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Convergence and stability of difference scheme for an elliptic system of non-linear differential-functional equations with boundary conditions of Dirichlet type

by Bogusław Bożek (Cracow) *

Abstract. We consider a system of second order differential-functional equations of elliptic type with boundary conditions of Dirichlet type. We propose an implicite scheme for this problem and under certain assumptions we show the convergence and stability of this scheme.

1. Introduction. Let $D = [0, X]^n \subset \boldsymbol{R}^n$, $X < +\infty$. We consider the following system of second order differential-functional equations of elliptic type

(1.1) $\quad f_l(x, u(x), (u_l)_x(x), (u_l)_{xx}(x), u) = 0 \quad$ for $x \in \operatorname{int} D$ $(l = 1, \ldots, p)$

with boundary conditions of Dirichlet type

(1.2) $$u_l(x) = \varphi_l(x) \quad \text{for} \quad x \in \partial D \ (l = 1, \ldots, p),$$

where

$$x = (x_i)_{i=1,\ldots,n}, \quad u = (u_\mu)_{\mu=1,\ldots,p}, \quad (u_l)_x = (\partial u_l/\partial x_i)_{i=1,\ldots,n},$$
$$(u_l)_{xx} = (\partial^2 u_l/\partial x_i\, \partial x_j)_{i,j=1,\ldots,n} \quad (l = 1, \ldots, p).$$

We will define an implicite difference scheme for problem (1.1), (1.2).

Under certain assumptions concerning the functions f_l, u_l $(l = 1, \ldots, p)$ and the mesh size h we show the convergence and stability of this scheme.

The difference approximation of the mixed derivatives of the solution and the approximation of the functional argument goes by the method adopted from [2].

Notations concerning difference operators used here and in [1], [2] come back to A. Pliś and seem to be most convenient for our purposes.

2. Assumptions H. We assume that

(1) The scalar functions f_l: $E \ni (x, y, q, w, z) \to f_l(x, y, q, w, z) \in \boldsymbol{R}$ $(l = 1, \ldots, p)$, where $E := D \times \boldsymbol{R}^p \times \boldsymbol{R}^n \times \boldsymbol{R}^{n^2} \times B(D)$, $x = (x_i)_{i=1,\ldots,n}$, $y = (y_\mu)_{\mu=1,\ldots,p}$, $q = (q_i)_{i=1,\ldots,n}$, $w = (w_{ij})_{i,j=1,\ldots,n}$,

* Stanisław Staszic University of Mining and Metallurgy, Institute of Mathematics.

$B(D) := \{z = (z_1, z_2, \ldots, z_p)\colon z_l\colon D \to R$ is a bounded function, $(l = 1, \ldots, p)\}$ are such that

$$(2.1)\quad f_l(x, y, q, w, z) - f_l(x, \bar{y}, \bar{q}, \bar{w}, \bar{z}) = \sum_{\mu=1}^{p} \alpha_{l\mu}(y_\mu - \bar{y}_\mu) + \sum_{i=1}^{n} \beta_{li}(q_i - \bar{q}_i) + \sum_{i,j=1}^{n} \gamma_{lij}(w_{ij} - \bar{w}_{ij}) + \varkappa_l \|z - \bar{z}\|$$

$(l = 1, \ldots, p)$ for any two points (x, y, q, w, z), $(x, \bar{y}, \bar{q}, \bar{w}, \bar{z}) \in E$, and $\alpha_{l\mu}$, β_{li}, γ_{lij}, $\varkappa_l$ $(l, \mu = 1, \ldots, p;\ i, j = 1, \ldots, n)$ are functions defined for all points $(x, y, \bar{y}, q, \bar{q}, w, \bar{w}, z, \bar{z}) \in D \times \boldsymbol{R}^{2p} \times \boldsymbol{R}^{2n} \times \boldsymbol{R}^{2n^2} \times [B(D)]^2$ and bounded in this set.

