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Differential inequalities of parabolic type in the Sobolev space

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Abstract. In this paper the following inequalities of parabolic type are investigated:

$$\frac{u^k}{\partial t} < \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i^k(t, x, u^k, u_x^k) + B^k(t, x, U, u_x^k),$$

$$\frac{v^k}{\partial t} > \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i^k(t, x, v^k, v_x^k) + B^k(t, x, V, v_x^k),$$

where $\{u^k(t, x)\}$ and $\{v^k(t, x)\}$ $(k = 1, \ldots, N)$ are defined in $[0, T]\times R_n$. The solutions $\{u^k\}$ and $\{v^k\}$ are assumed to belong to the Sobolev space W_2^1. The main result of the paper gives conditions under which the initial inequalities $u^k(0, x) \leqslant V^k(0, x)$ imply $u^k(t, x) \leqslant v^k(t, x)$ in $[0, T]\times R_n$. As an application the maximum principle and the uniqueness of the Cauchy problem are obtained. Subject Classification. Primary 35 19, Secondary 46 38. Key words and phrases. Maximum principle. Generalized solutions. Energy estimates.

1. In this paper we investigate systems of parabolic partial differential inequalities of the form

$$\frac{\partial u^k}{\partial t} \leqslant \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i^k(t, x, u^k, u_x^k) + B^k(t, x, U, u_x^k), \tag{1}$$

$$\frac{\partial v^k}{\partial t} \geqslant \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i^k(t, x, v^k, v_x^k) + B^k(t, x, V, v_x^k) \qquad (k = 1, \ldots, N), \tag{2}$$

where

$$u_x^k(t, x) = \{u_{x_1}^k(t, x), \ldots, u_{x_n}^k(t, x)\}, \qquad v_x^k(t, x) = \{v_{x_1}^k(t, x), \ldots, v_{x_n}^k(t, x)\},$$
$$U(t, x) = \{u^1(t, x), \ldots, u^N(t, x)\}, \qquad V(t, x) = \{v^1(t, x), \ldots, v^N(t, x)\}.$$

We also discuss the maximum principle and uniqueness of the Cauchy problem. We use the technique of the proofs and the notation from [1].

Before formulating the main result we introduce some definitions and notations.

DEFINITION 1. A function $u(t, x)$ defined and measurable in an open domain D in $(0, T)\times R_n$ is said *to belong to the space* $W_2^{1,1}(D)$ if u possesses generalized derivatives u_t u_{x_i} $(i = 1, \ldots, n)$ (see [3]) such that

$$\int_D \Big[u(t, x)^2 + u_t(t, x)^2 + \sum_{i=1}^{n} u_{x_i}(t, x)^2\Big]\, dt\, dx < \infty.$$

In [3] it is shown that every function $u \in W_2^{1,1}\big((0, T)\times(|x| < R)\big)$ (where $|x|^2 = \sum_{i=1}^{n} x_i^2$) has boundary values $u(0, x)$ and $u(T, x)$ belonging to $L_2(|x| < R)$ in the following sense:

$$\lim_{t\to 0} \int_{|x|<R} [u(t, x) - u(0, x)]^2 dx = 0, \qquad \lim_{t\to T} \int_{|x|<R} [u(t, x) - u(T, x)]^2 dx = 0.$$

DEFINITION 2. A function $u(t, x)$ defined and measurable in $(0, T)\times R_n$ is said *to belong to the class* E_2 if there exists a number $\alpha > 0$ such that

$$\int_0^T \int_{R_n} u(t, x)^2 e^{-\alpha|x|^2}\, dt\, dx < \infty.$$

DEFINITION 3. A system of functions $\{u^k(t, x)\}$ $(k = 1, \ldots, N)$ is said to be a *weak solution* of the system of differential inequalities (1) (or (2)) in $(0, T)\times R_n$ if $u^k \in W_2^{1,1}\big((0, T)\times(|x| < R)\big)$ $(k = 1, \ldots, N)$ for all $R > 0$ and $\{u^k\}$ $(k = 1, \ldots, N)$ satisfies

$$(3) \qquad \int_0^T \int_{R_n} \Big[\varphi^k u_t^k + \sum_{i=1}^{n} \varphi_{x_i}^k A_i^k(t, x, u^k, u_x^k)\Big] dt\, dx \leqslant \int_0^T \int_{R_n} \varphi^k B^k(t, x, U, u_x^k)\, dt\, dx$$

