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On the convergence of iterates

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§ 1. Let x be a real or complex variable and let $f(x)$ be a function defined in a neighbourhood of a point a such that $f(a) = a$. We assume that the derivative $f'(a)$ exists and we write $f'(a) = s$. For a given x_0 we define the sequence of iterates x_n by

$$x_{n+1} = f(x_n) \,. \tag{1}$$

Let

$$R = \{x:\ |x-a| < r\}\,,$$
$$R^* = \{x:\ 0 < |x-a| < r\}\,.$$

A part of results contained in Theorems 2.12 and 2.13 in [1] may be formulated as follows:

Let $0 \leqslant |s| \leqslant 1$ *and let*

$$|f(x)-a| < |x-a| \quad \textit{for} \quad x \in R^* \ (^1)\,. \tag{2}$$

Then the limit

$$\lim_{n\to\infty} \frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|} \tag{3}$$

exists (where the ratio under the lim *sign is defined as zero whenever* $x_n = x_{n+1}$*) and, moreover,*

$$\lim_{n\to\infty} \frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|} = |s|\,. \tag{4}$$

The proof is based on the following scheme. Put

$$s_n = \frac{x_{n+1}-a}{x_n-a} \tag{5}$$

($s_n = 0$ if $x_{n+1} = a$); then $\lim\limits_{n\to\infty} s_n = f'(a) = s$. We have

$$\frac{x_{n+1}-x_n}{x_n-x_{n-1}} = \frac{(x_{n+1}-a)-(x_n-a)}{(x_n-a)-(x_{n-1}-a)} \tag{6}$$
$$= \frac{(s_n-1)(x_n-a)}{(s_{n-1}-1)(x_{n-1}-a)} = \frac{(s_n-1)s_{n-1}}{s_{n-1}-1}\,,$$

(1) (2) is authomatically fulfilled if $|s| < 1$ and r is small enough.

whence

$$\lim_{n\to\infty}\left|\frac{x_{n+1}-x_n}{x_n-x_{n-1}}\right| = \lim_{n\to\infty}\left|\frac{(s_n-1)s_{n-1}}{s_{n-1}-1}\right| = |s| .$$

Now, this inference evidently fails if $s = 1$. What is worse, the result itself is not true either. It is the purpose of the present note to show that in the case $s = 1$:

(i) *Limit* (3) *need not exist.*

(ii) *If limit* (3) *exists, then relation* (4) *holds.*

§ 2. Let x be a real variable. We define the sequence c_n by

$$c_n = \begin{cases} 2^{-n} & \text{for odd } n , \\ n^{-1} & \text{for even } n , \end{cases} \qquad n = 1, 2, 3, \ldots ,$$

and we put

$$x_0 = 1 ,$$

(7)
$$x_n = \prod_{i=1}^{n}(1-c_i) \qquad \text{for} \qquad n = 1, 2, 3, \ldots$$

The sequence x_n is strictly decreasing, moreover, since the series $\sum_{n=1}^{\infty} c_n$ diverges, $\lim_{n\to\infty} x_n = 0$.

We define the function $f(x)$ on $\langle -1, 1\rangle$ by the following conditions: (1) holds for the points of sequence (7), $f(x)$ is linear on the intervals $\langle x_{n+1}, x_n\rangle$, $n = 0, 1, 2, \ldots$, and $f(x)$ is odd. Because of this last condition it is enough to consider the function $f(x)$ only for $x \in \langle 0, 1\rangle$.

It follows from the above conditions that

(8)
$$x_{n+1} < f(x) \leqslant x_n \qquad \text{for} \qquad x \in (x_n, x_{n-1}\rangle ,$$

whence $f(x) \leqslant x_n < x$ for $x \in (x_n, x_{n-1}\rangle$, $n = 1, 2, 3, \ldots$, i.e.

(9)
$$f(x) < x \qquad \text{in} \qquad (0, 1\rangle .$$

Note that in our case $a = 0$ and consequently (9) shows that (2) holds. Further it follows from (8) that

$$\frac{x_{n+1}}{x_{n-1}} \leqslant \frac{f(x)}{x} \leqslant \frac{x_n}{x_n} = 1 \qquad \text{for} \qquad x \in (x_n, x_{n-1}\rangle ,$$

whence

$$f'(0) = \lim_{x\to 0}\frac{f(x)}{x} = 1 ,$$

since

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_{n-1}}=\lim_{n\to\infty}(1-c_{n+1})(1-c_n)=1\,.$$

Thus $f(x)$ fulfils the conditions of the preceding section. Further

$$s_n=\frac{x_{n+1}}{x_n}=1-c_{n+1}$$

and

$$\left|\frac{(s_n-1)s_{n-1}}{s_{n-1}-1}\right|=\frac{c_{n+1}(1-c_n)}{c_n}$$

has no limit as $n\to\infty$. This proves our assertion (i).

§ 3. The example of § 2 is instructive and suggests a proof of assertion (ii). So let us suppose that the function $f(x)$ is defined in a set R, fulfils (2), $s=f'(a)=1$ and limit (3) exists. We may write (with notation (5))

(10) $$c_n=1-s_{n-1},\qquad n=1,2,3,\ldots$$

We have by (5) and (10)

(11) $$|x_n-a|=|x_0-a|\prod_{i=1}^{n}|1-c_i|.$$

By (2) the sequence $|x_n-a|$ tends to zero. (If $|x_n-a|$ had a positive limit, then a subsequence of x_n would converge to an $x^*\in R^*$, and we would have $|f(x^*)-a|=|x^*-a|$, contrary to (2).) Therefore also the product in (11) must tend to zero.

Now, since

(12) $$\lim_{n\to\infty}s_n=s=1\,,$$

we have in view of (10)

(13) $$\lim_{n\to\infty}c_n=0\,.$$

By (12) and (6) the existence of limit (3) is equivalent to the existence of the limit

$$g=\lim_{n\to\infty}\left|\frac{s_n-1}{s_{n-1}-1}\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|,$$

and the thing to show is that $g=1$. In fact, $g>1$ is impossible in view of (13). On the other hand, if we had $g<1$, the series $\sum_{n=1}^{\infty}|c_n|$ would converge and consequently also the product $\prod_{n=1}^{\infty}(1-c_n)$ would converge. Then also the product $\prod_{n=1}^{\infty} |1-c_n| = |\prod_{n=1}^{\infty} (1-c_n)|$ would converge, which contradicts the condition $\lim_{n\to\infty} |x_n - a| = 0$. This completes the proof of assertion (ii).

Reference

[1] H. J. Hamilton, *Roots of equations by functional iteration*, Duke Math. J. 13 (1946), pp. 113-121.

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