ANNALES
POLONICI MATHEMATICI
XXXIII (1976)
CONFERENCE ON ANALYTIC FUNCTIONS

Analytic families of quasi-conformal mappings

by Olli Lehto (Helsinki)

Abstract. Suitably normalized quasi-conformal mappings $z \mapsto f(z, w)$ of the plane are known to depend analytically on the complex parameter w if their complex dilatation $z \mapsto \mu(z, w)$ is a holomorphic function of w for almost all z. This leads to a general majorization principle which has applications to various problems ([2]).

The situation is different if $z \mapsto f(z, w)$ are quasi-conformal mappings of a Jordan domain A onto a Jordan domain A'. It is not difficult to prove that, with $w \mapsto \mu(z, w)$ holomorphic, the function $w \mapsto f(z, w)$ is analytic for every $z \in A$ if and only if $w \mapsto f(z, w)$ is constant for every z lying on the boundary of A.

The majorization principle can, however, be saved in many cases in which the mappings f are not necessarily quasi-conformal in the whole plane. The idea is to approximate f by mappings f_n which are quasi-conformal in the plane, apply the majorization principle to f_n, and then proceed to the limit in order to obtain a result for f.

1. Mappings of Jordan domains. Let A and A' be Jordan domains in the extended plane. We denote the boundary of A by ∂A and the closure by $\bar{A}$, and use similar notation for other sets.

Lemma 1. *Let $z \mapsto f(z, w)$ be a continuous mapping of $\bar{A}$ into $\bar{A}'$ for every w lying in the unit disc D. If $w \mapsto f(z, w)$ is analytic in D for every $z \in E \subset \bar{A}$, then $w \mapsto f(z, w)$ is analytic also for $z \in \bar{E}$.*

Proof. Assuming that $\bar{E} - E \neq \emptyset$, let $\zeta \in \bar{E} - E$ and choose points $z_n \in E$, $n = 1, 2, \ldots$, so that $\lim z_n = \zeta$. Then $\{f(z_n, \ )\}$ is a normal family. It follows that $f(\zeta, \ )$ is the uniform limit of analytic functions, and hence analytic.

Lemma 2. *Let $z \mapsto f(z, w)$ be a continuous mapping of $\bar{A}$ into $\bar{A}'$ for every $w \in D$ and let $\zeta \in \partial A$. If $w \mapsto f(\zeta, w)$ is analytic and there is a $w_0 \in D$ so that $f(\zeta, w_0) \in \partial A'$, then $w \mapsto f(\zeta, w)$ is constant.*

Proof. If $w \mapsto f(\zeta, w)$ is not constant, then the image of D under $f(\zeta, \ )$ is an open set of the plane. On the other hand, the image lies in $\bar{A}'$ and contains the boundary point $f(\zeta, w_0)$. This is a contradiction.

Assume next that to every $w \in D$ there corresponds a complex-valued function $z \mapsto \mu(z, w)$, measurable in A and with $\|\mu(\ , w)\|_\infty < 1$. Let $z \mapsto$

$\mapsto f(z, w)$ now be a quasi-conformal mapping of A onto A' which has the complex dilatation $z \mapsto \mu(z, w)$ for almost all $z \in A$. Such a mapping exists always. Furthermore, it admits a homeomorphic extension to the closure of A; we use the same notation for the extended mapping.

THEOREM 1. *Let $w \mapsto \mu(z, w)$ be differentiable in D for almost every $z \in A$. If $w \mapsto f(z, w)$ is analytic in D for every $z \in A$, then $w \mapsto \mu(z, w)$ is holomorphic in D for almost every $z \in A$.*

Proof. By the Lemmas 1 and 2, $w \mapsto f(z, w)$ is constant for every $z \in \partial A$. Hence, with one mapping $f(\ , w)$ given, all the others are uniquely determined by their complex dilatation.

There is no loss of generality to assume that $A = A' = D$. By a result of Ahlfors and Bers [1], the functions $w \mapsto f_z(z, w)$ and $w \mapsto f_{\bar{z}}(z, w)$ are differentiable in D for almost every $z \in A$, and $f_{z\bar{w}} = f_{\bar{w}z}$, $f_{\bar{z}\bar{w}} = f_{\bar{w}\bar{z}}$. Because $w \mapsto f(z, w)$ is analytic, it follows that $\mu_{\bar{w}}(z, w) = \big(f_{\bar{z}}(z, w)/f_z(z, w)\big)_{\bar{w}} = 0$ in D for almost all $z \in A$. Since $w \mapsto \mu(z, w)$ is differentiable in D, this implies that $w \mapsto \mu(z, w)$ is holomorphic in D for almost all $z \in A$.