The norm $\|\ \|$ in the space $B(D)$ is defined by the formula

$$(2.2)\qquad \|z\| := \max_{1 \leqslant l \leqslant p} \{\sup_{x \in D} |z_l(x)|\} \quad \text{for } z \in B(D).$$

(2) The functions $\alpha_{l\mu}$, β_{li}, γ_{lij}, $\varkappa_l$ $(l, \mu = 1, \ldots, p;\ i, j = 1, \ldots, n)$ satisfy the following conditions (for all admissible arguments):

$$(2.3)\qquad \alpha_{ll} \leqslant L < 0, \quad 0 \leqslant \alpha_{l\mu} \leqslant J \quad (l \neq \mu),$$

$$(2.4)\qquad |\beta_{li}| \leqslant \Gamma,$$

$$(2.5)\qquad 0 < g \leqslant \gamma_{lii} - \sum_{\substack{i,j=1 \\ i \neq j}}^{n} |\gamma_{lij}|,$$

$$(2.6)\qquad \gamma_{lij} = \gamma_{lji},$$

(2.7) for each pair of indices i, j $(i \neq j)$ the function γ_{lij} is either always non-positive, or always non-negative,

$$(2.8)\qquad |\varkappa_l| \leqslant K.$$

(3) There exists a solution $u(x)$ of problem (1.1), (1.2) such that $u \in C^2(D)$.

(4) The inequality

$$(2.9)\qquad L + J(p-1) + K < 0$$

is satisfied.

3. We introduce the uniform net in the cube D. Given a sequence of indices $M = (m_1, \ldots, m_n)$, $m_i = 0, 1, \ldots, N$ $(i = 1, \ldots, n)$, we denote by x^M the nodal point with the coordinates $x^M = (x_1^{m_1}, \ldots, x_n^{m_n})$, where $x_i^{m_i} = m_i h$ $(i = 1, \ldots, n)$, $0 < h = X/N$ and $N \geqslant 2$.

We introduce the following denotations:

$$
\begin{aligned}
&Z := \{M\colon\ 0 \leqslant m_i \leqslant N,\ i = 1, \ldots, n\},\\
(3.1)\qquad &Z_1 := \{M\colon\ 1 \leqslant m_i \leqslant N,\ i = 1, \ldots, n\},\\
&Z_2 := \{M\colon\ 0 \leqslant m_i \leqslant N-1,\ i = 1, \ldots, n\},
\end{aligned}
$$

and we introduce special symbols for the nodal points

$$
(3.2)\qquad
\begin{aligned}
-i(M) &:= (m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n) \quad (M \in Z_1),\\
i(M) &:= (m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n) \quad (M \in Z_2)
\end{aligned}
$$

$(i = 1, \ldots, n)$.

For any net function a: $Z \ni M \to a^M \in \boldsymbol{R}$ the following operators are defined:

$$
\begin{aligned}
&a^{Mi} = 0.5\,h^{-1}(a^{i(M)} - a^{-i(M)}),\\
(3.3)\quad &a^{-Mij} = 0.5\,h^{-2}(a^{i(M)} + a^{j(M)} + a^{-i(M)} + a^{-j(M)} - 2a^M - a^{i(-j(M))} - a^{-i(j(M))}),\\
&a^{+Mij} = 0.5\,h^{-2}(-a^{i(M)} - a^{j(M)} - a^{-i(M)} - a^{-j(M)} + 2a^M + a^{i(j(M))} + a^{-i(-j(M))})\\
&\qquad\qquad (M \in Z_1 \cap Z_2;\ i, j = 1, \ldots, n)
\end{aligned}
$$

Every function $b = (b_1, \ldots, b_p) \in B(D)$ is approximated by $b^* = (b_1^*, \ldots, b_p^*) \in B(D)$, where

$$
(3.4)\qquad b_\mu^*(x) := \sum_{M \in Z} \chi_M(x)\, b_\mu^M,
$$

$$
(3.5)\qquad
\begin{aligned}
&\chi_M(x) := \begin{cases} 1 & \text{for} \quad x \in I_M,\\ 0 & \text{for} \quad x \notin I_M, \end{cases}\\
&I_M := \{x \in D\colon\ m_i h \leqslant x_i < (m_i + 1)h,\ i = 1, \ldots, n\},
\end{aligned}
$$

$$
(3.6)\qquad b_\mu^M := b_\mu(x^M) \quad (\mu = 1, \ldots, p;\ M \in Z).
$$

In the same way, for every net function c: $Z \ni M \to c^M \in \boldsymbol{R}^p$ we define c^*:

$$
(3.4')\qquad c_\mu^*(x) := \sum_{M \in Z} \chi_M(x)\, c_\mu^M \quad (\mu = 1, \ldots, p),
$$

where $\chi_M(x)$ and I_M are defined by (3.5).

