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A note on boundary value problems for second order differential equations

by L. H. Erbe* (Edmonton, Canada) and
H. W. Knobloch (Würzburg, Germany)

Zdzisław Opial in memoriam

Abstract. We consider questions of existence-uniqueness and smooth dependence on parameters for second order differential systems. We illustrate our results for the scalar second order BVP: $x'' = f(t, x)$, $\dot{x}(0) = x_0$, $x(T) = x_1$.

1. Introduction. Consider the second order differential system

$$x'' = A(t)x' + f(t, x), \tag{1.1.}$$

where $x = (x_1, \ldots, x_n)^T$ and $f(t, x)$ are real n-dimensional column vectors and $A(t)$ is a real $n \times n$ matrix-valued function which is defined and continuous on $[0, T]$. We assume also that $f(t, x)$ and the Jacobian matrix of $f(t, x)$ with respect to x, denoted by $F(t, x)$ are continuous on a bounded open set Ω in (t, x) space containing the set

$$\mathscr{P} = \{(t, x)\colon\ 0 \leqslant t \leqslant T,\ x^T P^{-1}(t) x \leqslant R^2\}. \tag{1.2}$$

Here $P(t)$ is a given symmetric positive definite matrix which is defined and elementwise of class C^2 on $[0, T]$ and $R > 0$ is a given real number.

The basic problem which we propose to investigate here is the following: Suppose there exists a solution $\hat{x} = \hat{x}(t)$ of (1.1) with $(t, \hat{x}(t)) \in \Omega$ on $[0, T]$. We then obtain conditions under which there will exist a solution of (1.1) which remains in the set

$$\hat{\mathscr{P}} = \{(t, x)\colon\ 0 \leqslant t \leqslant T,\ (x - \hat{x}(t))^T P^{-1}(t)(x - \hat{x}(t)) \leqslant R^2\} \tag{1.3}$$

and which satisfies certain two-point boundary conditions. Moreover, we are also able to make certain statements concerning the uniqueness of the solution, a certain maximum principle, and continuous (in fact, smooth) dependence on boundary data. In Section 2 below we recall some previous results of [2]

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and then illustrate how one may apply these to obtain a result for the scalar second order nonlinear equation. The results may be considered as extending the existence-uniqueness (and estimates for the solutions) for the second order equation

$$x'' = kx, \quad k > 0, \tag{1.4}$$

or more generally

$$x'' = f(t, x), \quad f_x(t, x) > 0, \tag{1.5}$$

with $x(0) = x_0$, $x(T) = x_1$. We illustrate our results by an example in Section 3. Although higher dimensional analogues of our results are possible, we shall concentrate on existence-uniqueness-smooth dependence statements for the scalar case. We refer to [1], [3]–[10] and the references therein for a further discussion of additional related results concerning existence-uniqueness and continuous dependence on boundary data.

2. We suppose that $P(t)$ is a given positive definite symmetric matrix which is elementwise of class C^2 on $[0, T]$ and suppose further that $x = \tilde{x}(t)$ is a given solution of

$$x'' = f(t, x) \tag{2.1}$$

with $(t, \tilde{x}(t)) \in \Omega$ on $[0, T]$. Consider the quadratic forms

$$\varphi(t, x) \equiv x^T P^{-1}(t) x, \quad \tilde{\varphi}(t, x) \equiv \varphi(t, x - \tilde{x}(t)) \tag{2.2}$$

and let $\mathscr{P}_\delta = \{(t, x)\colon 0 < t < T,\ \tilde{\varphi}(t, x) < \delta^2\}$. We may then rephrase Theorem 2 of [8] as follows:

THEOREM 2.1. *Let $P(t)$ be as above and assume that*

$$F(t, x)P(t) + P(t)F(t, x)^T - P''(t) > 0, \quad (t, x) \in \Omega,\ 0 \leqslant t \leqslant T \tag{2.3}$$

and assume $\bar{P}_\delta \subseteq \Omega$, $\tilde{\varphi}(0, x_0) \leqslant \delta^2$, $\tilde{\varphi}(T, x_1) \leqslant \delta^2$. Then the boundary value problem (BVP)

$$x'' = f(t, x), \quad x(0) = x_0, \quad x(T) = x_1 \tag{2.4}$$

has a solution $x = x(t)$ with $\tilde{\varphi}(t, x(t)) \leqslant \delta^2$ on $[0, T]$.