The converse is not true in general:

THEOREM 2. *Let $w \mapsto \mu(z, w)$ be holomorphic in D for almost every $z \in A$. Then the function $w \mapsto f(z, w)$ is analytic in D for every $z \in A$ if and only if $w \mapsto f(z, w)$ is constant for every $z \in \partial A$.*

Proof. The necessity of the condition follows directly from the Lemmas 1 and 2. In order to prove the sufficiency, suppose that $w \mapsto f(z, w)$ is constant for every $z \in \partial A$. Again, there is no loss of generality in assuming that $A = A' = D$. Let g denote a quasi-conformal extension of $z \mapsto f(z, 0)$ to the complement of D. Set $h(z, w) = f(z, w)$ if $z \in D$, $h(z, w) = g(z)$ if z lies outside D. Then $z \mapsto h(z, w)$ is a quasi-conformal mapping of the plane whose complex dilatation depends holomorphically on w for almost every z. It follows that $w \mapsto h(z, w)$ is analytic for every z ([1], [2]). Consequently, $z \mapsto f(z, w)$ is analytic for every $z \in D$.

There are two natural ways to normalize the mappings $f(\ , w)$ so that complex dilatation determines them uniquely. We can require that

$$f(a, w) = a', \quad f(b, w) = b', \quad a \in A,\ a' \in A',\ b \in \partial A,\ b' \in \partial A', \tag{1}$$

or that

$$f(c_i, w) = c_i', \quad c_i \in \partial A,\ c_i' \in \partial A',\ i = 1, 2, 3, \tag{2}$$

where the a's, b's and c's are independent of w. In view of Theorem 2, it is not surprising that for the function $w \mapsto f(z, w)$ to be analytic, condition (1) is more restrictive than (2): Let $w \mapsto \mu(z, w)$ be holomorphic in D for almost every $z \in A$, and $z \mapsto f_i(z, w)$, $i = 1, 2$, quasi-conformal mappings of A onto A' with complex dilatation $z \mapsto \mu(z, w)$ a.e., where f_1 satisfies condition (1) and f_2 condition (2). If $w \mapsto f_1(z, w)$ is analytic in D, then so is $w \mapsto f_2(z, w)$. Conversely, if $w \mapsto f_2(z, w)$ is analytic in D for every $z \in A$, then $w \mapsto f_1(z, w)$ need not be analytic in D for any $z \in A$, $z \neq a$.

In order to establish these statements, suppose first that $w \mapsto f_1(z, w)$ is analytic in D for every $z \in A$. By Theorem 2, there is a conformal self-mapping φ of A' which is independent of w so that $f_2(\ , w) = \varphi \circ f_1(\ , w)$. It follows that $w \mapsto f_2(z, w)$ is analytic in D for every $z \in A$.

The following example shows that the analyticity of $w \mapsto f_2(z, w)$ does not imply that $w \mapsto f_1(z, w)$ is analytic. Let A and A' be upper half-planes, $c_1 = c_1' = 0$, $c_2 = c_2' = 1$, $c_3 = c_3' = \infty$. If $\mu(z, w) = w$, then

$$f_2(z, w) = \frac{z + w\bar{z}}{1 + w}.$$

Hence, $w \mapsto f_2(z, w)$ is analytic for every z. On the other hand, if $a = a' = i$, $b = b' = \infty$, then

$$f_1(z, w) = \frac{(1+\overline{w})(z + w\bar{z}) + i(w - \overline{w})}{1 - |w|^2}.$$

From

$$\frac{\partial f_1(z, w)}{\partial \overline{w}} = \frac{(1+w)\big(z - i + w(\bar{z} + i)\big)}{(1 - |w|^2)^2}$$

it follows that $w \mapsto f_1(z, w)$ is analytic in D only if $z = i$.

The following simple example shows that, with $w \mapsto \mu(z, w)$ holomorphic, it is very exceptional for $w \mapsto f(z, w)$ to be analytic. Let A and A' be upper half-planes and φ a quasi-symmetric function with the properties $\varphi(0) = 0$, $\varphi(1) = 1$. Let h be a quasi-conformal self-mapping of A with boundary values φ. If ν denotes the complex dilatation of h, we set $\mu(z, w) = w\nu(z)/\|\nu\|_\infty$. Let $z \mapsto f(z, w)$ be the quasi-conformal self-mapping of A which fixes $0, 1, \infty$ and has the complex dilatation $z \mapsto \mu(z, w)$. For x real, we then have $f(x, \|\nu\|_\infty) = \varphi(x)$, while $f(x, 0) = x$. We conclude from Theorem 2 that $w \mapsto f(z, w)$ is analytic for every $z \in A$ only if $\varphi(x) = x$. Even this very restrictive condition is not sufficient (cf. Section 2).

