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Dependence of a differential equation on the first eigenvalue of a suitable problem

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Abstract. In this paper a new version of the known results on the inverse problem of Sturm Liouville type is given. The main result of this paper is contained in Theorem 4, whose advantage is that its assumptions concerns only the first eigenvalue, contrary to the previous theorems, where the assumptions involve the whole sequence of eigenvalues. The method used in this paper is also new and differ from the previous methods.

Introduction. The purpose of this paper is to give an alternative from of the known results on the so-called inverse problem of Sturm Liouville type (see [1], [3], [4], [7], [8]). This problem mainly consists in determining the dependence of the differential equation on the set of its eigenvalues. All the known (so far) results concerning this problem require the knowledge of the entire set of eigenvalues. This is a disadvantage from the point view of applications.

Let D be a bounded domain in the space E^m. We assume that the boundary ∂D of D is a surface of class C_σ^1 (for the definition of a surface of class C_σ^1 see [6], p. 148). In the sequel we denote by $X = (x_1, \ldots, x_m)$ point of E^m.

In the domain D we shall consider the problem of eigenvalues and eigenfunctions for the differential equation of the form

$$\Delta u + [\lambda - tq(X)]u = 0, \quad t \in R \tag{1}$$

with the boundary condition

$$\frac{du}{dn} - h(X)u = 0 \quad \text{on } \partial D - \Gamma, \quad u = 0 \quad \text{on } \Gamma, \tag{2}$$

n being the interior normal to ∂D and Γ denoting an $(m-1)$ dimensional part of ∂D (Γ being connected or not). We assume that h is a non-negative continuous function defined on $\bar{D}$. The boundary condition (2) may be taken in the sense of generalization (cf. [2]).

1. Dependence of the first eigenvalue of problem (1), (2) on the parameter t. Let us denote by $\lambda_1(t)$ the first eigenvalue of problem (1), (2) for a fixed $t \in R$, and let φ_t denote the first eigenfunction of this problem corresponding to the eigenvalue $\lambda_1(t)$. The eigenvalues and eigenfunctions of problem (1), (2) will be defined variationally (cf. [5] or [2]). Accordingly, the first eigenvalue $\lambda_1(t)$ of problem (1), (2) is defined as

$$(3) \qquad \lambda_1(t) = \min_{\varphi\in K}\Big\{\int_D [\mathrm{grad}^2\varphi + tq(X)\varphi^2]\,dX + \int_{\partial D-\Gamma} h(X)\varphi^2\,dS\Big\},$$

where K is the set of functions φ of class C^1_σ in D such that $\|\varphi\|_{L^2(D)} = 1$, $\varphi = 0$ on Γ (the definition of functions of class C^1_σ is given in [6], II, p. 300); the first eigenfunction φ_t is that at which minimum (3) is attained. As is known (cf. [5]), from the assumptions on the regularity of the domain D it follows that the minimum (3) is realized by a function φ_t, which is of class C^2 in D and satisfies equation (1) and the boundary condition (2). It is also known (cf. [2]) that the function φ_t is uniquely and φ_t preserves its sign in the domain D.

We shall prove the following

THEOREM 1. *If the domain D satisfies the assumptions formulated in the introduction and if q is a continuous function in the closure $\bar{D}$ of D, then the first eigenvalue of problem* (1), (2) $\lambda_1 = \lambda_1(t)$ *is a continuous function and satisfies the Lipschitz condition for $t \in R$.*

Proof. Let $t \in R$ be a fixed number and let $k \in R$, $k \neq 0$. For every function $\varphi \in K$ we have the following equality:

$$(4) \qquad \int_D [\mathrm{grad}^2\varphi + (t+k)q(X)\varphi^2]\,dX + \int_{\partial D-\Gamma} h(X)\varphi^2\,dS$$
$$= \int_D [\mathrm{grad}^2\varphi + tq(X)\varphi^2]\,dX + \int_{\partial D-\Gamma} h(X)\varphi^2\,dS + k\int_D q(X)\varphi^2\,dX.$$