A similar statement holds for the BVP

$$x'' = A(t)x' + f(t, x), \quad x(0) = x_0, \quad x(T) = x_1 \tag{2.5}$$

provided condition (2.3) *is replaced by*

$$C(t, x) \equiv F(t, x)P(t) + P(t)F(t, x)^T - P''(t) + P'(t)A(t)^T + A(t)P'(t) - \tfrac{1}{2}A(t)P(t)A^T(t) > 0 \quad \text{on } \Omega. \tag{2.6}$$

As a consequence we have

COROLLARY 2.2. *Consider the one-parameter system*

$$x'' = A(t, \lambda)x' + f(t, x, \lambda), \tag{2.7}$$

where $\lambda \in \Lambda$ is a given parameter set. Assume that (2.6) holds for all $(t, x) \in \Omega$ and all $\lambda \in \Lambda$ and that A, f are sufficiently smooth functions of all variables. Then the solution of (2.7) is a differentiable (hence continuous) function of the parameter λ.

Proof. We introduce the extended system

$$\begin{gathered} x'' = A(t, \lambda)x' + f(t, x, \lambda), \quad x(0) = x_0, \quad x(T) = x_1, \\ \lambda'' = 0, \quad \lambda(0) = \lambda(T) = \lambda_0. \end{gathered} \tag{2.8}$$

Existence and uniqueness of solutions to (2.8) is clear. The differentiability will follow if one knows that the linearization of (2.8) around some solution yields a differential equation for which the two-point BVP is uniquely solvable. The linearized equation for (2.8) is

$$\begin{aligned} &\xi'' = A(t, \lambda)\xi' + f_x(t, x, \lambda)\xi + \big(A_\lambda(t, \lambda)x' + f_\lambda(t, x, \lambda)\big)\eta, \\ &\eta'' = 0. \end{aligned} \tag{2.9}$$

Thus we have a system of the form (2.5) with

$$A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad F \to \begin{pmatrix} f_x & b \\ 0 & 0 \end{pmatrix}, \tag{2.10}$$

where $b = \big(A_\lambda(t, \lambda)x' + f_\lambda(t, x, \lambda)\big)$. If we choose

$$\hat{P} = \begin{pmatrix} P & 0 \\ 0 & \alpha \end{pmatrix},$$

then condition (2.6) takes the form

$$\hat{C} = \begin{pmatrix} C & \alpha b \\ \alpha b^T & -\alpha'' \end{pmatrix} > 0. \tag{2.11}$$

This will be true for an appropriate choice of α. (We may take $\alpha_0 \in C^2[0, T]$ with $\alpha_0 > 0$, $\alpha_0'' < 0$ on $[0, T]$ and then put $\alpha = \varepsilon\alpha_0$, $\varepsilon > 0$ sufficiently small.)

As an application we consider the scalar case.

PROPOSITION 2.3. *Let x, ξ be scalar and consider the two point* BVP *for the system*

$$x'' = f(t, x, \xi) + a_1(t)x' + a_2(t)\xi', \quad \xi'' = b_1(t)\xi + b_2(t)\xi' \tag{2.12}$$

together with the boundary condition

$$x(0) = x_0, \quad x(T) = x_1, \quad \xi(0) = \xi(T) = 0. \tag{2.13}$$

Assume that for the coupled system (2.12), (2.13) *the hypotheses of Theorem* 2.1 *hold. Then $\xi(t) = 0$ for all $0 \leqslant t \leqslant T$.*

Proof. Let $\tilde{\xi}(t)$, $\tilde{x}(t)$ be a given solution of (2.12), (2.13). We introduce a parameter λ which varies near T. Consider the BVP which is defined in terms of equation (2.12) and the conditions

$$x(0) = \tilde{x}(0), \quad x(\lambda) = \tilde{x}(T), \quad \xi(0) = \xi(\lambda) = 0. \tag{2.14}$$

Call this solution $x(t, \lambda)$, $\xi(t, \lambda)$. We claim that this solution depends continuously upon λ. To see this, note that by means of a time transformation the problem can be restated in such a way that the terminal time becomes T (instead of λ). The variable λ appears as a parameter in the differential equation and for $\lambda = T$ we obtain the given equation (2.12). For all λ sufficiently close to T condition (2.6) holds for this system. Hence by Corollary 2.2 the solutions are continuous in the variables t, λ. Now $\xi = \xi(t, \lambda)$ is a solution of the BVP

$$\xi'' = b_1(t)\xi + b_2(t)\xi', \quad \xi(0) = \xi(\lambda) = 0. \tag{2.15}$$

Since the eigenvalues of this problem are discrete, we have $\xi(t, \lambda) = 0$ for $0 \leqslant t \leqslant \lambda$ and all $\lambda < T$, with λ sufficiently close to T. Therefore, by continuity we have $\xi(t, T) = 0$ for all t.

