φ in formula (32) is monotonous and subadditive with respect to each variable separately, then the function ϱ_7 satisfies the triangle inequality.

From definitions (25)-(31) of the functions ϱ_i it follows that if the metric space (E, ϱ) is a Cartesian n-dimensional space with the usual metric, then for any regular arcs A and B beginning at the point p we have

(4.1) $(A, B) \in \mathrm{T}_{\varrho_i}(a, b, k, p)$ if and only if $(A, B) \in \mathrm{T}_{\varrho_j}(a, b, k, p)$ for $i, j = 0, 1, \ldots, 6$,

where a and b satisfy the condition

$$\frac{a(r)}{r} \xrightarrow[r \to 0+]{} 0 \quad \text{or else} \quad \frac{b(r)}{r} \xrightarrow[r \to 0+]{} 0.$$

The sentence (4.1) is not true in general, i. e. when the sets A and B are not assumed to be regular arcs in the Cartesian space or when the metric space (E, ϱ) is not a Cartesian space with the usual metric. It appears a natural problem to study some of the connections between the relation $\mathrm{T}_{\varrho_i}(a, b, k, p)$ and $\mathrm{T}_{\varrho_j}(a, b, k, p)$ for $i, j = 0, 1, \ldots, 6$. This problem is not considered in the present paper. We note only that the function ϱ_0 allows us to describe the concept of tangency in a metric space treated in papers [1], [2], [5], [6], [7]. In those papers the set A is said to be tangent to the set B at the point p in the metric space (E, ϱ) iff p is a cluster point of the set A and

$$\frac{\varrho(x; B)}{\varrho(x, p)} \xrightarrow[\substack{x \to p \\ x \in A}]{} 0. \tag{33}$$

It is easy to state that condition (33) is equivalent to the following one:

$$\frac{1}{r}\sup\{\varrho(x; B);\ x\in A \ \textit{and}\ \varrho(x, p) = r\} \xrightarrow[r\to 0^+]{} 0.$$

This condition can be written in the form

$$\frac{1}{r}\varrho_0\big(A\cap S(p, r), B\big) \xrightarrow[r\to 0^+]{} 0.$$

This, setting $a(r) = 0$ and $b(r) = r$ for $r > 0$, we may write in the following manner:

$$\frac{1}{r}\varrho_0\big(A\cap S(p, r)_{a(r)}, B\cap S(p, r)_{b(r)}\big) \xrightarrow[r\to 0^+]{} 0. \tag{34}$$

Here the sphere $S(p, r)$ is the sphere of the radius r and the centre at p in the metric space (E, ϱ). According to the notion of sphere defined in Section 1, this sphere is identical with the sphere $S_l(p, r)$, where $l = \varrho_0$. Let us remark that the function l_0 defined by formula (1) for $l = \varrho_i$, $i = 0, 1, \ldots, 6$, does not depend on i, because $\varrho_i(\{x\}, \{y\}) = \varrho(x, y)$ for any $x, y\in E$, $i = 0, 1, \ldots, 6$. Condition (34) for the sets A and B such that p is their cluster point states that the set A is (a, b)-tangent of order 1 to the set B at the point p.

As a corollary from Theorem 3.3 we obtain

4.1. *If (E, ϱ) is a metric space, then for the space (E, ϱ_0) the relation $T_{\varrho_0}(a, b, k, p)$, where ϱ_0 is defined by formula* (25), *$a(r) = 0$ and $b(r) = r$ for $r > 0$, is transitive.*

References

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