Analog with the system of differential-functional equations (1.1), (1.2) we consider the system of difference equations

$$
(3.7)\qquad f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*) = 0 \quad (M \in Z_1 \cap Z_2;\ l = 1, \ldots, p)
$$

with the boundary conditions

$$
(3.8)\qquad v_l^M = \varphi_l^M := \varphi_l(x^M) \quad \text{for } M \in Z \backslash (Z_1 \cap Z_2)\ (l = 1, \ldots, p),
$$

where $v^M = (v_\mu^M)_{\mu = 1, \ldots, p}$, $v_l^{MI} = (v_l^{Mi})_{i = 1, \ldots, n}$, $v_l^{MIJ} = (v_l^{Mij})_{i,j = 1, \ldots, n}$,

$$
v_l^{Mij} := \begin{cases} v_l^{-Mij} & \text{for } i = j \text{ or } \gamma_{lij} \leqslant 0,\\ v_l^{+Mij} & \text{for } i \neq j \text{ and } \gamma_{lij} \geqslant 0. \end{cases}
$$

The operators v_l^{Mi}, v_l^{-Mij}, v_l^{+Mij} $(l = 1, \ldots, p;\ i, j = 1, \ldots, n)$ and the function v^* are defined by (3.3), (3.4'), (3.5), (3.6).

System (3.7) and the boundary conditions (3.8) are generated by system (1.1) with the boundary conditions (1.2). Let us denote by v and u the solutions of these problems, respectively. The solution u of problem (1.1), (1.2) is not, in general, a solution of problem (3.7), (3.8) (with $u^M := u(x^M)$ for $M \in Z$). So, we can write

$$(3.9)\qquad f_l(x^M, u^M, u_l^{MI}, u_l^{MIJ}, u^*) = \varepsilon_l^M(h) \qquad (M \in Z_1 \cap Z_2;\ l = 1, \ldots, p).$$

From assumption (1), the regularity of the functions u_l $(l = 1, \ldots, p)$ and the definition of u^* (see (3.4)–(3.6)) we have

$$(3.10)\qquad \lim_{h\to 0} \varepsilon_l^M(h) = 0 \quad \text{for } M \in Z_1 \cap Z_2 \text{ and } l = 1, \ldots, p.$$

Let us write

$$(3.11)\qquad r^M := u^M - v^M = (r_\mu^M)_{\mu=1,\ldots,p} \qquad (M \in Z);$$

then

$$(3.12)\qquad r_\mu^{Mi} = u_\mu^{Mi} - v_\mu^{Mi}, \quad r_\mu^{Mij} = u_\mu^{Mij} - v_\mu^{Mij}, \quad r_\mu^* = u_\mu^* - v_\mu^*$$
$$(\mu = 1, \ldots, p;\ i, j = 1, \ldots, n;\ M \in Z_1 \cap Z_2).$$

Formulas (3.12) are true also for arbitrary net functions for which equation (3.11) is fulfilled.

4. THEOREM 1. *Under assumptions* H, *if the mesh size* h *satisfies the inequality*

$$(4.1)\qquad g/h - 0.5\,\Gamma \geqslant 0,$$

then the difference method (3.7), (3.8) *is convergent and we have the following estimation for the error*:

$$(4.2)\qquad |r_l^M| \leqslant -\frac{\varepsilon(h)}{L + J(p-1) + K} \qquad (M \in Z;\ l = 1, \ldots, p),$$

where

$$(4.3)\qquad \varepsilon(h) := \max_{\substack{M \in Z_1 \cap Z_2 \\ 1 \leqslant l \leqslant p}} |\varepsilon_l^M(h)|.$$

Proof. The sets Z and $\{1, \ldots, p\}$ are finite, so there exists $A \in Z$ and $k \in \{1, \ldots, p\}$ such that

$$(4.4)\qquad |r_k^A| = \max_{\substack{1 \leqslant l \leqslant p \\ M \in Z}} |r_l^M|,$$

where r is the function defined by (3.11).