2. Remarks on mappings of the disc. Let $z \mapsto f(z, w)$ be a quasi-conformal self-mapping of the unit disc D which keeps invariant three fixed boundary points c_i, $i = 1, 2, 3$. We assume that its complex dilatation $z \mapsto \mu(z, w)$ is a holomorphic function of w in D for almost all $z \in D$ and that there is a $w_0 \in D$ such that $\mu(z, w_0) = 0$ for every z. Then $f(z, w_0) = z$.

From the well-known representation formula for $f(z, w)$ in terms of $\mu(z, w)$ we obtain easily a quasi-explicit condition for $w \mapsto f(z, w)$ to be analytic. We extend μ by setting $\mu(z, w) = 0$ whenever $|z| > 1$, and define $\varphi_1\big(\mu(\ , w)\big) = \mu(\ , w)$, $\varphi_n\big(\mu(\ , w)\big) = \mu(\ , w)\, S\varphi_{n-1}\big(\mu(\ , w)\big)$, $n = 2, 3, \ldots$, where S is the Hilbert-transformation.

Let H denote the Hilbert space consisting of all complex-valued functions L^2-integrable in D, with the inner product

$$(f, g) = \iint_D f(z)\bar{g}(z)\,dx\,dy.$$

In H the elements which are holomorphic in D form a closed subspace. Let O be its orthogonal complement with respect to H. The following result is then true:

The function $w \mapsto f(z, w)$ is holomorphic in D for every $z \in D$ if and only if

$$\sum_{i=1}^{\infty} \bar{\varphi}_i(\mu(\ , w)) \in O \tag{1}$$

for every $w \in D$.

Proof. Let h be the quasi-conformal mapping of the plane which has complex dilatation ν, is conformal in $|z| > 1$, and has the expansion $h(z) = z + \sum_1^{\infty} b_n z^{-n}$. Then ([2])

$$b_n = \frac{1}{\pi} \sum_{i=1}^{\infty} \iint_{|z|<1} \varphi_i(\nu)(z) z^{n-1}\,dx\,dy, \qquad n = 1, 2, \ldots \tag{2}$$

Suppose first that $w \mapsto f(z, w)$ is holomorphic. By Theorem 2, the function $z \mapsto h(z, w)$, defined by $h(z, w) = f(z, w)$ if $z \in D$, $h(z, w) = z$ if z lies outside D, is a quasi-conformal mapping of the plane. Hence, by (2), $\sum \bar{\varphi}_i(\mu(\ , w))$ is orthogonal to every power z^{n-1}. Since $1, z, z^2, \ldots$ is a complete orthogonal system in the subspace of the holomorphic elements of H, (1) follows.

Assume conversely that (1) holds. Let $z \mapsto h(z, w)$ now be the quasi-conformal mapping of the plane which has the complex dilatation $z \mapsto \mu(z, w)$ and the property $h(z, w) - z = o(1)$ as $z \to \infty$. From (1) and (2) we deduce that $h(z, w) = z$ for $|z| \geqslant 1$. Then $h(\ , w)|D \circ f(\ , w)^{-1}$ is a conformal self-mapping of D which keeps three boundary points fixed. Hence $f(z, w) = z$ if $z \in \partial D$, and it follows from Theorem 2 that $w \mapsto f(z, w)$ is holomorphic.

Let us assume, in particular, that $\mu(z, w) = a(w)\varkappa(z)$, where a is a holomorphic function in D, $a(0) = 0$, $|a(w)| < 1$, and $\varkappa$ is measurable in the plane, $\|\varkappa\|_\infty \leqslant 1$, $\varkappa(z) = 0$ if $|z| > 1$. Then $\varphi_n(a(w)\varkappa) = a(w)^n \varphi_n(\varkappa)$, $n = 1, 2, \ldots$ From the above proof we thus obtain the following result: The function $w \mapsto f(z, w)$ is holomorphic in D for every $z \in D$ if and only if

$$\bar{\varphi}_i(\varkappa) \in O, \qquad i = 1, 2, \ldots$$

Condition

$$\sum_{i=1}^{\infty} \bar{\varphi}_i(\varkappa) \in O \tag{3}$$

is not sufficient for $w \mapsto f(z, w)$ to be holomorphic. For let h be an arbitrary quasi-conformal self-mapping of D which keeps every boundary point fixed and $\varkappa$ its complex dilatation. From the above proof it follows that (3) holds. On the other hand, it is well known that quasi-conformal self-mappings of D which keep three boundary points fixed and have complex dilatation $t\varkappa$, $0 < t < 1$, do not necessarily fix every boundary point.