Next suppose we are given a scalar equation

$$x'' = f(t, x), \quad t \in [0, T], \tag{2.16}$$

We extend (2.16) to the system

$$x'' = f(t, x) + \gamma(t)\xi + \delta(t)\xi', \quad \xi'' = \alpha(t)\xi + \beta(t)\xi'. \tag{2.17}$$

We shall always impose the boundary conditions $\xi(0) = \xi(T) = 0$ and hence it is clear by Proposition 2.3 that solutions of (2.17) are of the form $(x(t), 0)$, where $x(t)$ is a solution of (2.16). We will apply Theorem 2.1 with

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{pmatrix}, \tag{2.18}$$

where

$$p_{11}(t) > 0, \quad p_{22}(t) > 0, \quad \Delta(t) = p_{11}(t)p_{22}(t) - p_{12}^2(t) > 0. \tag{2.19}$$

Since ξ is always 0, the conclusion of the theorem is then that the BVP (2.4) is solvable and the solution satisfies the maximum principle:

$$\frac{p_{22}(t)}{\Delta(t)}(x(t) - \tilde{x}(t))^2 \text{ assumes its maximum on } [0, T] \text{ for } t = 0 \text{ or } t = T. \tag{2.20}$$

In the statement of the maximum principle, as it stands, α, β, γ, δ, p_{ij} are functions of t. However, in the subsequent analysis, they will be regarded as functions of t, x. The idea of the theorem is first applied with Ω being a sufficiently small neighbourhood of $(t, \tilde{x}(t))$ and the α, β, γ, δ, p_{ij} appearing in (2.17), (2.19) being actually $\alpha(t, \tilde{x}(t))$, $\beta(t, \tilde{x}(t))$, etc. Existence of the solution in case x_0, x_1, are not close to $\tilde{x}(0)$, $\tilde{x}(T)$ is then established using intermediate steps. That is, we solve successively the problem with boundary condition

$$x(0) = \tilde{x}(0)+\frac{i}{N}(x_0-\tilde{x}(0)), \qquad x(T) = \tilde{x}(T)+\frac{i}{N}(x_1-\tilde{x}(T))$$

for $i = 1,\dots, N$, with N sufficiently large. Inequality (2.20) is also then modified (now with $p_{22} = p_{22}(t, x)$, $\Delta = \Delta(t, x)$) and becomes

$$|x(t)-\tilde{x}(t)| \leqslant K\cdot\max\{M(0)|x(0)-\tilde{x}(0)|,\ M(T)|x(T)-\tilde{x}(T)|\}, \tag{2.21}$$

where

$$\frac{1}{K^2} = \min\left\{\frac{p_{22}(t, x)}{\Delta(t, x)}\colon (t, x)\in\Omega\right\} \tag{2.22}$$

and

$$M(t) = \max\left\{\left(\frac{p_{22}(t, x)}{\Delta(t, x)}\right)^{1/2}\colon (t, x)\in\Omega\right\}. \tag{2.23}$$

Accordingly, one needs to make the following assumption concerning the boundary values x_0, x_1 (cf. (2.4))

(2.24) $(t, x)\in\Omega$ whenever $0 \leqslant t \leqslant T$ and

$$|x-\tilde{x}(t)| \leqslant K\cdot\max\{M(0)|\tilde{x}(0)-x_0|, M(T)|\tilde{x}(T)-x_1|\}.$$

Henceforth, we shall regard p_{ij} as functions of t, x and α, β, γ, δ as functions of t, x, x'. We have then ($p = p_{ij}, \cdot = d/dt$)

$$\begin{aligned} \dot{p}(t, x) &= p_t(t, x)+p_x(t, x)x', \\ \ddot{p}(t, x) &= p_{tt}(t, x)+2p_{tx}(t, x)x'+p_{xx}(t, x)x'^2+p_x(t, x)f(t, x). \end{aligned} \tag{2.25}$$