We shall show that

$$|r_k^A| \leqslant -\frac{\varepsilon(h)}{L+J(p-1)+K}. \tag{4.5}$$

If $A \in Z\backslash(Z_1 \cap Z_2)$, then $r_k^A = 0$, because of the boundary conditions (1.2) and (3.8). Thus (4.5) holds in this case.

Now we assume that $A \in Z_1 \cap Z_2$. On account of (3.9), (3.7), assumption (2.1) and formulas (3.12) we have

$$\begin{aligned} \varepsilon_l^M(h) &= f_l(x^M, u^M, u_l^{MI}, u_l^{MIJ}, u^*) \\ &= f_l(x^M, u^M, u_l^{MI}, u_l^{MIJ}, u^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*) \\ &= \sum_{\mu=1}^{p} \alpha_{l\mu} r_\mu^M + \sum_{i=1}^{n} \beta_{li} r_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij} r_l^{Mij} + \varkappa_l \|r^*\|. \end{aligned} \tag{4.6}$$

We also have

$$\|r^*\| = \max_{1 \leqslant l \leqslant p} \{\sup_{x \in D} |\sum_{M \in Z} \chi_M(x) r_l^M|\} = \max_{1 \leqslant l \leqslant p} \{\max_{M \in Z} |r_l^M|\} = |r_k^A|, \tag{4.7}$$

in view of definition (2.2) of the norm $\|\ \|$.

Now we can write (4.6) in the form

$$\varepsilon_l^M(h) = \sum_{\mu=1}^{p} \alpha_{l\mu} r_\mu^M + \sum_{i=1}^{n} \beta_{li} r_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij} r_l^{Mij} + \varkappa_l |r_k^A| \tag{4.8}$$

$(M \in Z_1 \cap Z_2;\ l = 1, \ldots, p)$.

For $M = A$ and $l = k$, from equality (4.8) and definition (4.3) of $\varepsilon(h)$ we obtain two inequalities:

$$\sum_{\mu=1}^{p} \alpha_{k\mu} r_\mu^A + \sum_{i=1}^{n} \beta_{ki} r_k^{Ai} + \sum_{i,j=1}^{n} \gamma_{kij} r_k^{Aij} + \varkappa_k |r_k^A| \leqslant \varepsilon(h), \tag{4.9}$$

$$\sum_{\mu=1}^{p} \alpha_{k\mu} r_\mu^A + \sum_{i=1}^{n} \beta_{ki} r_k^{Ai} + \sum_{i,j=1}^{n} \gamma_{kij} r_k^{Aij} + \varkappa_k |r_k^A| \geqslant -\varepsilon(h). \tag{4.10}$$

Now we shall examine the two cases:

(i) We assume that $r_k^A \geqslant 0$. Then we can repeat the argument of the proof of Theorem 1 in [1] and we obtain the inequality

$$\sum_{i=1}^{n} \beta_{ki} r_k^{Ai} + \sum_{i,j=1}^{n} \gamma_{kij} r_k^{Aij} \leqslant 0. \tag{4.11}$$

From (4.10) we have

$$(4.12)\qquad -\varepsilon(h) \leqslant \sum_{\mu=1}^{p} \alpha_{k\mu} r_\mu^A + \varkappa_k r_k^A = \alpha_{kk} r_k^A + \sum_{\substack{\mu=1\\ \mu\neq k}}^{p} \alpha_{k\mu} r_\mu^A + \varkappa_k r_k^A$$

$$\leqslant Lr_k^A + J(p-1) r_k^A + Kr_k^A = \big(L+J(p-1)+K\big) r_k^A$$

in view of (4.11), (2.3), (2.8) and (i). From the above and (2.9) we get (4.5).