$(k = 1, \ldots, N)$ or

$$(3') \qquad \int_0^T \int_{R_n} \Big[\varphi^k u_t^k + \sum_{i=1}^{n} \varphi_{x_i}^k A_i^k(t, x, u^k, u_x^k)\Big] dt\, dx \geqslant \int_0^T \int_{R_n} \varphi^k B^k(t, x, U, u_x^k)\, dt\, dx$$

for any system of non-negative functions $\{\varphi^k\}$ $(k = 1, \ldots, N)$ of the class $W_2^{1,1}\big((0, T)\times R_n\big)$ with compact supports in x and vanishing for $t = T$. The functions A_i^k and B^k are assumed to be measurable.

DEFINITION 4. Let a system of functions $\{B^k(t, x, Z, P)\}$ $(k = 1, \ldots, N)$ be defined for $(t, x) \in (0, T)\times R_n$ and for arbitrary $Z = (z_1, \ldots, z_N)$ and $P = (p_1, \ldots, p_n)$. System $\{B^k\}$ is said *to satisfy condition* W with respect to Z if for every index k the inequalities

$$z_j \leqslant \bar{z}_j \qquad (j \neq k)$$

imply

$$B^k(t, x, Z, P) \leqslant B^k(t, x, \bar{z}_1, \ldots, \bar{z}_{k-1}, z_k, \bar{z}_{k+1}, \ldots, \bar{z}_N, P)$$

for all $(t, x) \in (0, T)\times R_n$ and P.

We are now able to formulate and prove the main result:

THEOREM 1. *Let the functions $A_i^k(t, x, s, P)$, $B^k(t, x, Z, P)$ $(k = 1, \ldots, N,$ $i = 1, \ldots, n)$ be defined and measurable for $(t, x) \in (0, T) \times R_n$ and for arbitrary $s \in (-\infty, +\infty)$, P, Z, and let the functions B^k satisfy condition* W (see Definition 4) *with respect to Z. Suppose further that the inequalities*

$$(4) \qquad \sum_{i=1}^{n} (p_i - \bar{p}_i)[A_i^k(t, x, s, P) - A_i^k(t, x, \bar{s}, \bar{P})] \geqslant a|P - \bar{P}|^2 - b(|x|^2+1)(s-\bar{s})^2$$

$(k = 1, \ldots, N)$,

$$(5) \qquad |A_i^k(t, x, s, P) - A_i^k(t, x, \bar{s}, \bar{P})| \leqslant a_1|P - \bar{P}| + b_1(|x|^2+1)^{\frac{1}{2}}|s - \bar{s}|$$

$(k = 1, \ldots, N,\ i = 1, \ldots, n)$,

$$(6) \qquad [B^k(t, x, Z, P) - B^k(t, x, \bar{Z}, \bar{P})]\operatorname{sgn}(z_k - \bar{z}_k) \leqslant c(|x|^2+1)\sum_{j=1}^{N}|z_j - \bar{z}_j| + d(|x|^2+1)^{\frac{1}{2}}|P - P| \ (^1) \qquad (k = 1, \ldots, N)$$

hold true, where a, b, a_1, b_1, c and d are positive constants.

If $\{u^k(t, x)\}$ and $\{v^k(t, x)\}$ $(k = 1, \ldots, N)$ satisfy the system of inequalities (1) *and* (2), *respectively* (see Definition 3), *and the conditions*

$$(7) \qquad u^k(0, x) \leqslant v^k(0, x) \qquad (k = 1, \ldots, N)$$

almost everywhere in R_n,

$$(u^k - v^k)_+ \in E_2 \qquad (k = 1, \ldots, N),$$

where $(u^k - v^k)_+ = \max(u^k - v^k, 0)$, then

$$u^k(t, x) \leqslant v^k(t, x) \qquad (k = 1, \ldots, N)$$

almost everywhere in $(0, T) \times R_n$.

