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A natural parameter of a curve in the symplectic space

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The object of this note is to define a natural parameter ϑ of a curve in the symplectic space G_{2n} of dimension $2n$, i.e. in the affine space of the same dimension with the symplectic group Sp_{2n} [4] of transformations. The parameter ϑ will be defined in a geometric way — as for example in [1], p. 8 the arc s of a curve in the Euclidean space is defined. The parameter s is such a parameter that the length of the tangent vector $\boldsymbol{x}'(s)$ of a curve $x = x(s)$ in the Euclidean space is equal to 1. But for two points in G_{2n} there is no invariant with respect to the symplectic transformations. We take therefore two vectors $\boldsymbol{x}^{(\varrho)}(\vartheta)$ and $\boldsymbol{x}^{(\varrho+1)}(\vartheta)$ attached to a point of a curve in G_{2n}; the (symplectic) scalar product of these two vectors is equal to 1; ϱ is the order of the derivative of the vector $\boldsymbol{x} = \boldsymbol{x}(\vartheta)$, where ϑ is our natural parameter and denotes the rank, defined in section 3, of the curve.

Parameter ϑ may be defined for a more extensive class of curves than the class with parameter s defined in [2] or [5].

Our definition of a curve is based on the same idea as the definition of Nomizu [3]. Nomizu defined a curve of class C^∞ as an equivalence class of parametrized curves of class C^∞. It is not required in this paper that the equivalent curves be of the same class.

1. Let the space G_{2n} be referred to a symplectic frame. Thus

$$\Omega = \sum_{i=1}^{n} [\boldsymbol{x}^i \boldsymbol{x}^{n+i}]$$

is its fundamental form. The value of the exterior form Ω for two vectors $\boldsymbol{x}_l = (\boldsymbol{x}_l^1, \ldots, \boldsymbol{x}_l^{2n})$, $l = 1, 2$, denoted by $\Omega[\boldsymbol{x}_1, \boldsymbol{x}_2]$ and equal to

$$\sum_{i=1}^{n} (\boldsymbol{x}_1^i \boldsymbol{x}_2^{n+i} - \boldsymbol{x}_1^{n+i} \boldsymbol{x}_2^i) ,$$

is the only invariant ([4], p. 122) of the pair of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ with respect to the transformations of Sp_{2n}. This invariant will be called the *scalar product of two vectors* and also denoted by $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ or $\sum_{i=1}^{n} [\boldsymbol{x}_1^i, \boldsymbol{x}_2^{n+i}]$.

The value of the kth exterior power of Ω, i.e. the value of the form

$$[\Omega^k] = \sum_{i_1, \ldots, i_k=1}^{n} [\Pi_{i_1} \ldots \Pi_{i_k}],$$

where $\Pi_i = [\boldsymbol{x}^i \boldsymbol{x}^{n+i}]$, for $2k$ vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2k}$ denoted by $\Omega^k[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2k}]$ or $[\boldsymbol{x}_1 \ldots \boldsymbol{x}_{2k}]$, is the only invariant of these vectors. This definition was used by Yaglom [5] to define the arc s of a curve in G_{2n} by the following formulas:

$$s = \begin{cases} \int [\boldsymbol{x}'\boldsymbol{x}'' \ldots \boldsymbol{x}^{(2n)}]^{1/n(2n+1)} dt & \text{for } n \text{ even}, \\ \int |[\boldsymbol{x}'\boldsymbol{x}'' \ldots \boldsymbol{x}^{(2n)}]|^{1/n(2n+1)} dt & \text{for } n \text{ odd}, \end{cases}$$

where the vector $\boldsymbol{x} = \boldsymbol{x}(t)$ defines a curve of G_{2n} and the superscripts denote differentiation. The volume of the parallelepiped constructed on vectors $\boldsymbol{x}'(s), \boldsymbol{x}''(s), \ldots, \boldsymbol{x}^{(2n)}(s)$, i.e. the value of $[\Omega^n]$ for these vectors, is equal to 1.

The author has defined [2] a parameter s of a curve in G_{2n} by the formula

$$s = \int [\boldsymbol{x}'\boldsymbol{x}'']^{1/3} dt \tag{1}$$

assuming that $[\boldsymbol{x}'\boldsymbol{x}''] \neq 0$. The scalar product $[\boldsymbol{x}'(s)\boldsymbol{x}''(s)]$ of two vectors $\boldsymbol{x}'(s)$ and $\boldsymbol{x}''(s)$, where s is the parameter defined by (1), is equal to 1.