(ii) We assume that $r_k^A \leqslant 0$. Now, we define the net function $\bar{r}$: $Z \ni M \to \bar{r}^M \in \boldsymbol{R}^p$, where

$$(4.13)\qquad \bar{r}^M := -r^M = v^M - u^M, \quad \bar{r}^M = (\bar{r}_\mu^M)_{\mu=1,\ldots,p}.$$

In this case we have

$$(4.14)\qquad |r_k^A| = |-r_k^A| = |\bar{r}_k^A| = \bar{r}_k^A,$$

since $\bar{r}_k^A \geqslant 0$. Multiplying inequality (4.9) by -1 we hence obtain

$$(4.15)\qquad \sum_{\mu=1}^{p} \alpha_{k\mu} \bar{r}_\mu^A + \sum_{i=1}^{n} \beta_{ki} \bar{r}_k^{Ai} + \sum_{i,j=1}^{n} \gamma_{kij} \bar{r}_k^{Aij} - \varkappa_k \bar{r}_k^A \geqslant -\varepsilon(h),$$

and we are in the situation of case (i) with $\bar{r}$ instead of r and $-\varkappa_k$ instead of $\varkappa_k$.

This completes the proof of the error estimate (4.2).

The difference method (3.7), (3.8) is convergent, because $\lim_{h\to 0} \varepsilon(h) = 0$ (see (3.10) and (4.3)), in view of the error estimate (4.2). Hence we have $\lim_{h\to 0} r_l^M = 0$ for $M \in Z$ and $l \in \{1, \ldots, p\}$.

The proof of Theorem 1 is now complete.

5. Along with the difference problem (3.7), (3.8) we consider here the perturbed difference problem

$$(5.1)\quad f_l(x^M, w^M, w_l^{MI}, w_l^{MIJ}, w^*) = \eta_l^M(h) \quad (M \in Z_1 \cap Z_2;\ l = 1, \ldots, p),$$

$$(5.2)\qquad w_l^M = \varphi_l^M + \theta_l^M(h) \quad (M \in Z \setminus (Z_1 \cap Z_2);\ l = 1, \ldots, p).$$

THEOREM 2. *If the assumptions of Theorem* 1 *are satisfied, then*

$$(5.3)\qquad \max_{\substack{M\in Z\\ 1\leqslant l\leqslant p}} |v_l^M - w_l^M| \leqslant Q(h),$$

where v is the solution of problem (3.7), (3.8), *w is the solution of* (5.1), (5.2),

$$(5.4)\qquad Q(h) := \max\left(\theta(h), \frac{-\eta(h)}{L+J(p-1)+K}\right),$$

$$(5.5)\qquad \theta(h) := \max_{\substack{M\in Z\setminus(Z_1\cap Z_2)\\ 1\leqslant l\leqslant p}} |\theta_l^M(h)|,$$

(5.6) $$\eta(h) := \max_{\substack{M\in Z_1\cap Z_2\\ 1\leqslant l\leqslant p}} |\eta_l^M(h)|.$$

Proof. We define the net function r: $Z \ni M \to r^M \in \boldsymbol{R}^p$, $r^M := v^M - w^M = (r_\mu^M)_{\mu=1,\ldots,p}$. There exist $A \in Z$ and $k \in \{1, \ldots, p\}$ such that $|r_k^A| = \max\limits_{\substack{M\in Z\\ 1\leqslant l\leqslant p}} |r_l^M|$.

Now we shall examine the two cases:

(i) We assume that $A \in Z\setminus(Z_1 \cap Z_2)$. Then

(5.7) $$\max_{\substack{M\in Z\\ 1\leqslant l\leqslant p}} |r_l^M| = |r_k^A| = |\theta_k^A(h)| \leqslant \theta(h) \leqslant Q(h).$$

(ii) Now, we assume that $A \in Z_1 \cap Z_2$. We repeat the argument of the proof of Theorem 1 (the case $A \in Z_1 \cap Z_2$) with $\eta(h)$ instead of $\varepsilon(h)$ and we obtain

(5.8) $$\max_{\substack{M\in Z\\ 1\leqslant l\leqslant p}} |r_l^M| \leqslant \frac{-\eta(h)}{L+J(p-1)+K} \leqslant Q(h).$$

This ends the proof of Theorem 2.

Remark 1. Theorem 2 yields the corollary: if $\lim\limits_{h\to 0}\eta(h) = 0$ and $\lim\limits_{h\to 0}\theta(h) = 0$, then $\lim\limits_{h\to 0} Q(h) = 0$. In consequence, $\lim\limits_{h\to 0} w_l^M = v_l^M$ for each $M \in Z$ and $l \in \{1, \ldots, p\}$.

References

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