Proof. Put

$$w_k(t, x) = [u^k(t, x) - v^k(t, x)]_+ \qquad (k = 1, \ldots, N).$$

By the standard properties of weak derivatives, $w_k \in W_2^{1,1}\big((0, T) \times (|x| < R)\big)$ for all $R > 0$ (Lemma 4.2, p. 99 [2]). We introduce the functions

$$\varphi^k(t, x) = \left[\zeta(x)\exp\left(-\frac{\alpha(|x|^2+1)}{1-\mu t}\right)\right]^2 w_k(t, x)\Phi(t),$$

where the functions ζ and Φ satisfy the following conditions:

$$\zeta \in C^2(R_n),\ 0 \leqslant \zeta(x) \leqslant 1 \text{ for } x \in R_n, \qquad \zeta(x) = 1 \text{ for } |x| \leqslant R,$$

(^{1}) $\operatorname{sgn} x$ denotes 1 if $x > 0$ and -1 if $x < 0$.

$\zeta(x) = 0$ for $|x| \geqslant R+1$, $|\zeta_x(x)|$ is bounded by a constant independent of R,

$$\Phi(t) = 1 \text{ for } 0 \leqslant t \leqslant \tau-\varepsilon, \quad \Phi(t) = \frac{\tau-t}{\varepsilon} \text{ for } \tau-\varepsilon \leqslant t \leqslant \tau,$$

$$\Phi(t) = 0 \text{ for } \tau \leqslant t,$$

where $\tau \in (0, 1/2\mu)$ and a constant μ will be chosen suitably. For convenience we will write

$$\varphi^k = \eta^2 w_k \Phi,$$

where

$$\eta = \zeta \exp\left(-\frac{a(|x|^2+1)}{1-\mu t}\right).$$

It follows from (7) that

$$\int\limits_{R_n} w_k(t,x)^2 \eta(t,x)^2 \Phi(t)\Big|_{t=0} dx = 0, \qquad \int\limits_{R_n} w_k(t,x)^2 \eta(t,x)^2 \Phi(t)\Big|_{t=T} dx = 0.$$

Hence we conclude that

$$\int\limits_0^T \int\limits_{R_n} \varphi^k (u^k - v^k)_t \, dt\, dx = \tfrac{1}{2} \int\limits_0^T \int\limits_{R_n} (w_k^2)_t \eta^2 \Phi \, dt\, dx$$

$$= \frac{1}{2\varepsilon} \int\limits_{\tau-\varepsilon}^{\varepsilon} \int\limits_{R_n} w_k^2 \eta^2 \, dt\, dx - \int\limits_0^T \int\limits_{R_n} w_k^2 \eta \eta_t \Phi \, dt\, dx.$$

Then letting $\varepsilon \to 0$ we obtain

$$\int\limits_0^\tau \int\limits_{R_n} w_k \eta^2 (u^k - v^k)_t \, dt\, dx = \tfrac{1}{2} \int\limits_{R_n} w_k(\tau,x)^2 \eta(\tau,x)^2 dx - \int\limits_0^\tau \int\limits_{R_n} w_k^2 \eta \eta_t \, dt\, dx \tag{8}$$

for almost all $\tau \in (0, 1/2\mu)$. Now, substituting into (3) and (3′) the functions φ^k defined above, subtracting (3′) from (3) and using (8), we can easily verify that

$$\tfrac{1}{2} \int\limits_{R_n} w_k(\tau,x)^2 \eta(\tau,x)^2 dx + \int\limits_0^\tau \int\limits_{R_n} \sum_{i=1}^n (\eta^2 w_k)_{x_i} [A_i^k(t,x,u^k,u_x^k) - \tag{9}$$

$$- A_i^k(t,x,v^k,v_x^k)] \, dt\, dx$$

$$\leqslant \int\limits_0^\tau \int\limits_{R_n} \eta^2 w_k [B^k(t,x,U,u_x^k) - B^k(t,x,V,v_x^k)] \, dt\, dx + \int\limits_0^\tau \int\limits_{R_n} w_k^2 \eta \eta_t \, dt\, dx$$

for almost all $\tau \in (0, 1/2\mu)$. Introducing the functions

$$\overline{w}_i = \max[-(u^i - v^i), 0] \quad \text{for } i \neq k$$

by condition W and assumption (6), we have the following inequality:

$$(10) \quad \int_0^\tau \int_{R_n} \eta^2 w_k [B^k(t, x, U, u_x^k) - B^k(t, x, V, v_x^k)] \, dt \, dx$$

$$\leqslant \int_0^\tau \int_{R_n} \eta^2 w_k [B^k(t, x, u^1 + \overline{w}_1, \ldots, u^{k-1} + \overline{w}_{k-1}, u^k, u^{k+1} +$$

$$+ \overline{w}_{k+1}, \ldots, u^N + \overline{w}_N, u_x^k) - B^k(t, x, V, v_x^k)] \, dt \, dx$$

$$\leqslant \int_0^\tau \int_{R_n} \eta^2 w_k \Big[c(|x|^2+1) \sum_{j=1}^n w_j + d(|x|^2+1)^{\frac{1}{2}} \times |u_x^k - v_x^k| \Big] \, dt \, dx.$$