2. By a *parametrized curve* $\mathfrak{K}(t)$ of class C^{μ_t}, where μ_t is zero, an integer, ∞ or ω, we mean a mapping of a certain interval $T = [a, b]$ into G_{2n} defined by $x^a = x^a(t)$, $a = 1, \ldots, 2n$, where $x^a(t)$ are functions of class C^{μ_t} on T. The parametrized curve $\mathfrak{K}(t)$ of class C^{μ_t} is said to be *equivalent* to the parametrized curve $\mathfrak{K}(\bar{t})$ of class $C^{\mu_{\bar{t}}}$ if there exists a strictly monotonic and continuous function $\bar{t} = \psi(t)$, $\bar{t} \in \bar{T}$, such that $x^a(t) = \bar{x}^a(\psi(t))$ for every $t \in T$. This mapping will be called an *admissible transformation* of the parameter. The maximal cardinal number of all numbers μ_t obtained by any admissible change of the parameter t is denoted by μ. The equivalence class of a parametrized curve $\mathfrak{K}(t)$ of class C^{μ_t} will be called the *curve* $\mathfrak{K}$ of class C^{μ}.

Instead of writing "a parametrized curve $\mathfrak{K}(t)$" we will write "a curve $\mathfrak{K}(t)$".

Two equivalent curves are the same sets of points in G_{2n}. Double points are to be counted twice, triple points — thrice and so on.

A curve $\mathfrak{K}$ is ***essentially imbedded*** in G_{2n} if neither the curve $\mathfrak{K}$ nor any of its arcs may be imbedded in a subspace of G_{2n}. We assume that the curve which we consider, except in section 5, is such a curve.

We assume that all derivatives that we use exist. For a curve $\mathfrak{K}(t) \in \mathfrak{K}$ of class C^{2n+1} there exist vectors $\boldsymbol{x}'(t), \ldots, \boldsymbol{x}^{(2n)}(t)$ continuous and independent.

3. The expressions

$$\sum_{i=1}^{n} \left[\frac{d^h x^i}{dt^h} \cdot \frac{d^k x^{n+i}}{dt^k}\right] = K_{h|k}(t)\,, \qquad h < k\,,\ k = 1, 2, \ldots\,, \tag{2}$$

are the differential invariants of order k with respect to the symplectic transformations of a curve $\mathfrak{K}(t)$. From Theorem I in [2] it follows that

$$K_{h|k}(t) = \sum_{p=1}^{m} (-1)^p \binom{k-h-p}{p-1} K_{h+p-1|h+p}^{(k-h-2p+1)}(t)\,,$$
$$m = \left[\frac{k-h+1}{2}\right], \qquad h = 1, \ldots, k-1\,,$$

where $(k-h-2p+1)$ denotes differentiation of order $k-h-2p+1$ and the brackets [ ] denote "entier".

Let $p(t)$ denote a point of a curve $\mathfrak{K}(t)$, i.e. the end of the vector $\boldsymbol{x}(t)$. If $\bar{t} = \psi(\bar{t})$ is an admissible change of parameter t of the curve, $p(\bar{t})$ is the same point. Denote by p all incident points of equivalent parametrized curves.

Assign to every point $p(t)$ of a curve $\mathfrak{K}(t)$ a set of plane elements $E_1(t) = \{\boldsymbol{x}'(t), \boldsymbol{x}''(t)\}$, $E_2(t) = \{\boldsymbol{x}''(t), \boldsymbol{x}'''(t)\}, \ldots, E_{2n}(t) = \{x^{(2n)}(t), x^{(2n+1)}(t)\}$ and a set of invariants

$$K_{1|2}(t)\,, \quad K_{2|3}(t)\,, \quad \ldots\,, \quad K_{2n|2n+1}(t)\,.$$

The plane element $E_k(t)$ is *involutive* if the value of the form Ω for vectors $\boldsymbol{x}^{(k)}(t)$ and $\boldsymbol{x}^{(k+1)}(t)$ is equal to zero.

A point $p(t) \in \mathfrak{K}(t)$ is said to be *of rank* ϱ if the elements $E_1(t), \ldots, E_{\varrho-1}(t)$ are involutive planes and the element $E_\varrho(t)$ is not an involutive plane. This definition is equivalent to the following one: a point $p(t) \in \mathfrak{K}(t)$ is said to be *of rank* ϱ if

$$K_{1|2}(t) = \ldots = K_{\varrho-1|\varrho}(t) \quad \text{and} \quad K_{\varrho|\varrho+1}(t) \neq 0\,.$$

We have the following

LEMMA 1. *If the rank of a point $p(t)$ of a curve $\mathfrak{K}(t)$ is ϱ, then the rank of the point $p(\bar{t})$ of the equivalent curve $\mathfrak{K}(\bar{t})$ is also ϱ.*