Putting in equality (4) $\varphi = \varphi_{t+k}$, we get

$$\lambda_1(t+k) = \int_D [\mathrm{grad}^2\varphi_{t+k} + tq(X)\varphi_{t+k}^2]\,dX + \int_{\partial D-\Gamma} h(X)\varphi_{t+k}^2\,dS +$$
$$+ k\int_D q(X)\varphi_{t+k}^2\,dX.$$

Therefore

$$\lambda_1(t+k) \geqslant \lambda_1(t) + k\int_D q(X)\varphi_{t+k}^2\,dX,$$

or

$$(5) \qquad \lambda_1(t+k) - \lambda_1(t) \geqslant k\int_D q(X)\varphi_{t+k}^2\,dX.$$

Analogously, from equality (4) we get

$$\lambda_1(t+k)-\lambda_1(t) \leqslant k \int_D q(X)\varphi_t^2\,dX. \tag{6}$$

From the assumptions imposed on the function q it follows that

$$\exists M \geqslant 0 \ \forall X \in \bar{D}\colon \ |q(X)| \leqslant M.$$

From this and from inequalities (5) and (6) we have

$$|\lambda_1(t+k)-\lambda_1(t)| \leqslant M|k|. \tag{7}$$

Inequality (7) is just the assertion of Theorem 1.

THEOREM 2. *Under the assumptions of Theorem* 1, *the function* $\lambda_1 = \lambda_1(t)$ *is differentiable almost everywhere and*

$$\lambda_1'(t) = \int_D q(X)\varphi_t^2\,dX \tag{8}$$

for almost every point $t \in R$.

Proof. The existence of a finite derivative of the function $\lambda_1(t)$, for almost every point $t \in R$, follows from inequality (7) (cf. [9], Chapter VII). To prove equality (8), observe that, owing to inequalities (5) and (6) we have

$$k \int_D q(X)\varphi_{t+k}^2\,dX \leqslant \lambda_1(t+k)-\lambda_1(t) \leqslant k \int_D q(X)\varphi_t^2\,dX. \tag{9}$$

Let k in inequality (9) be a positive number; then

$$\int_D q(X)\varphi_{t+k}^2\,dX \leqslant \frac{\lambda_1(t+k)-\lambda_1(t)}{k} \leqslant \int_D q(X)\varphi_t^2\,dX. \tag{10}$$

If $k < 0$, then

$$\int_D q(X)\varphi_t^2\,dX \leqslant \frac{\lambda_1(t+k)-\lambda_1(t)}{k} \leqslant \int_D q(X)\varphi_{t+k}^2\,dX. \tag{11}$$

From (10) and (11) we get

$$\lim_{k\to 0^+} \frac{\lambda_1(t+k)-\lambda_1(t)}{k} \leqslant \int_D q(X)\varphi_t^2\,dX \leqslant \lim_{k\to 0^-} \frac{\lambda_1(t+k)-\lambda_1(t)}{k}. \tag{12}$$

Inequality (12) yields the assertion of Theorem 2.

THEOREM 3. *Under the assumptions of Theorem* 1, *if there exist numbers* $t_1, t_2 \in R$ *such that* $\lambda_1(t_1) = \lambda_1(t_2)$, *then*

$$(t_2-t_1) \int_D q(X)\varphi_{t_1}^2\,dX \geqslant 0. \tag{13}$$

Proof. According to the definition,

$$\lambda_1(t_1) = \int_D [\mathrm{grad}^2\varphi_{t_1} + t_1 q(X)\varphi_{t_1}^2]dX + \int_{\partial D - \Gamma} h(X)\varphi_{t_1}^2 dS,$$

where φ_{t_1} is the only function which realizes the minimum in (3) for $t = t_1$, and so

$$\int_D [\mathrm{grad}^2\varphi_{t_1} + t_2 q(X)\varphi_{t_1}^2]dX + \int_{\partial D - \Gamma} h(X)\varphi_{t_1}^2 dS \geqslant \lambda_1(t_2) = \lambda_1(t_1)$$
$$= \int_D [\mathrm{grad}^2\varphi_{t_1} + t_1 q(X)\varphi_{t_1}^2]dX + \int_{\partial D - \Gamma} h(X)\varphi_{t_1}^2 dS.$$

Hence

$$(t_2 - t_1)\int_D q(X)\varphi_{t_1}^2 dX \geqslant 0.$$

The proof is complete.