The last inequality together with (4), (5) and (9) gives

$$(11) \quad \tfrac{1}{2} \int_{R_n} w_k(\tau, x)^2 \eta(\tau, x)^2 \, dx + \int_0^T \int_{R_n} \eta^2 [a|(w_k)_x|^2 - b(|x|^2+1) w_k^2] \, dt \, dx -$$

$$- \int_0^\tau \int_{R_n} 2n^{\frac{1}{2}} \eta |\eta_x| w_k [a_1 |(w_k)_x| + b_1 (|x|^2+1)^{\frac{1}{2}} w_k] \, dt \, dx$$

$$\leqslant \int_0^\tau \int_{R_n} \eta^2 w_k \Big[c(|x|^2+1) \sum_{j=1}^n w_j + d(|x|^2+1)^{\frac{1}{2}} |(w_k)_x| \Big] \, dt \, dx + \int_0^\tau \int_{R_n} w_k^2 \eta \eta_t \, dt \, dx.$$

Note that

$$2a_1 n^{\frac{1}{2}} \eta |\eta_x| w_k |(w_k)_x| \leqslant \frac{a}{4} \eta^2 |(w_k)_x|^2 + \frac{4na_1^2}{a} |\eta_x|^2 w_k^2,$$

$$d(|x|^2+1)^{\frac{1}{2}} w_k |(w_k)_x| \leqslant \frac{a}{4} |(w_k)_x|^2 + \frac{d^2}{a} (|x|^2+1) w_k^2.$$

By the use of these inequalities it follows from (11) that

$$(12) \quad \tfrac{1}{2} \int_{R_n} w_k(\tau, x)^2 \eta(\tau, x)^2 \, dx + \frac{a}{2} \int_0^\tau \int_{R_n} \eta^2 |(w_k)_x|^2 \, dt \, dx$$

$$\leqslant \int_0^\tau \int_{R_n} \Big[\Big(b + \frac{d^2}{a} \Big) (|x|^2+1) \eta^2 w_k^2 + c(|x|^2+1) \eta^2 w_k \sum_{j=1}^N w_j +$$

$$+ \frac{4na_1^2}{a} |\eta_x|^2 w_k^2 + 2b_1 (|x|^2+1)^{\frac{1}{2}} \eta |\eta_x| w_k^2 \Big] \, dt \, dx + \int_0^\tau \int_{R_n} w_k^2 \eta \eta_t \, dt \, dx,$$

and summing inequalities (12) over k from 0 to N we have

$$(13)\qquad \tfrac{1}{2}\sum_{k=1}^{N}\int_{R_n} w_k(\tau,x)^2\eta(\tau,x)^2\,dx+\frac{a}{2}\sum_{k=0}^{N}\int_0^{\tau}\int_{R_n}\eta^2|(w_k)_x|^2\,dt\,dx$$
$$\leqslant\int_0^{\tau}\int_{R_n}\Big[\Big(b+\frac{d^2}{a}+Nc\Big)(|x|^2+1)\eta^2+\frac{4na_1^2}{a}|\eta_x|+$$
$$+2b_1(|x|^2+1)^{\frac{1}{2}}\eta\,|\eta_x|+\eta\eta_t\Big]\sum_{k=1}^{N}w_k^2\,dt\,dx.$$