In fact, we have

$$(4)\qquad \bar{K}_{\sigma-1|\sigma}(\bar{t}) = \sum_{k,h=1}^{\sigma} \beta_{hk}(t) K_{h|k}(t)\,, \qquad \sigma = 1, \ldots, h < k\,,$$

where $\beta_{hk}(t)$ are known functions. From the assumptions of the lemma and from (3) it follows that, in (4), $K_{h|k}(t) = 0$ for $k \leqslant \varrho$. Thus $\bar{K}_{1|2}(\bar{t}) = \ldots = \bar{K}_{\varrho-1|\varrho}(\bar{t}) = 0$. But

$$(5)\qquad \bar{K}_{\varrho|\varrho+1}(\bar{t}) = \left(\frac{dt}{d\bar{t}}\right)^{2\varrho+1} K_{\varrho|\varrho+1}(t) \neq 0\,,$$

which proves the lemma.

A point $p \in \mathfrak{K}$ is said to be *of rank* ϱ if $p(t) \in \mathfrak{K}(t)$ is of rank ϱ. A curve $\mathfrak{K}$ is *of rank* ϱ if its every point is of rank ϱ. In what follows we assume that the curve $\mathfrak{K}$ is of rank ϱ, i.e. that

$$(6)\qquad K_{\varrho|\varrho+1}(t) \neq 0$$

for every $t \in T$.

LEMMA 2. *The rank of any curve $\mathfrak{K}$ imbedded in G_{2n} is not greater than n.*

In fact, if $K_{1|2} = K_{2|3} = \ldots = K_{n|n+1} = 0$ then for a curve $\mathfrak{K}(t) \in \mathfrak{K}$, as follows from (3), $K_{h|k}(t) = 0$ $(h < k)$ for $k = 1, \ldots, n+1$. Hence there exists a set of $n+1$ independent vectors $\boldsymbol{x}'(t), \ldots, \boldsymbol{x}^{(n+1)}(t)$ such that the linear space expanded on these vectors would be an involutive subspace of dimension $n+1$, which is impossible ([4], p. 84). Thus Lemma 2 is proved.

LEMMA 3. *If a curve $\mathfrak{K}(t)$ is of rank ϱ, then, for every $t \in \operatorname{int} T$, there exists such a neighbourhood U that the inequality $[\boldsymbol{x}^{(\varrho)}(t)\boldsymbol{x}^{(\varrho)}(\tau)] \neq 0$ holds for every $\tau \in U$ except $\tau = t$.*

The statement follows from

$$\left(\frac{d}{d\tau}[\boldsymbol{x}^{(\varrho)}(t)\boldsymbol{x}^{(\varrho)}(\tau)]\right)_{\tau=t} = K_{\varrho|\varrho+1}(t) \neq 0 \quad \text{and} \quad [\boldsymbol{x}^{(\varrho)}(t)\boldsymbol{x}^{(\varrho)}(\tau)]_{\tau=t} = 0\,.$$

For the end-points of curve $\mathfrak{K}(t)$ there exist right-hand and left-hand neighbourhoods that satisfy the statement made in Lemma 3.

4. Let $\varepsilon_0(t)$ denote the length (i.e. the difference $|t_2 - t_1|$, where t_1 and t_2 are the end-points of an interval) of the largest neighbourhood of a point $t \in \operatorname{int} T$ for which Lemma 3 is satisfied. Particularly, $\varepsilon_0(t)$ may be equal to the length of T. It is easy to see that $\inf \varepsilon_0(t) \neq 0$. By ε_1 and ε_2 we denote the lengths of the largest right-hand and left-hand neighbourhoods of the end-points of T, respectively.

Choose a set of points $a = t_0, t_1, \ldots, t_m = b$ belonging to T in such a way that $|t_i - t_{i-1}| < \varepsilon/2$ $(i = 1, \ldots, m)$, where $\varepsilon = \min\big(\inf \varepsilon_0(t), \varepsilon_1, \varepsilon_2\big)$. From assumption (6) it follows that the function $K_{\varrho|\varrho+1}(t)$ does not change its sign and, as follows from Lemma 3,

$$\text{sign}[\boldsymbol{x}^{(\varrho)}(t_{i-1})\boldsymbol{x}^{\varrho}(t_i)] = \text{const} \neq 0 \tag{7}$$

for every i.