2. The generalization of Ambarzumian's theorem. We shall now formulate and prove the main result of this paper.

THEOREM 4. *If the domain D satisfies the assumptions formulated in the introduction, q is a continuous function in the closure $\bar{D}$ of D and if there exist real numbers a, β such that*

$$\lambda_1(0) = \lambda_1(a) = \lambda_1(\beta), \tag{14}$$

where $a > \beta > 0$, then for every point $X \in D$ we have $q(X) = 0$.

Proof. According to inequality (13) with $t_1 = \beta$, $t_2 = 0$, we have

$$-\beta \int_D q(X)\varphi_\beta^2 dX \geqslant 0,$$

and so

$$\int_D q(X)\varphi_\beta^2 dX \leqslant 0. \tag{15}$$

Putting in (13) $t_1 = \beta$, $t_2 = a$, we get

$$(a - \beta)\int_D q(X)\varphi_\beta^2 dX \geqslant 0;$$

therefore

$$\int_D q(X)\varphi_\beta^2 dX \geqslant 0. \tag{16}$$

Inequalities (15) and (16) result in:

$$\int_D q(X)\varphi_\beta^2 dX = 0. \tag{17}$$

From equality (17) we get

$$\int\limits_{D} \operatorname{grad}^2 \varphi_\beta \, dX + \int\limits_{\partial D - \Gamma} h(X) \varphi_\beta^2 \, dS = \lambda_1(\beta). \tag{18}$$

By assumption $\lambda_1(0) = \lambda_1(\beta)$. Since

$$\lambda_1(0) = \min_{\varphi \in K} \Big\{ \int\limits_{D} \operatorname{grad}^2 \varphi \, dX + \int\limits_{\partial D - \Gamma} h(X) \varphi^2 \, dS \Big\}, \tag{19}$$

it follows from (18) and (19) that the function φ_β realizes the minimum in (19). On the other hand, the function φ_β is an eigenfunction of problem (1), (2) with $t = \beta$. As is known (cf. [2] or [5]), the first eigenfunction of problem (1), (2) is determined uniquely, satisfies equation (1) with $t = \beta$ and the boundary condition (2). This means that the function φ_β satisfies equation (1) with $t = 0$ and $t = \beta \neq 0$. Therefore, for every point $X \in D$ we have the equality

$$\beta q(X) \varphi_\beta(X) = 0. \tag{20}$$

Since $\beta \neq 0$ and $\varphi_\beta(X) \neq 0$ for the every point $X \in D$, the assertion of Theorem 4 follows from equality (20).

Remark 1. Theorem 4 is just a new version of Kuzniecov's theorem (see [7]) and of the results of papers [3] and [4]. The difference lies in that the assumptions of Theorem 4 concern the first eigenvalue of problem (1), (2), whereas the assumptions of previous theorems involve the whole sequence of eigenvalues of problem (1), (2).

Theorem 4 may be formulated in the following form:

THEOREM 4′. *If the domain D satisfies the assumptions formulated in the introduction, q is a continuous function in the closure $\bar{D}$ of D and if the first eigenvalues of the equations*

$$\Delta u + \lambda u = 0, \qquad \Delta u + [\lambda - q(X)] u = 0, \qquad \Delta u + [\lambda - t q(X)] u = 0,$$

$t > 0$ and $t \neq 1$, with boundary condition (2), are equal, then $q(X) = 0$ for every point $X \in D$.

Remark 2. All the results of this paper may be carried to the case of a more general equation of form (1), where the Laplace operator Δ is replaced by any operator L, where

$$Lu = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left[a_{ij}(X) \frac{\partial u}{\partial x_j} \right],$$

such that $a_{ij}(X) = a_{ij}(X)$, $i, j = 1, \ldots, m$, are of class $C^1(D)$, and the quadratic form

$$\sum_{i,j=1}^{m} a_{ij}(X) \xi_i \xi_j$$

is positive definite in the domain D.

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