Notice that $w_k\in E_2$ (see Definition 2), and hence the limit passage $R\to\infty$ in inequality (13) implies

$$(14)\qquad \tfrac{1}{2}\sum_{k=1}^{N}\int_{R_n} w_k(\tau,x)^2H(\tau,x)^2\,dx\leqslant\int_0^{\tau}\int_{R_n}\Big[\Big(b+\frac{d^2}{a}+cN\Big)(|x|^2+1)H^2+$$
$$+\frac{4na_1^2}{a}|H_x|^2+2b_1(|x|^2+1)H|H_x|+HH_t\Big]\sum_{k=1}^{N}w_k^2\,dt\,dx,$$

where

$$H(t,x)=\exp\Big(-a\frac{|x|^2+1}{1-\mu t}\Big).$$

Now choose μ sufficiently large so that

$$(15)\quad \Big(b+\frac{d^2}{a}+cN\Big)(|x|^2+1)H^2+\frac{4na_1^2}{a}|H_x|^2+2b_1(|x|^2+1)^{\frac{1}{2}}H|H_x|+HH_t\leqslant 0$$

for $(t,x)\in(0,1/2\mu)\times R_n$. From (14) and (15) it follows that

$$\sum_{k=1}^{N}\int_{R_n} w_k(\tau,x)^2H(\tau,x)^2\,dx=0$$

for almost all $\tau\in(0,1/2\mu)$; hence $w_k(t,x)=0$ ($k=1,\dots,N$) almost everywhere in $(0,1/2\mu)\times R_n$. If $1/2\mu=T$, this completes the proof; otherwise the proof can be completed by a finite number of applications of the same argument on $(1/2\mu,1/\mu)\times R_n$, $(1/\mu,\frac{3}{2}\mu)\times R_n$, etc.

As an immediate consequence of Theorem 1 we obtain the following corollaries:

COROLLARY 1 (Maximum principle). *Let the function A_k^i, B^k ($k=1,\dots,N$; $i=1,\dots,n$) satisfy all the hypotheses of Theorem 1. Assume that*

$$A_i^k(t,x,m_k,0)=0,\qquad B^k(t,x,M,0)\leqslant 0\qquad (k=1,\dots,N;\ i=1,\dots,n)$$

for almost all $(t, x) \in (0, T) \times R_n$, *where* $M = (m_1, \ldots, m_N)$ *and* m_i *are constants.*

If $\{u^k(t, x)\}$ $(k = 1, \ldots, N)$ *satisfies the system of inequalities* (1) *and the conditions*

$$u^k(0, x) \leqslant m_k \quad (k = 1, \ldots, n)$$

almost everywhere in R_n,

$$(u^k - m_k)_+ \in E_2 \quad (k = 1, \ldots, N),$$

then

$$u^k(t, x) \leqslant m_k \quad (k = 1, \ldots, N)$$

almost everywhere in $(0, T) \times R_n$.

Before formulatnig the next corollary, we introduce the following definition:

DEFINITION 5. A system of functions $\{u^k(t, x)\}$ $(k = 1, \ldots, N)$ is said to be a *weak solution of the Cauchy problem*

$$u_t^k = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i^k(t, x, u^k, u_x^k) + B^k(t, x, U, u_x^k) \quad (k = 1, \ldots, N) \tag{16}$$

with the initial condition

$$u^k(0, x) = \Psi^k(x) \quad (k = 1, \ldots, N)$$

almost everywhere in $(0, T) \times R_n$, where Ψ^k are given functions in $L^2_{\mathrm{loc}}(R_n)$, if $u^k \in W_2^{1,1}\big((0, T) \times (|x| < R)\big)$ $(k = 1, \ldots, N)$ for all $R > 0$ and $\{u^k\}$ satisfies

$$\int_0^T \int_{R_n} \Big[\varphi^k u_t^k + \sum_{i=1}^{n} \varphi_{x_i}^k A_i^k(t, x, u^k, u_x^k)\Big]\, dt\, dx = \int_0^T \int_{R_n} \varphi^k B^k(t, x, U, u_x^k)\, dt\, dx$$
$$(k = 1, \ldots, N)$$

for any system of functions $\{\varphi^k\}$ $(k = 1, \ldots, N)$ of the class $W_2^{1,1}\big((0, T) \times R_n\big)$ with compact supports in x and vanishing for $t = T$.

COROLLARY 2 (Uniqueness criterion). *Suppose that the functions* A_i^k, B^k $(k = 1, \ldots, N,\ i = 1, \ldots, n)$ *satisfy all the assumptions of Theorem* 1.

Then the Cauchy problem for the system of partial differential equations (16) *admits in* $(0, T) \times R_n$ *at most one solution of class* E_2.