We now wish to write down $C(t, x)$ (cf. (2.6)). We have

$$F = \begin{pmatrix} f_x & \gamma \\ 0 & \alpha \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \delta \\ 0 & \beta \end{pmatrix},$$

$$\begin{aligned} AP'+P'A^T &= \begin{pmatrix} \delta\dot{p}_{12} & \delta\dot{p}_{22} \\ \beta\dot{p}_{12} & \beta\dot{p}_{22} \end{pmatrix} + \begin{pmatrix} \delta\dot{p}_{12} & \beta\dot{p}_{12} \\ \delta\dot{p}_{22} & \beta\dot{p}_{22} \end{pmatrix} \\ &= \begin{pmatrix} 2\delta\dot{p}_{12} & \delta\dot{p}_{22}+\beta\dot{p}_{12} \\ \delta\dot{p}_{22}+\beta\dot{p}_{12} & 2\beta\dot{p}_{22} \end{pmatrix} \end{aligned}$$

and

$$APA^T = \begin{pmatrix} 0 & \delta \\ 0 & \beta \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \delta & \beta \end{pmatrix} = p_{22}\begin{pmatrix} \delta \\ \beta \end{pmatrix}(\delta\ \beta) = p_{22}\Lambda\Lambda^T,$$

where $\Lambda = \begin{pmatrix}\delta\\ \beta\end{pmatrix}$. Hence, we have

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix},$$

where

$$\begin{aligned} c_{11} &= 2(f_x p_{11} + \gamma p_{12}) - \ddot{p}_{11} + 2\delta \dot{p}_{12} - \tfrac{1}{2}\delta^2 p_{22}, \\ c_{12} &= f_x p_{12} + \gamma p_{22} + \alpha p_{12} - \dot{p}_{12} + \delta \dot{p}_{22} + \beta \dot{p}_{12} - \delta\beta p_{22}, \\ c_{22} &= 2\alpha p_{22} - \ddot{p}_{22} + 2\beta \dot{p}_{22} - \tfrac{1}{2}\beta^2 p_{22}. \end{aligned} \tag{2.26}$$

If we impose the condition $c_{12} = 0$, then we have

$$\gamma = -\frac{1}{p_{22}}(f_x p_{12} + \alpha p_{12} - \dot{p}_{12} + \delta \dot{p}_{22} + \beta \dot{p}_{12}) + \delta\beta. \tag{2.27}$$

Furthermore from the last relation in (2.26) we have

$$\alpha = \frac{1}{2p_{22}}(c_{22} + \ddot{p}_{22} - 2\beta \dot{p}_{22} + \tfrac{1}{2}\beta^2 p_{22}). \tag{2.28}$$

From (2.25) we may consider c_{11} as a quadratic form in $\dot{x}$, say $c_{11} = a_0 + a_1\dot{x} + a_2\dot{x}^2$. We observe from (2.25)–(2.26) that the coefficient of $\dot{x}^2$ is

$$a_2 = -(p_{11})_{xx} - \left(\frac{p_{12}}{p_{22}}\right)^2 (p_{22})_{xx}. \tag{2.29}$$

Also, some calculation gives

$$\begin{aligned} a_1 = &-\left(\frac{p_{12}}{p_{22}}\right)^2 [2(p_{22})_{tx} - 2\beta(p_{22})_x] \\ &- \frac{2p_{12}}{p_{22}}[-(p_{12})_x + \delta(p_{22})_x + \beta(p_{12})_x] + 2\delta(p_{12})_x - 2(p_{11})_{tx} \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} a_0 = {} & 2f_x\left[p_{11} - \frac{(p_{12})^2}{p_{22}}\right] - \left(\frac{p_{12}}{p_{22}}\right)^2 [(p_{22})_{tt} + (p_{22})_x f - 2\beta(p_{22})_t + \tfrac{1}{2}\beta^2 p_{22}] \\ & - 2\frac{p_{12}}{p_{22}}[(\beta - 1)(p_{12})_t + \delta(p_{22})_t + p_{22}\beta\delta] + 2\delta(p_{12})_t - \tfrac{1}{2}\delta^2 p_{22} - (p_{11})_{tt} - (p_{11})_t f. \end{aligned} \tag{2.31}$$

We may choose δ, β so that $a_1 = 0$ therefore $c_{11} > 0$ in case $a_0 > 0$ and $a_2 \geqslant 0$. Since α is determined by (2.28), we can guarantee that $c_{22} > 0$ by an appropriately chosen α. As a consequence, we conclude local existence and uniqueness for equation (2.16) under assumption (2.24) (and the above conditions which guarantee that $c_{11} > 0$, $c_{22} > 0$, $c_{12} = 0$).