Nor is condition $\bar{\varkappa} \in O$ sufficient for $w \mapsto f(z, w)$ to be analytic. A simple counterexample is obtained if we take $\varkappa(z) = z$. Then $\bar{\varkappa} \in O$, but $\varphi_2(\varkappa)(z) = |z|^2$ so that $\bar{\varphi}_2 \notin O$.

Reich and Strebel have studied systematically Teichmüller mappings of D which keep every boundary point invariant (see e.g. [3]).

We conclude this section by giving two examples of mappings $f(\ , w)$: $D \to D$, fixing 0 and 1, which depend analytically on w. If

$$\mu(z, w) = \frac{w}{2+w} \cdot \frac{z}{\bar{z}},$$

then

$$f(z, w) = z|z|^{w}.$$

We see that $w \mapsto f(z, w)$ is analytic and, in accordance with Theorem 2 $f(z, w) = z$ if $z \in \partial D$.

In the second example we take $\mu(z, w) = w(z/\bar{z})^2$. Then

$$f(z, w) = 2|z|^2\big[\bar{z} + wz + \big((\bar{z}+wz)^2 - 4w|z|^4\big)^{1/2}\big]^{-1}.$$

Again, $w \mapsto f(z, w)$ is analytic.

3. Majorant principle. In this section we consider mappings f quasi-conformal in the domain $A = \{z \mid |z| > r\}$, $r < 1$, conformal in $B = \{z \mid |z| > 1\}$, and satisfying the condition

$$f(z) - z = o(1) \quad \text{as } z \to \infty. \tag{1}$$

Let F_k denote the class of all such mappings, with the additional property that their complex dilatations satisfy the inequality $\|\mu\|_\infty \leqslant k$, $k < 1$. If $r > 0$, the mappings belonging to F_k are not necessarily quasi-conformal in the whole plane, and it is not easy to directly study how $f(z)$ depends on μ. However, we shall now briefly indicate, without giving detailed proofs, how the use of suitable approximation leads to a majorant principle similar to the one holding for mappings quasi-conformal in the whole plane.

Let $\tilde{F}_k$ be the subclass of F_k whose functions are restrictions to A of quasi-conformal mappings of the plane. Then, for every $f \in F_k$, there are mappings $f_n \in \tilde{F}_k$ such that

$$f(z) = \lim_{n \to \infty} f_n(z), \tag{2}$$

uniformly on every compact subset of A.

Let F_1 be the class of all conformal mappings of B with the normalization (1), and φ a holomorphic functional defined in F_1. This means that $\varphi(f) = \omega(\zeta_1, \ldots, \zeta_n)$, where ω is a complex-valued holomorphic function of the variables ζ_i, each ζ_i being the value of f or of its m-th derivative, $m = 1, 2, \ldots$, at a fixed point z_i of B or a coefficient of the power series expansion of f at infinity. If $M(k) = \sup |\varphi(f)|$ in F_k, $0 \leqslant k \leqslant 1$, then it follows from (2) that

$$M(k) = \sup_{f \in \tilde{F}_k} |\varphi(f)|. \tag{3}$$

Using (3) and modifying suitably the proof of Theorem I. 3.1 in [2], we obtain the desired majorant for $M(k)$:

THEOREM 3. *For a holomorphic functional in* F_1,

$$M(k) \leqslant \frac{kM(1) + M(0)}{1 + kM(0)/M(1)}.$$

Suppose that a mapping $f \in F_k$ has the expansion

$$f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$$

in B. As an application of Theorem 3 we choose $\varphi(f) = b_n$. Then $M(0)/M(1) = r^{n+1}$. Since $\max |b_1| = 1$, $\max |b_2| = 2/3$ in F_1, it follows that

$$|b_1| \leqslant \frac{k + r^2}{1 + kr^2}, \qquad |b_2| \leqslant \frac{2}{3} \frac{k + r^3}{1 + kr^3}.$$

Both estimates are sharp.

References

[1] L. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. Math. 72 (1960).

[2] O. Lehto, *Quasiconformal mappings and singular integrals*, Symposia Mathematica, Istituto Nazionale di Alta Matematica Roma (to appear).

[3] E. Reich and K. Strebel, *Einige Klassen Teichmüllerscher Abbildungen, die die Randpunkte festhalten*, Ann. Acad. Sci. Fenn. Ser. A I Math. 457 (1970).

UNIVERSITY OF HELSINKI