EXAMPLE. To illustrate Theorem 1, let us consider a system of linear equations

$$u_t^k = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} [a_{ij}^k(t, x) u_{x_j}^k + a_i^k(t, x) u^k] + \sum_{i=1}^{n} b_i^k(t, x) u_{x_i}^k + \sum_{l=1}^{N} c_l^k(t, x) u^l$$

$(k = 1, \ldots, N)$ with the coefficients defined and measurable in $(0, T)\times R_n$. Suppose that the coefficients satisfy the conditions

$$\beta_1|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}^k(t,x)\xi_i\xi_j \leqslant \beta_2|\xi|^2, \qquad |a_i^k(t,x)| \leqslant A(|x|^2+1)^{\frac{1}{2}},$$

$$|b_i^k(t,x)| \leqslant B(|x|^2+1)^{\frac{1}{2}}, \qquad |c_i^k(t,x)| \leqslant C(|x|^2+1),$$

$$c_l^k(t,x) \geqslant 0 \qquad (k \neq l)$$

for all $\xi \in R_n$ and $(t,x)\in(0,T)\times R_n$, where A, B, C and β_i are positive constants. It is easy to verify that the functions

$$A_i^k(t,x,s,P) = \sum_{j=1}^{n} a_{ij}^k(t,x)p_j + a_i^k(t,x)s,$$

$$B^k(t,x,Z,P) = \sum_{i=1}^{n} b_i^k(t,x)p_i + \sum_{l=1}^{n} c_l^k(t,x)z_l$$

satisfy all the assumptions of Theorem 1.

2. In the uniqueness criterion (see Corollary 2) we assumed the system of functions B^k to satisfy condition W and inequality (6). These assumptions can be replaced by a Lipschitz condition. More precisely, we have the following theorem:

THEOREM 2. *Let the functions A_i^k $(k = 1, \ldots, N,\ i = 1, \ldots, n)$ satisfy all the assumptions of Theorem 1. Suppose that the functions B^k $(k = 1, \ldots, N)$ satisfy the inequalities*

$$|B^k(t,x,Z,P) - B^k(t,x,\bar{Z},\bar{P})|$$
$$\leqslant c(|x|^2+1)\sum_{j=1}^{n}|z_j-\bar{z}_j| + d(|x|^2+1)^{\frac{1}{2}}\times|P-\bar{P}| \qquad (k = 1, \ldots, N).$$

Then the Cauchy problem for the system of partial differential equations (16) *admits in $(0,T)\times R_n$ at most one solution of class E_2.*

Proof. Suppose that $U(t,x)$ and $V(t,x)$ are two such solutions. It is obvious that it suffices to prove the inequality

$$\sum_{k=1}^{N}\int_{R_n} w_k(\tau,x)^2 H(\tau,x)^2\,dx$$
$$\leqslant \int_0^{\tau}\int_{R_n}[C_1(|x|^2+1)H^2 + C_2|H_x|^2 + C_3(|x|^2+1)^{\frac{1}{2}}H|H_x| + HH_t]\sum_{k=1}^{n} w_k^2\,dt\,dx$$

for almost all $\tau\in(0, 1/2\mu)$, where $w_k = u^k - v^k$, H is the function defined in the proof of Theorem 1, C_1, C_2 and C_2 are positive constants. To derive this estimate we take

$$\varphi^k(t, x) = [\zeta(x) H(t, x)]^2 w_k(t, x) \Phi(t)$$

as the test function, where ζ and Φ are the functions defined in Theorem 1. Proceeding as in the proof of Theorem 1, we check that

$$\tfrac{1}{2}\int_{R_n} w_k(\tau, x)^2 \eta(\tau, x)^2 dx + \int_0^\tau \int_{R_n} \sum_{i=1}^n (\eta^2 w_k)_{x_i} [A_i^k(t, x, u^k, u_x^k) - A_i^k(t, x, v^k, v_x^k)] dt\, dx$$
$$= \int_0^\tau \int_{R_n} \eta^2 w_k [B^k(t, x, U, u^k) - B^k(t, x, V, v_x^k)] dt\, dx + \int_0^\tau \int_{R_n} w_k^2 \eta \eta_t dt\, dx \qquad (k = 1, \ldots, N)$$

for almost all $\tau \in (0, 1/2\mu)$, where $\eta = \zeta H$. Now, as in Theorem 1, we derive the required estimate.

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