If we denote $Q(f)(x') = a_0 + a_1 x' + a_2 x'^2$, where a_0, a_1, a_2 are given in (2.29)–(2.31) and if we know that $Q(f_1)(x') > 0$ and $Q(f_2)(x') > 0$ for $f_i = f_i(t, x)$, then from relation (2.31) it follows that $a_0(f_1) > 0$, $a_0(f_2) > 0$ and hence for all $0 \leqslant \varrho \leqslant 1$ we have $Q(\varrho f_1 + (1-\varrho) f_2)(x') > 0$. Therefore any convex combination of f_1 and f_2 will satisfy the hypotheses for local existence and uniqueness of solutions to the two point BVP provided (2.24) holds, where $\tilde{x}(t)$ is to be interpreted as a common reference solution for f_1 and f_2. In an application one may choose f_1 to be linear, say $f_1(t, x) = q(t)x$ (so that $\tilde{x} = 0$ is a reference solution). Furthermore, a simple continuity argument shows that we will have a global uniqueness statement – that is; given the BVP (2.4) and a reference solution $x = \tilde{x}(t)$ such that (2.24) holds (for appropriately chosen p_{ij}), it follows that the solution to (2.32) is unique.

3. We wish to illustrate the application of the previous results by means of an example in this section. We consider the equation

$$x'' = a(t)x + b(t)x^2 \tag{3.1}$$

and we take $\Omega_M = \{(t, x)\colon 0 \leqslant t \leqslant 1,\ 0 \leqslant x \leqslant M\}$, where $M > 0$ is arbitrary. Since $x \equiv 0$ is a solution of (3.1) we wish to find conditions under which (3.1) has a solution satisfying

$$x(0) = x_0, \qquad x(1) = x_1 \qquad \text{with } x_0, x_1 \geqslant 0 \tag{3.2}$$

and $x(t) \geqslant 0$ on $[0, 1]$. We choose $P(t) = (p_{ij}(t))_{2\times 2}$ with $p_{12}(t) \equiv 0$ $(\equiv p_{21}(t))$, $p_{11}(t) = p_0(t) + p_1 x$ ($p_1 = \text{const}$), $\delta \equiv 0$, and $p_{22}(t) > 0$ arbitrary. We have then from (2.29)–(2.31) that

$$a_2 = 0, \ a_1 = 0$$

and

$$a_0 = \big(2a(t)p_0(t) - p_0''(t)\big) + (4b(t)p_0(t) + p_1 a(t))x + 3p_1 b(t)x^2.$$

Therefore, $a_0 > 0$ provided the following conditions hold:

$$2a(t)p_0(t) - p_0''(t) \geqslant 0, \qquad 4b(t)p_0(t) + p_1 a(t) \geqslant 0 \tag{3.3}$$
$$3p_1 b(t) > 0, \qquad 0 \leqslant t \leqslant 1.$$

If (3.3) holds with $p_1 > 0$, $b(t) > 0$ on $[0, 1]$, then $\psi(t) \equiv M > 0$ is an upper solution of the differential equation for sufficiently large M and hence there exists a solution $x = \hat{x}(t)$ of (3.1) satisfying (3.2) for any $x_0, x_1 \geqslant 0$ (cf. [1]); furthermore $0 \leqslant \hat{x}(t) \leqslant M$ on $[0, 1]$. Existence follows also from a slight modification of Theorem 2.1. Uniqueness on the other hand, does not seem to be a consequence of known results in the literature nor does the smooth dependence on boundary conditions, as far as the authors are aware.

If, for example, the second order linear equation

$$y'' + 2a(t)y = 0 \tag{3.4}$$

is disconjugate on $[0, 1]$, $p_0(t)$ may be taken to be a positive solution of (3.4). Hence, with $b(t) > 0$ then it follows that (3.3) holds since one may take $p_1 > 0$ sufficiently small. In particular, (3.4) is disconjugate on $[0, 1]$ if

$$\int_0^1 a_+(t)dt \leqslant 2, \qquad a_+(t) = \max\{0, a(t)\}$$

(cf. [5], p. 346). One may further generalize this example to equations of the form

$$x'' + a(t)x + q(t, x) = 0 \tag{3.5}$$

with appropriate assumptions on $q(t, x)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA

MATHEMATISCHES INSTITUT DER UNIVERSITÄT
AM HUBLAND
WÜRZBURG, FEDERAL REPUBLIC OF GERMANY

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