Let $\Delta\vartheta_i$ denote the invariant

$$\Delta\vartheta_i = \{[\boldsymbol{x}^{(\varrho)}(t_{i-1})\boldsymbol{x}^{(\varrho)}(t_i)](\Delta t_i)^{2\varrho}\}^{1/(2\varrho+1)}, \tag{8}$$

where $\Delta t_i = t_i - t_{i-1}$. By the mean value theorem we have

$$\boldsymbol{x}^{(\varrho)}(t_i) = \boldsymbol{x}^{(\varrho)}(t_{i-1}) + \boldsymbol{x}^{(\varrho+1)}(\theta_i)\Delta t_i\,, \tag{9}$$

where $t_{i-1} < \theta_i < t_i$. From (8) and (9) we have

$$\Delta\vartheta_i = [\boldsymbol{x}^{(\varrho)}(t_{i-1})\boldsymbol{x}^{(\varrho+1)}(\theta_i)]^{1/(2\varrho+1)}\Delta t_i\,.$$

The integral

$$\lim_{m\to\infty}\sum_{i=1}^{m}\Delta\vartheta_i = \int_a^b [\boldsymbol{x}^{(\varrho)}(t)\boldsymbol{x}^{(\varrho+1)}(t)]^{1/(2\varrho+1)}dt \tag{10}$$

is called the *symplectic length of the curve* $\mathfrak{K}$. From (7) it follows that

$$\vartheta = \int_a^t [\boldsymbol{x}^{(\varrho)}(t)\boldsymbol{x}^{(\varrho+1)}(t)]^{1/(2\varrho+1)}dt \tag{11}$$

is a strictly monotonic function of t. From (11) and (5) we get

$$\left(\frac{d\vartheta}{dt}\right)^{2\varrho+1} = K_{\varrho|\varrho+1}(t) = \left(\frac{d\vartheta}{dt}\right)^{2\varrho+1} K_{\varrho|\varrho+1}(\vartheta)\;;$$

thus for the curve $\mathfrak{K}(\vartheta)$ we have

$$[\boldsymbol{x}^{(\varrho)}(\vartheta)\boldsymbol{x}^{(\varrho+1)}(\vartheta)] = 1\,. \tag{12}$$

The parameter for which equation (12) is satisfied will be called the *natural parameter* of the curve $\mathfrak{K}$ of rank ϱ.

The natural parameter is defined up to an additive constant.

Integral (10) does not change its value by any admissible transformation of the parameter of the curve $\mathfrak{K}$. Thus the natural parameter of a curve $\mathfrak{K}$ is an invariant with respect to such transformations.

The expression

$$d\vartheta^{2\varrho+1} = [d^{\varrho}\boldsymbol{x}\, d^{\varrho+1}\boldsymbol{x}]$$

is called the *element of the symplectic arc* of a curve $\mathfrak{K}$ of rank ϱ.

The class of differentiability of function (11) is lower than that of the curve $\mathfrak{K}(t)$ and the difference is equal to ϱ. Thus, if a curve $\mathfrak{K}$ of rank ϱ if of class C^{μ}, then the curve $\mathfrak{K}(\vartheta)$ is of class $C^{\mu-\varrho}$. If the class of a curve $\mathfrak{K}$ is equal to C^{2n+1}, then the invariants $K_{1|2}, K_{2|3}, \ldots, K_{2n|2n+1}$ and the vectors $\boldsymbol{x}', \boldsymbol{x}'', \ldots, \boldsymbol{x}^{(2n+1)}$ exist and are continuous. But the natural parameter may be defined on the curve of class C^{μ} $(\mu \geqslant 2\varrho+1)$ and then the vectors $\boldsymbol{x}'(\vartheta), \ldots, \boldsymbol{x}^{(\varrho+1)}(\vartheta)$ are continuous.

Therefore we have the following

THEOREM. *If a curve $\mathfrak{K}$, essentially imbedded in G_{2n}, is of rank ϱ and of class at least C^{μ} $(\mu \geqslant 2\varrho+1)$, then there exists one and only one (up to an additive constant) parametrized curve $\mathfrak{K}(\vartheta) \in \mathfrak{K}$ which is of class $C^{\varrho+1}$ and $[\boldsymbol{x}^{(\varrho)}(\vartheta)\boldsymbol{x}^{(\varrho+1)}(\vartheta)] = 1$.*

5. Denote by $G_{2\gamma;\beta}$ $(\gamma = 1, \ldots, n, 2\gamma+\beta < 2n)$ a linear subspace of G_{2n} of dimension $2\gamma+\beta$, where β is the number of independent and distinguished vectors of G_{2n}. A vector of a linear subspace of the symplectic space is called a *distinguished* one if it is in involution with all other vectors of the subspace ([4], p. 86).

The theorem of section 4 may be formulated in a more general form:

If a curve $\mathfrak{K}$ of rank ϱ is essentially imbedded in a $G_{2\gamma;\beta}$ and if it is of class at least $C^{2\varrho+1}$, then there exists one and only one (up to an additive constant) parametrized curve $\mathfrak{K}(\vartheta) \in \mathfrak{K}$ which is of class $C^{\varrho+1}$ and $K_{\varrho|\varrho+1} = 1$.

The rank of a curve $\mathfrak{K}$ essentially imbedded in $G_{2\gamma;\beta}$ is not greater than γ.

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